Topics in Occupation Times and Gaussian Free Fields

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Foreword

The following notes grew out of the graduate course "Special topics in probability", which I gave at ETH Zurich during the Spring term 2011. One of the objectives was to explore the links between occupation times, Gaussian free fields, Poisson gases of Markovian loops, and random interlacements. The stimulating atmosphere during the live lectures was an encouragement to write a fleshed-out version of the handwritten notes, which were handed out during the course. I am immensely grateful to Pierre-François Rodriguez, Artëm Sapozhnikov, Balázs Ráth, Alexander Drewitz, and David Belius, for their numerous comments on the successive versions of these notes.

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0 Introduction

This set of notes explores some of the links between occupation times and Gaussian processes. Notably they bring into play certain isomorphism theorems going back to Dynkin [4], [5] as well as certain Poisson point processes of Markovian loops, which originated in physics through the work of Symanzik [26]. More recently such Poisson gases of Markovian loops have reappeared in the context of the "Brownian loop soup" of Lawler and Werner [16] and are related to the so-called "random interlacements", see Sznitman [27]. In particular they have been extensively investigated by Le Jan [17], [18].

A convenient set-up to develop this circle of ideas consists in the consideration of a finite connected graph E endowed with positive weights and a non-degenerate killing measure. One can then associate to these data a continuous-time Markov chain $\overline{X}_t$, $t \ge 0$, on E, with variable jump rates, which dies after a finite time due to the killing measure, as well as

(0.1) $$\text{the Green density } g(x,y),\ x,y \in E,$$

(which is positive and symmetric),

(0.2) $$\text{the local times } \overline{L}^x_t = \int_0^t 1\{\overline{X}_s = x\}\, ds,\ t \ge 0,\ x \in E.$$

In fact $g(\cdot,\cdot)$ is a positive definite function on $E \times E$, and one can define a centered Gaussian process φ_x, $x \in E$, such that

(0.3) $$\mathrm{cov}(\varphi_x, \varphi_y) (= E[\varphi_x \varphi_y]) = g(x,y), \text{ for } x,y \in E.$$

This is the so-called Gaussian free field.

It turns out that $\frac{1}{2}\,\varphi^2_z$, $z \in E$, and $\overline{L}^z_\infty$, $z \in E$, have intricate relationships. For instance Dynkin's isomorphism theorem states in our context that for any $x,y \in E$,

(0.4) $$\big(\overline{L}^z_\infty + \frac{1}{2}\,\varphi^2_z\big)_{z \in E} \ \text{ under } P_{x,y} \otimes P^G,$$

has the "same law" as

(0.5) $$\frac{1}{2}\,(\varphi^2_z)_{z \in E} \ \text{ under } \varphi_x \varphi_y\, P^G,$$

where $P_{x,y}$ stands for the (non-normalized) h-transform of our basic Markov chain, with the choice $h(\cdot) = g(\cdot, y)$, starting from the point x, and P^G for the law of the Gaussian field φ_z, $z \in E$.

Eisenbaum's isomorphism theorem, which appeared in [7], does not involve h-transforms and states in our context that for any $x \in E$, $s \neq 0$,

(0.6) $$\big(\overline{L}^z_\infty + \frac{1}{2}\,(\varphi_z + s)^2\big)_{z \in E} \ \text{ under } P_x \otimes P^G,$$

has the "same law" as

(0.7) $$\big(\frac{1}{2}\,(\varphi_z + s)^2\big)_{z \in E} \ \text{ under } \big(1 + \frac{\varphi_x}{s}\big)\, P^G.$$

The above isomorphism theorems are also closely linked to the topic of theorems of Ray-Knight type, see Eisenbaum [6], and chapters 2 and 8 of Marcus-Rosen [19]. Originally, see [13, 21], such theorems came as a description of the Markovian character in the space variable of Brownian local times evaluated at certain random times. More recently, the Gaussian aspects and the relation with the isomorphism theorems have gained prominence, see [8], and [19].

Interestingly, Dynkin's isomorphism theorem has its roots in mathematical physics. It grew out of the investigation by Dynkin in [4] of a probabilistic representation formula for the moments of certain random fields in terms of a Poissonian gas of loops interacting with Markovian paths, which appeared in Brydges-Fröhlich-Spencer [2], and was based on the work of Symanzik [26].

