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Théorie des groupes/*Group Theory*

Finitely presented simple groups and products of trees

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Abstract. We construct lattices in $\mathrm{Aut}\, T_n \times \mathrm{Aut}\, T_m$ which are finitely presented, torsion free, simple groups.

Groupes simples de présentation finie et produits d'arbres

Résumé. *Nous construisons des réseaux de* $\mathrm{Aut}\, T_n \times \mathrm{Aut}\, T_m$ *qui sont de présentation finie, simples et sans torsion.*

Version française abrégée

Nous construisons une famille infinie de groupes simples, sans torsion et de présentation finie. Pour chaque couple (n, m) d'entiers pairs suffisamment grands, nous construisons un complexe carré $\mathcal{Y}_{n,m}$ dont le revêtement universel est un produit $T_n \times T_m$ d'arbres réguliers, et dont le groupe fondamental $\Gamma_{n,m}$ jouit des propriétés suivantes :

THÉORÈME

(1) *Le groupe* $\Gamma_{n,m}$ *est simple, de présentation finie, isomorphe au produit libre amalgamé* $F *_G F$ *où* F, G *sont des groupes libres de type fini.*

(2) *Si* $\Gamma_{n,m}$ *est isomorphe à* $\Gamma_{k,l}$, *les complexes associés* $\mathcal{Y}_{n,m}$ *et* $\mathcal{Y}_{k,l}$ *sont isomorphes et* $\{n, m\} = \{k, l\}$.

REMARQUES

(1) L'énoncé (1) répond à une question de P.M. Neumann ([10] p. 5 et [7] problème 4.45).

(2) Dans la décomposition $\Gamma_{n,m} = F *_G F$, les deux injections de G dans F sont conjuguées par un automorphisme de G, d'ordre fini dans $Out(G)$.

(3) Toute décomposition non triviale de $\Gamma_{n,m}$ en produit amalgamé $L *_K H$ provient de l'action de $\Gamma_{n,m}$ sur T_n et T_m et l'on a $[L : K] = [H : K] = n$ et m respectivement.

(4) Le groupe $\Gamma_{n,m}$ est automatique (*voir* [6]).

Note présentée par Jacques TITS.

0. Introduction

We construct an infinite family of torsion free finitely presented simple groups. For every pair (n, m) of sufficiently large even integers, we construct a finite square complex $\mathcal{Y}_{n,m}$ whose universal covering is the product $T_n \times T_m$ of regular trees of respective degrees n and m and whose fundamental group $\Gamma_{n,m}$ enjoys the following properties:

THEOREM

(1) *The group $\Gamma_{n,m}$ is simple, finitely presented and isomorphic to a free amalgam $F *_G F$ where F, G are finitely generated free groups.*

(2) *If $\Gamma_{n,m}$ is isomorphic to $\Gamma_{k,l}$, then the corresponding complexes $\mathcal{Y}_{n,m}$ and $\mathcal{Y}_{k,l}$ are isomorphic, and $\{n, m\} = \{k, l\}$.*

REMARKS.

(1) Part (1) of the theorem answers a question of P.M. Neumann (*see* [10]) p. 5, *see* also [7] problem 4.45).

(2) The two embeddings of G into F, in the decomposition $\Gamma_{n,m} = F *_G F$, are conjugate by an automorphism of G, of finite order in $\mathrm{Out}(G)$.

(3)Any nontrivial decomposition of $\Gamma_{n,m}$ as a free amalgam $L *_K H$ comes from the action of $\Gamma_{n,m}$ on either T_n or T_m, and then $[L : K] = [H : K]$ is n or m respectively. The groups involved are finitely generated free groups.

(4) It follows from [8] that $\Gamma_{n,m}$ is automatic.

These examples grew out of an ongoing study of the structure and classification of lattices in the group of automorphisms of a product of trees (*see* [3] and [4]). In particular one has, for a certain class of lattices, a geometric rigidity theorem from which Theorem (2) and Remark (3) follow.

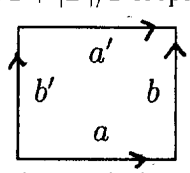
M. Bhattacharjee, motivated by P.M. Neumann's question, constructed in [1] a free amalgam $L *_K H$ of finitely generated free groups having no finite index subgroup. From a more geometric viewpoint D. Wise, [13], constructed a finite square complex covered by a product of two regular trees which does not admit any nontrivial finite covering.

1. CONSTRUCTION. - We first construct a square complex $\mathcal{X}_{n,m}$ with one vertex. The complex $\mathcal{Y}_{n,m}$ will be a finite covering of $\mathcal{X}_{n,m}$. A square complex $\mathcal{Y}$ on one vertex whose universal covering is $T_n \times T_m$ is given by two sets A,B ($n = |A|$, $m = |B|$) each endowed with a fixed point free involution (denoted $c \to c^{-1}$) and a subset $\mathcal{R} \subset A \times B \times B \times A$ such that:

(i) When $(a, b, b', a') \in \mathcal{R}$, then (a^{-1}, b', b, a'^{-1}), $(a'^{-1}, b'^{-1}, b^{-1}, a^{-1})$, (a', b^{-1}, b'^{-1}, a) belong to $\mathcal{R}$, and all four 4-tuples are distinct.

(ii) Each of the four projections of $\mathcal{R}$ to a subproduct of the form $A \times B$ or $B \times A$ is bijective.

We shall write $ab\Box b'a'$ when $(a, b, b', a') \in \mathcal{R}$.

The square complex $\mathcal{Y}$ is constructed by taking a bouquet of $|A|/2 + |B|/2$ loops whose set of oriented loops is $A \sqcup B$ and by constructing one "geometric square" 

for every four 4-tuples as in (i). Note that the link of the vertex is a complete bipartite graph; hence its universal covering is $T_n \times T_m$.

This description gives a finite presentation $\langle A \cup B \mid ab = b'a' \text{ whenever } ab\Box b'a' \rangle$ for $\Gamma = \pi_1(\mathcal{Y}) < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$. The subgroup generated by A (resp. B) is isomorphic to the stabilizer of a vertex x in T_m (resp. T_n). Labelling the edges $E(x)$ of T_m (resp. T_n), originating at

x, by the elements of B (resp. A), an element $a \in A$ (resp. $b' \in B$) induces on $E(x)$ the permutation $\sigma_a \in S_B$ (resp. $\tau_{b'} \in S_A$) defined by $a\sigma_a(b')\Box b'a'$ (resp. $ab\Box b'\tau_{b'}(a)$).

We will use the following square complexes on one vertex:

(i) For every pair of primes $p, q \equiv 1 \bmod 4$, one constructs, using the Hamilton quaternions, an irreducible torsion free lattice $\Lambda_{p,q} < PGL(2, \mathbf{Q}_p) \times PGL(2, \mathbf{Q}_q)$ acting freely transitively on the vertices of the associated Bruhat-Tits building $T_{p+1} \times T_{q+1}$ (*see* [8]). Let $\mathcal{A}(p, q) = \Lambda_{p,q} \backslash T_{p+1} \times T_{q+1}$ be the corresponding square complex.

(ii) Given a one vertex square complex $\mathcal{Y}$ defined by sets A, B and $\mathcal{R}$ as above, one constructs a new one vertex square complex $\mathcal{Y}^{\otimes 2}$ defined by $A^{\otimes 2} = A \times A$, $B^{\otimes 2} = B \times B$ and $(b', d')(a', c')\Box(a, c)(b, d)$ whenever $b'a'\Box ab$ and $d'c'\Box cd$.

(iii) For every even $k, l \geq 4$, let $\mathcal{C}(k, l)$ be the one vertex square complex defined by $A = \{a_i^\epsilon : \epsilon \in \{\pm 1\},\ 1 \leq i \leq k/2\}$, $B = \{b_j^\epsilon : \epsilon \in \{\pm 1\},\ 1 \leq j \leq l/2\}$ and $a_i^\epsilon b_{j+\epsilon}^\delta \Box b_j^\delta a_{i+\delta}^\epsilon$, $\epsilon, \delta \in \{\pm 1\}$, $1 \leq i \leq k/2$, $1 \leq j \leq l/2$.