The Poisson point gas of loops in question is a Poisson point process on the state space of loops on E modulo time-shift. Its intensity measure is a multiple $\alpha\mu^*$ of the image μ^* of a certain measure $\mu_{\rm rooted}$, under the canonical map for the equivalence relation identifying rooted loops γ that only differ by a time-shift. This measure $\mu_{\rm rooted}$ is the σ-finite measure on rooted loops defined by

$$\mu_{\rm rooted}(d\gamma) = \sum_{x\in E} \int_0^\infty Q^t_{x,x}(d\gamma)\,\frac{dt}{t}, \tag{0.8}$$

where $Q^t_{x,x}$ is the image of $1\{X_t = x\}\,P_x$ under $(X_s)_{0\le s\le t}$, if $X_.$ stands for the Markov chain on E with jump rates equal to 1 attached to the weights and killing measure we have chosen on E.

The random fields on E alluded to above, are motivated by models of Euclidean quantum field theory, see [11], and are for instance of the following kind:

$$\langle F(\varphi)\rangle = \int_{\mathbb{R}^E} F(\varphi)\,e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in E} h\Big(\frac{\varphi_x^2}{2}\Big)\,d\varphi_x \Big/ \int_{\mathbb{R}^E} e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in E} h\Big(\frac{\varphi_x^2}{2}\Big)\,d\varphi_x \tag{0.9}$$

with

$$h(u) = \int_0^\infty e^{-vu} d\nu(v),\ u \ge 0, \text{ with } \nu \text{ a probability distribution on } \mathbb{R}_+,$$

and $\mathcal{E}(\varphi,\varphi)$ the energy of the function φ corresponding to the weights and killing measure on E (the matrix $\mathcal{E}(1_x,1_y)$, $x,y\in E$ is the inverse of the matrix $g(x,y)$, $x,y\in E$ in (0.3)).

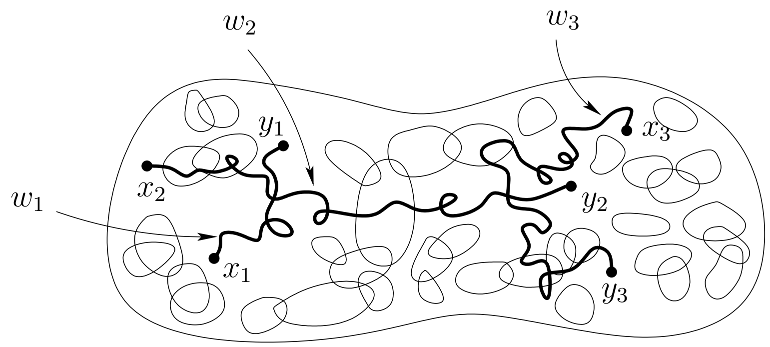


Fig. 0.1: The paths $w_1,\dots,w_k$ in E interact with the gas of loops through the random potentials.

The typical representation formula for the moments of the random field in (0.9) looks like this: for $k \geq 1$, $z_1, \dots, z_{2k} \in E$,

$$\langle \varphi_{z_1} \dots \varphi_{z_{2k}} \rangle = \sum_{\substack{\text{pairings} \\ \text{of } z_1, \dots, z_{2k}}} \frac{P_{x_1,y_1} \otimes \dots \otimes P_{x_k,y_k} \otimes \mathbb{Q}\big[e^{-\sum_{x \in E} v_x(\mathcal{L}_x + \overline{L}^x_\infty(w_1) + \dots + \overline{L}^x_\infty(w_k))}\big]}{\mathbb{Q}\big[e^{-\sum_{x \in E} v_x \mathcal{L}_x}\big]}, \tag{0.10}$$

where the sum runs over the (non-ordered) pairings (i.e. partitions) of the symbols $z_1, z_2, \dots, z_{2k}$ into $\{x_1, y_1\}, \dots, \{x_k, y_k\}$. Under $\mathbb{Q}$ the $v_x, x \in E$, are i.i.d. ν-distributed (random potentials), independent of the $\mathcal{L}_x$, $x \in E$, which are distributed as the total occupation times (properly scaled to take account of the weights and killing measure) of the gas of loops with intensity $\frac{1}{2}\,\mu$, and the P_{x_i,y_i}, $1 \leq i \leq k$ are defined just as below (0.4), (0.5).

The Poisson point process of Markovian loops has many interesting properties. We will for instance see that when $\alpha = \frac{1}{2}$ (i.e. the intensity measure equals $\frac{1}{2}\,\mu$),

(0.11) $(\mathcal{L}_x)_{x \in E}$ has the same distribution as $\frac{1}{2}\,(\varphi_x^2)_{x \in E}$, where $(\varphi_x)_{x \in E}$ stands for the Gaussian free field in (0.3).

The Poisson gas of Markovian loops is also related to the model of random interlacements [27], which loosely speaking corresponds to "loops going through infinity". It appears as well in the recent developments concerning conformally invariant scaling limits, see Lawler-Werner [16], Sheffield-Werner [24]. As for random interlacements, interestingly, in place of (0.11), they satisfy an isomorphism theorem in the spirit of the generalized second Ray-Knight theorem, see [28].

1 Generalities

In this chapter we describe the general framework we will use for the most part of these notes. We introduce finite weighted graphs with killing and the associated continuous-type Markov chains $X_.$, with constant jump rate equal to 1, and $\overline{X}_.$, with variable jump rate. We also recall various notions related to Dirichlet forms and potential theory.

1.1 The set-up

We introduce in this section the general set-up, which we will use in the sequel, and recall some classical facts. We also refer to [14] and [10], where the theory is developed in a more general framework. We assume that

(1.1) $$E \text{ is a finite non-empty set}$$

endowed with **non-negative weights**

(1.2) $$c_{x,y} = c_{y,x} \geq 0, \text{ for } x, y \in E, \text{ and } c_{x,x} = 0, \text{ for } x \in E,$$

so that

(1.3) E, endowed with the edge set consisting of the pairs $\{x, y\}$ such that $c_{x,y} > 0$, is a connected graph.

We also suppose that there is a **killing measure** on E:

(1.4) $$\kappa_x \geq 0, \; x \in E,$$

and that

(1.5) $$\kappa_x \neq 0, \text{ for at least some } x \in E.$$

We also consider a

(1.6) **cemetery state** Δ not in E

(we can think of κ_x as $c_{x,\Delta}$).

With these data we can define a measure on E:

(1.7) $$\lambda_x = \sum_{y \in E} c_{x,y} + \kappa_x, \; x \in E \text{ (note that } \lambda_x > 0, \text{ due to (1.2) - (1.5)).}$$

We can also introduce the **energy** of a function on E, or **Dirichlet form**

(1.8) $$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y \in E} c_{x,y}(f(y) - f(x))^2 + \sum_{x \in E} \kappa_x \, f^2(x),$$

for $f : E \to \mathbb{R}$.

Note that $(c_{x,y})_{x,y \in E}$ and $(\kappa_x)_{x \in E}$ determine the Dirichlet form. Conversely, the Dirichlet form determines $(c_{x,y})_{x,y \in E}$ and $(\kappa_x)_{x \in E}$. Indeed, one defines, by polarization, for $f, g : E \to \mathbb{R}$,

(1.9) $$\begin{aligned} \mathcal{E}(f, g) &= \frac{1}{4} \left[\mathcal{E}(f + g, f + g) - \mathcal{E}(f - g, f - g)\right] \\ &= \frac{1}{2} \sum_{x,y \in E} c_{x,y}(f(y) - f(x))(g(y) - g(x)) + \sum_{x \in E} \kappa_x f(x) g(x), \end{aligned}$$

and one notes that

$$\begin{aligned} \mathcal{E}(1_x, 1_y) &= -c_{x,y}, \text{ for } x \neq y \text{ in } E, \\ \mathcal{E}(1_x, 1_x) &= \sum_{y \in E} c_{x,y} + \kappa_x = \lambda_x, \text{ for } x \in E, \end{aligned} \tag{1.10}$$

so that the Dirichlet form uniquely determines the weights $(c_{x,y})_{x,y \in E}$ and the killing measure $(\kappa_x)_{x \in E}$. Observe also that by (1.3), (1.5), (1.8), (1.9), the Dirichlet form defines a positive definite quadratic form on the space $\mathcal{F}$ of functions from E to $\mathbb{R}$, see also (1.39) below.

We denote by $(\cdot, \cdot)_\lambda$ the scalar product in $L^2(d\lambda)$:

$$(f, g)_\lambda = \sum_{x \in E} f(x) g(x) \lambda_x, \text{ for } f, g : E \to \mathbb{R}\,. \tag{1.11}$$

The weights and the killing measure induce a **sub-Markovian transition probability** on E:

$$p_{x,y} = \frac{c_{x,y}}{\lambda_x}, \text{ for } x, y \in E, \tag{1.12}$$

which is λ-reversible:

$$\lambda_x \, p_{x,y} = \lambda_y \, p_{y,x}, \text{ for all } x, y \in E. \tag{1.13}$$

One then extends $p_{x,y}, x, y \in E$ to a transition probability on $E \cup \{\Delta\}$ by setting

$$p_{x,\Delta} = \frac{\kappa_x}{\lambda_x}, \text{ for } x \in E, \text{ and } p_{\Delta,\Delta} = 1, \tag{1.14}$$

so the corresponding discrete-time Markov chain on $E \cup \{\Delta\}$ is absorbed in the cemetery state Δ once it reaches Δ. We denote by

(1.15) $Z_n, n \geq 0$, the canonical discrete Markov chain on the space of discrete trajectories in $E \cup \{\Delta\}$, which after finitely many steps reaches Δ and from then on remains at Δ,

and by

(1.16) P_x the law of the chain starting from $x \in E \cup \{\Delta\}$.