Given even $n \geq 218$ and $m \geq 350$ let ${}^1\mathcal{Y} = \mathcal{A}(13, 17)$, ${}^2\mathcal{Y} = {}^1\mathcal{Y}^{\otimes 2}$, ${}^3\mathcal{Y} = \mathcal{C}(k_3, l_3)$ and ${}^4\mathcal{Y} = \mathcal{C}(k_4, l_4)$, where $14 + 196 + k_3 + k_4 = n$ and $18 + 324 + l_3 + l_4 = m$. Denote by ${}^sA = \{a_i^\epsilon\}$, ${}^sB = \{b_j^\epsilon\}$, $1 \leq s \leq 4$, the corresponding sets of oriented loops. We construct a complex $\mathcal{X}_{n,m}$ by mating the complexes ${}^s\mathcal{Y}$, $1 \leq s \leq 4$. We let $A = \sqcup^s A$, $B = \sqcup^s B$ and add the following square to the existing ones:

(1) for (i, j) not $(1, 1)$ or $(2, 2)$ and every $\epsilon, \delta \in \{\pm 1\}$, $1 \leq r \neq s \leq 4$, add the square ${}^ra_i^\epsilon\, {}^sb_j^\delta \Box {}^sb_j^{\delta\, r}a_i^\epsilon$;

(2) for every $\epsilon, \delta \in \{\pm 1\}$, $1 \leq r \neq s \leq 4$, we add a square ${}^ra_1^\epsilon\, {}^sb_1^\delta \Box {}^{\rho_r^\epsilon(s)}b_1^{\delta\, r}a_1^\epsilon$, where $\rho_r : \{1, 2, 3, 4\} \setminus \{r\} \to \{1, 2, 3, 4\} \setminus \{r\}$ is a fixed cyclic permutation;

(3) for every $\epsilon, \delta \in \{\pm 1\}$, $1 \leq r \neq s \leq 4$, we add a square ${}^ra_2^\epsilon\, {}^sb_2^\delta \Box {}^sb_2^{\delta\, \rho_s^\delta(r)}a_2^\epsilon$, where $\rho_s : \{1, 2, 3, 4\} \setminus \{s\} \to \{1, 2, 3, 4\} \setminus \{s\}$ is a fixed cyclic permutation.

The geometric description of $\mathcal{X}_{n,m}$. – First join the four complexes ${}^s\mathcal{Y}$, identifying their vertices. Then, for each pair $({}^ra_i, {}^sb_j)$, $r \neq s$, consisting of a "horizontal" and a "vertical" geometric loop belonging to different complexes, construct a torus by gluing. This corresponds to applying rule (1) above to *all* pairs (i, j). The resulting complex is covered by $T_n \times T_m$. Now consider the three torii involving, say, the loop ${}^1a_1$ and the loops ${}^2b_1$, ${}^3b_1$, ${}^4b_1$. Perform a "surgery" by cutting the torii along the loops ${}^2b_1$, ${}^3b_1$, ${}^4b_1$ and gluing them back according to a cyclic permutation. This operation preserves the link of the vertex. Applying this procedure to each loop ${}^sa_1$ versus the loops ${}^tb_1$, $t \neq s$, and similarly to each loop ${}^tb_2$ versus the loops ${}^ta_2$, $t \neq s$, we obtain $\mathcal{X}_{n,m}$.

One verifies that the permutation group $\mathcal{P}(A) = \langle \sigma_a : a \in A \rangle$ (resp. $\mathcal{P}(B) = \langle \tau_b : b \in B \rangle$) act 2-transitively on B (resp. A). This uses: (i) For ${}^1\mathcal{Y}$ the corresponding groups $\mathcal{P}({}^1A)$ and $\mathcal{P}({}^1B)$ are 2-transitive. (ii) For ${}^2\mathcal{Y}$ the corresponding groups $\mathcal{P}({}^2A)$ and $\mathcal{P}({}^2B)$ are transitive. (iii) For ${}^3\mathcal{Y}$, ${}^4\mathcal{Y}$ the corresponding groups have exactly two orbits corresponding to the two orientations. (iv) For ${}^1\mathcal{Y}$, by choosing a_1, a_2, b_1 and b_2 so that $a_1b_1\Box b_1a_1$ and $a_2b_2\Box b_2a_2$, we guarantee that this complex contains a square $a_k^\alpha b_j^\beta \Box b_1^\gamma a_i^\delta$ with $i, j, k \neq 1$ and a square $a_2^\alpha b_{j'}^\beta \Box b_{k'}^\gamma a_{i'}^\delta$ with $i', j', k' \neq 2$ (we omitted the left superscript 1 to simplify notation).

The above choices (in particular of $p = 13$, $q = 17$), imply that $\mathcal{P}(A) \subset A_m$ and $\mathcal{P}(B) \subset A_n$. The classification of finite 2-transitive group actions (*see* [5]) implies that $\mathcal{P}(A) = A_m$ and $\mathcal{P}(B) = A_n$, except for a set $(n, m) \in \mathbf{N}^2$ of density zero. Result of A. Bochert [2], *cf.* [12], asserts that a 2-transitive group of permutation of a set of d elements containing an element whose set of fixed point is of size $> \frac{2}{3}d + \frac{2}{3}\sqrt{d}$ must be either A_d or S_d. We may apply this result to deduce that $\mathcal{P}(A) = A_m$, by observing for example that the permutation $\sigma_{{}^1a_3}$ fixes at least $m - 18$ ($> \frac{2}{3}m + \frac{2}{3}\sqrt{m}$ as $m \geq 350$), and similarly verify that $\mathcal{P}(B) = A_n$.

Let $\Gamma = \pi_1(\mathcal{X}_{n,m})$ and $H_1 = \overline{\mathrm{pr}_{\mathrm{Aut}\, T_n}(\Gamma)}$, $H_2 = \overline{\mathrm{pr}_{\mathrm{Aut}\, T_m}(\Gamma)}$. Since $\pi_1({}^1\mathcal{Y})$ is irreducible, H_1, H_2 are non-discrete. Thus (by Lemma 3) $H_1 = U(A_n)$ and $H_2 = U(A_m)$. The group $\pi_1({}^2\mathcal{Y})$ is not residually finite (Corollary 2). Thus, by Theorem 4 and Remark 5, the intersection $\Gamma_{n,m}$ of all subgroups of finite index in Γ is of finite index in Γ and satisfies $\overline{\mathrm{pr}_{\mathrm{Aut}\, T_n}(\Gamma_{n,m})} = U(A_n)^+$, $\overline{\mathrm{pr}_{\mathrm{Aut}\, T_m}(\Gamma_{n,m})} = U(A_m)^+$. It follows that $\Gamma_{n,m}$ is simple and that its actions on the trees T_n and T_m give two decompositions of $\Gamma_{n,m}$ as in Theorem (1). The complex $\mathcal{Y}_{n,m} := \Gamma_{n,m}\backslash T_n \times T_m$ is the unique maximal finite covering of $\mathcal{X}_{n,m}$. There exists a halting Turing machine which, given (n,m), computes $\mathcal{Y}_{n,m}$.

(2) Let $\mathcal{T} = (X, Y)$ be a locally finite tree, $H < \mathrm{Aut}\,\mathcal{T}$, and for $x \in X$, $\underline{H}(x)$ be the finite permutation group on the sphere $S(x,1)$ of radius 1 induced by the stabilizer $H(x)$ of x. We say that H is locally primitive if $\underline{H}(x)$ is a primitive permutation group for all $x \in X$. In particular, H is transitive on the set of geometric edges of $\mathcal{T}$. For a totally disconnected group H, let $H^{(\infty)} \lhd H$ denote the intersection of all closed normal cocompact subgroups of H and $QZ(H) = \cup Z_H(U)$, where the union is over all open subgroups $U < H$.