We will attach to the Dirichlet form (1.8) (or, equivalently, to the weights and the killing measure), two continuous-time Markov chains on $E \cup \{\Delta\}$, which are time change of each other, with discrete skeleton corresponding to Z_n, $n \geq 0$. The first chain $X_.$ will have a unit jump rate, whereas the second chain $\overline{X}_.$ (defined in Section 1.6) will have a variable jump rate governed by λ.

1.2 The Markov chain $X_.$ (with jump rate 1)

We introduce in this section the continuous-time Markov chain on $E \cup \{\Delta\}$ (absorbed in the cemetery state Δ), with discrete skeleton described by Z_n, $n \geq 0$, and exponential holding times of parameter 1. We also bring into play some of the natural objects attached to this Markov chains.

The canonical space D_E for this Markov chain consists of right-continuous functions with values in $E \cup \{\Delta\}$, with finitely many jumps, which after some time enter Δ and from then on remain equal to Δ. We denote by

(1.17) $X_t, t \geq 0$, the canonical process on D_E,

$\theta_t, t \geq 0$, the canonical shift on D_E: $\theta_t(w)(\cdot) = w(\cdot + t)$, for $w \in D_E$,

P_x the law on D_E of the Markov chain starting at $x \in E \cup \{\Delta\}$.

Remark 1.1. Whenever convenient we will tacitly enlarge the canonical space D_E and work with a probability space on which (under P_x) we can simultaneously consider the discrete Markov chain Z_n, $n \geq 0$, with starting point a.s. equal to x, and an independent sequence of positive variables T_n, $n \geq 1$, the "jump times", increasing to infinity, with increments $T_{n+1} - T_n$, $n \geq 0$, i.i.d. exponential with parameter 1 (with the convention $T_0 = 0$). The continuous-time chain X_t, $t \geq 0$, will then be expressed as

$$X_t = Z_n, \text{ for } T_n \leq t < T_{n+1}, \; n \geq 0.$$

Of course, once the discrete-time chain reaches the cemetery state Δ, the subsequent "jump times" T_n are only fictitious "jumps" of the continuous time chain. □

Examples:

1) Simple random walk on the discrete torus killed at a constant rate

$$E = (\mathbb{Z}/N\mathbb{Z})^d, \text{ where } N > 1, \; d \geq 1,$$

endowed with the graph structure, where x, y are neighbors if exactly one of their coordinates differs by ± 1, and the other coordinates are equal. We pick

$$c_{x,y} = 1_{\{x,\, y \text{ are neighbors}\}}, \; x, y \in E,$$
$$\kappa_x = \kappa > 0\,.$$

So X_t, $t \geq 0$, is the simple random walk on $(\mathbb{Z}/N\mathbb{Z})^d$ with exponential holding times of parameter 1, killed at each step with probability $\frac{\kappa}{2d+\kappa}$, when $N > 2$, and probability $\frac{\kappa}{d+\kappa}$, when $N = 2$.

2) Simple random walk on $\mathbb{Z}^d$ killed outside a finite connected subset of $\mathbb{Z}^d$, that is:

E is a finite connected subset of $\mathbb{Z}^d$, $d \geq 1$.

$c_{x,y} = 1_{\{|x-y|=1\}}$, for $x, y \in E$,

$\kappa_x = \sum_{y \in \mathbb{Z}^d \setminus E} 1_{\{|x-y|=1\}}$, for $x \in E$,

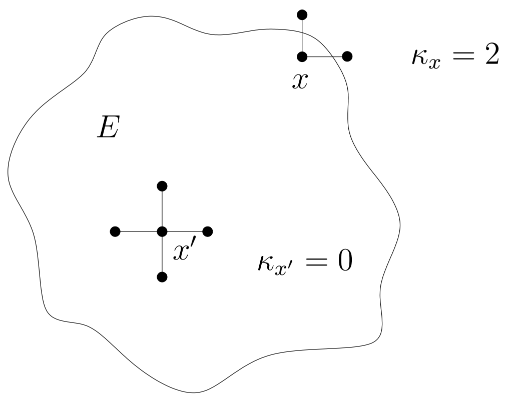


Fig. 1.1

X_t, $t \ge 0$, when starting in $x \in E$, corresponds to the simple random walk in $\mathbb{Z}^d$ with exponential holding times of parameter 1 killed at the first time it exists E. $\square$

Our next step is to introduce some natural objects attached to the Markov chain $X_{\textbf{.}}$, such as the transition semi-group, and the Green function.