Proposition 1. – *Let $H < \mathrm{Aut}\,\mathcal{T}$ be closed, locally primitive and non-discrete. Then $H^{(\infty)}$ is cocompact in H, the group $QZ(H)$ acts freely on X and for any closed normal subgroup $N \lhd H$, one has either $N \supset H^{(\infty)}$ or $N \subset QZ(H)$.*

Proof. – If $N \lhd H$ does not act freely on X, then N admits either an edge or a complete star as a fundamental domain. If N acts freely, it is discrete and hence $N \subset QZ(H)$. If $QZ(H)$ were not acting freely on X we could choose a finitely generated subgroup $\Lambda < QZ(H)$ with $\Lambda\backslash X$ finite, and hence find $U < H$ open commuting with Λ. This would imply that U is trivial and H is discrete. Thus $QZ(H)$ acts freely on X. Let $\mathcal{N}(H)$ be the set of closed normal subgroups of H which do not act freely on X; $\mathcal{N}(H)$ is ordered by inclusion and, by Zorn's lemma, the set $M(H)$ of minimal elements of $\mathcal{N}(H)$ is nonempty. Observe that $\mathcal{N}(H)$ is precisely the set of normal cocompact subgroups of H. For $M \in M(H)$ and $N \in \mathcal{N}(H)$ with $M \not\subset N$, $[N, M] \subset N \cap M$ is discrete; for $\Lambda \subset N$ finitely generated with $\Lambda\backslash X$ finite, there exists $U < M$ open with $[U, \Lambda] = e$ and hence $U = \{e\}$ contradicting the non-discreteness of M. Thus $M(H) = \{H^{(\infty)}\}$. Q.E.D.

Notation. – for $\Gamma < \mathrm{Aut}\,\mathcal{T}_1 \times \mathrm{Aut}\,\mathcal{T}_2$, set $\Lambda_1 := \Gamma \cap (\mathrm{Aut}\,\mathcal{T}_1 \times e)$, $\Lambda_2 := \Gamma \cap (e \times \mathrm{Aut}\,\mathcal{T}_2)$.

Corollary 2. – *Let $\Gamma < \mathrm{Aut}\,\mathcal{T}_1 \times \mathrm{Aut}\,\mathcal{T}_2$ be a cocompact lattice such that $H_i := \overline{\mathrm{pr}_i(\Gamma)}$ is locally primitive and non-discrete. If Γ' is commensurable to Γ, then Γ' contains $\left[H_1^{(\infty)} \times e, \Lambda_1\right] \cdot \left[e \times H_2^{(\infty)}, \Lambda_2\right]$. In particular if $\Lambda_1\Lambda_2 \neq e$, then Γ is not residually finite.*

Proof. – For $\Gamma'' \lhd \Gamma$ of finite index and $\Lambda_1'' := \Gamma'' \cap (H_1 \times e)$, the subgroups Λ_1 and Λ_1'' are discrete and normal in H_1. The kernel of the associated action of H_1 on the finite group Λ_1/Λ_1'' contains $H_1^{(\infty)}$. If $\Lambda_1 \neq e$, then $\left[H_1^{(\infty)} \times e, \Lambda_1\right] \neq e$, since $H_1^{(\infty)}$ is cocompact in $\mathrm{Aut}\,\mathcal{T}_1$ (Proposition 1). Q.E.D.

Example. – An alternative description of the complex ${}^2\mathcal{Y} = \mathcal{A}(p,q)^{\otimes 2}$ is as follows: For $r = p, q$, let $\mathcal{D}_r = (X_r, Y_r)$ denote the graph whose set of vertices X_r is the set of vertices of $T_{r+1} \times T_{r+1}$ and whose set of edges Y_r is the set of pairs of opposite vertices of squares in $T_{r+1} \times T_{r+1}$. Let $\mathcal{D}_r^+$, $\mathcal{D}_r^-$ be the two connected components of $\mathcal{D}_r$. Let $\Gamma_1 < \Lambda_{p,q} \times \Lambda_{p,q}$ be the stabilizer of $\mathcal{D}_p^+ \times \mathcal{D}_q^+$. Then ${}^2\mathcal{Y} = \Gamma_1\backslash\mathcal{D}_p^+ \times \mathcal{D}_q^+$. Observe that ${}^2\mathcal{Y}$ admits an involution σ. Its fundamental group $\Gamma' = \pi_1({}^2\mathcal{Y})$ is an extension of Γ_1 by $\pi_1(\mathcal{D}_p^+) \times \pi_1(\mathcal{D}_q^+)$ (a product of countably generated free groups) and is normalized by any lift $\tilde{\sigma} \in \mathrm{Aut}\, T_{(p+1)^2} \times \mathrm{Aut}\, T_{(q+1)^2}$. One verifies that $\Gamma := \langle \Gamma_1, \tilde{\sigma} \rangle$ together with Γ' satisfy the assumptions of corollary 2 and are therefore not residually finite.

3. Let $F < S_d$ be a transitive permutation group, $T_d = (X, Y)$ the d-regular tree, $E(x) \subset Y$ the set of edges with origin $x \in X$ and $\iota : Y \to \{1, 2, \ldots, d\}$ a legal coloring, namely a map which assigns the same color to the two orientations of every edge and whose restrictions to the sets $E(x)$ are bijective. Then the group $U(F) = \{g \in \mathrm{Aut}\, T_d : \iota_{|_{E(gx)}} g \iota^{-1}_{|_{E(x)}} \in F,\ \forall x \in X\} < \mathrm{Aut}\, T_d$ is well defined up to conjugation and applying of Theorem 4.5 [11] one shows that $U(F)^+$ is simple. If $H < \mathrm{Aut}\, T_d$ is vertex transitive and $\underline{H}(x)$ is permutation isomorphic to F, then we have $H < U(F)$ for a suitable legal coloring. Recall that $H < \mathrm{Aut}\, T_d$ is locally ∞-transitive if for all $x \in X$ and $n \geq 1$, $H(x)$ acts transitively on the sphere $S(x, n)$ centered at x and with radius n. If moreover H is vertex transitive, then H is said to be ∞-transitive. In this terminology we observe that $U(F)$ is ∞-transitive iff $F < S_d$ is 2-transitive.

LEMMA 3. – *Let $d \geq 6$ and $H < \mathrm{Aut}\, T_d$ be a closed, non-discrete, vertex transitive subgroup such that $\underline{H}(x) \cong A_d$ in its usual action. Then $H = U(A_d)$.*

Proof. – Let $H_n(x) = \{h \in H(x) : h_{|_{S(x,n)}} = id\}$ and $c(n) := |S(x, n)|$. We have to show that for every $n \geq 1$, $H_n(x)/H_{n+1}(x) \cong A_{d-1}^{c(n)}$ $(*)$.

NOTATION. – For $z \in S(x, n)$, let $p(z) \in S(x, n-1)$ be the vertex adjacent to z. Let $H_x(z) := (H(z) \cap H(p(z))/H_1(z)$. For $y \in S(x, 1)$, set $s_n(x, y) := \{z \in S(x, n) : d(z, x) = d(z, y) + 1\}$ and $a(n) := |s_n(x, y)|$.

(a) If $H_n(x) = H_{n+1}(x)$ for some $n \geq 1$, then $H_n(x) = e$. Thus for every $n \geq 1$ there exists $z \in S(x, n)$ such that the image of $H_n(x)$ in $H_x(z) \cong A_{d-1}$ is non-trivial, subnormal and hence equal to $H_x(z)$. Thus we deduce that H is ∞-transitive.

(b) Using the ∞-transitivity of H, one shows that if $\{B_i\}$ is a block decomposition for the $H(x)$ action on $S(x, n)$ $(n \geq 1)$, with $|B_i \cap s_n(x, y)| \leq 1$ for all i and $y \in S(x, 1)$, then $|B_i| = 1$. (c) Let $z, y \in S(x, 1)$, $z \neq y$; using the fact that H is vertex transitive, locally primitive and non-discrete, one shows that $H_1(x) \cap H_1(y) \not\subset H_1(z)$. This implies that $H_1(x)/H_2(x) \cong A_{d-1}^{c(1)}$. (d) Assertion $(*)$ is now established by induction on n. Let $n \geq 2$. For $y \in S(x, 1)$, $H_n(x) \triangleleft H_{n-1}(y)$, and by the induction hypothesis, the image of $H_{n-1}(y)$ in $\prod_{z \in s_n(x,y)} H_x(z)$ is $A_{d-1}^{a(n)}$; hence so is the image of $H_n(x)$. Finally, the image of $H_n(x)$ in $\prod_{y \in S(x,1)} \prod_{z \in s_n(x,y)} H_x(z)$ is isomorphic to a product of subdiagonals in $A_{d-1}^{c(n)}$ and hence induces a block decomposition of $S(x, n)$ as in (b). Q.E.D.

4. We need the following analogue of Margulis' normal subgroup theorem:

THEOREM 4. – *Let $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ be a cocompact lattice such that $H_i := \overline{\mathrm{pr}_i(\Gamma)}$ is locally ∞-transitive and $H_i^{(\infty)}$ is of finite index in H_i. Then any non-trivial normal subgroup of Γ has finite index.*

REMARK 5. – It follows that, if moreover Γ is not residually finite, then the intersection $\Gamma^{(\infty)}$ of all finite index normal subgroups of Γ is simple, of finite index in Γ and $\overline{\mathrm{pr}_i(\Gamma^{(\infty)})} = H_i^{(\infty)}$.

For a closed subgroup $H < \mathrm{Aut}\, T$, the following are equivalent: H is locally ∞-transitive; H is noncompact and transitive on the boundary $T(\infty)$; H is 2-transitive on $T(\infty)$. This implies using Proposition 1, that $H^{(\infty)}$ is 2-transitive on $T(\infty)$ and topologically simple. Moreover for $\xi \in T(\infty)$, $H(\xi)$ is a maximal subgroup of H. We sketch a proof of Theorem 4. For $\Gamma' := \Gamma \cap (H_1^{(\infty)} \times H_2^{(\infty)})$, we have $\overline{\mathrm{pr}_i(\Gamma')} = H_i^{(\infty)}$. Thus we may assume $H_i = H_i^{(\infty)}$. For $e \neq N \triangleleft \Gamma$, we have $\overline{\mathrm{pr}_i(N)} = H_i$, and since $(H_i, H_i(x))$ is a Gelfand pair, the quotient Γ/N has property (T) (*see* [9], Theorem IV 3.9). Thus either Γ/N is amenable and therefore finite, or Γ/N is not amenable and (*see* [9], IV 4.7) there exists an infinite Γ-invariant subalgebra $\mathcal{B}$ in the algebra $\mathcal{M}(T_1(\infty) \times T_2(\infty))$ of measurable sets, such that $nB = B$ for every $n \in N$ and $B \in \mathcal{B}$. Identifying $T_1(\infty) \times T_2(\infty)$ with $H_1/P_1 \times H_2/P_2$, where $P_i = H_i(\xi_i)$, $\xi_i \in T_i(\infty)$, we have (in the notation of [9] IV section 2) the following possibilities for $\mathcal{B}$: $\mathcal{M}(H_1/P_1 \times H_2/P_2)$, $\mathcal{M}(H_1/P_1 \times H_2/P_2, H_i)$, $i = 1, 2$. The proof of this statement follows verbatim the argument of Margulis' factor theorem (*see* [9] Theorem IV 2.11). The only point which needs to be verified is that if $g \in H_1 \times H_2$ generates an unbounded subgroup $A = \langle g \rangle$, then A acts ergodically on $\Gamma \backslash H_1 \times H_2$. This follows from the following analogue of Howe-Moore's theorem:

PROPOSITION 5. – *Let $G < \mathrm{Aut}\, T$ be a closed, locally ∞-transitive subgroup and $(\pi, \mathcal{H})$ a continuous unitary representation of G with no nonzero $G^{(\infty)}$-invariant vectors. Then all matrix coefficients of π vanish at infinity.*

Returning to the sketch of the proof of theorem 4, we conclude that N acts trivially on $H_1 \times H_2/Q$ where $Q \in \{P_1 \times P_2, P_1 \times H_2, H_1 \times P_2\}$ and thus $N \subset \cap_{h \in H_1 \times H_2} hQh^{-1}$; as both H_1, H_2 are topologically simple, the latter group is $e \times e$, $H_1 \times e$ or $e \times H_2$. If for example $N \subset H_1 \times e$, then $\mathrm{pr}_1(N) \triangleleft \overline{\mathrm{pr}_1(\Gamma)} = H_1$, and hence $N = e$. Q.E.D.

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