Transition semi-group and transition density:

Unless otherwise specified, we will tacitly view real-valued functions on E, as functions on $E \cup \{\Delta\}$, which vanish at the point Δ.

The **sub-Markovian transition semi-group** of the chain X_t, $t \ge 0$, on E is defined for $t \ge 0$, $f: E \to \mathbb{R}$, by

$$\begin{aligned} R_t f(x) &= E_x[f(X_t)] \quad \text{for } x \in E \\ &= \sum_{n \ge 0} e^{-t} \frac{t^n}{n!} E_x[f(Z_n)] \\ &= \sum_{n \ge 0} e^{-t} \frac{t^n}{n!} P^n f(x) = e^{t(P-I)} f(x), \end{aligned} \tag{1.18}$$

where I denotes the identity map on $\mathbb{R}^E$, and for $f\colon E \to \mathbb{R}$, $x \in E$,

$$Pf(x) = \sum_{y \in E} p_{x,y} f(y) \overset{(1.15)}{=} E_x[f(Z_1)]. \tag{1.19}$$

As a result of (1.13) and (1.18)

(1.20) P and R_t (for any $t \ge 0$) are bounded self-adjoint operators on $L^2(d\lambda)$,

(1.21) $R_{t+s} = R_t R_s$, for $t, s \ge 0$ (semi-group property).

We then introduce the **transition density**

$$r_t(x,y) = (R_t 1_y)(x) \frac{1}{\lambda_y}, \text{ for } t \ge 0,\ x, y \in E. \tag{1.22}$$

It follows from the self-adjointness of R_t, cf. (1.20), that

$$r_t(x,y) = r_t(y,x), \text{ for } t \ge 0,\ x, y \in E \text{ (symmetry)} \tag{1.23}$$

and from the semi-group property, cf. (1.21), that for $t, s \geq 0$, $x, y \in E$,

(1.24) $$r_{t+s}(x,y) = \sum_{z \in E} r_t(x,z)\, r_s(z,y)\, \lambda_z \text{ (Chapman-Kolmogorov equations).}$$

Moreover due to (1.3), (1.12), (1.18), we see that

(1.25) $$r_t(x,y) > 0, \text{ for } t > 0,\ x, y \in E.$$

Green function:

We define the Green function (or Green density):

(1.26) $$g(x,y) = \int_0^\infty r_t(x,y)\, dt \overset{(1.18),(1.22)}{\underset{\text{Fubini}}{=}} E_x\Big[\int_0^\infty 1\{X_t = y\} dt\Big] \frac{1}{\lambda_y}, \text{ for } x, y \in E.$$

Lemma 1.2.

(1.27) $g(x,y) \in (0,\infty)$ *is a symmetric function on* $E \times E$.

Proof. By (1.23), (1.25) we see that $g(\cdot,\cdot)$ is positive and symmetric. We now prove that it is finite. By (1.1), (1.3), (1.5) we see that for some $N \geq 0$, and $\varepsilon > 0$,

(1.28) $$\inf_{x \in E} P_x[Z_n = \Delta, \text{ for some } n \leq N] \geq \varepsilon > 0.$$

As a result of the simple Markov property at times, which are multiple of N, we find that

$$\begin{aligned}(P^{kN} 1_E)(x) &= P_x[Z_n \neq \Delta, \text{ for } 0 \leq n \leq kN] \\ &\underset{(1.28)}{\overset{\text{simple Markov}}{\leq}} (1-\varepsilon)^k, \text{ for } k \geq 1.\end{aligned}$$

It follows by a straightforward interpolation that with suitable $c, c' > 0$,

(1.29) $$\sup_{x \in E} (P^n 1_E)(x) \leq c\, e^{-c'n} \text{ for } n \geq 0.$$

As a result inserting this bound in the last line of (1.18) gives:

(1.30) $$\sup_{x \in E} (R_t 1_E)(x) \leq c\, e^{-t} \sum_{n \geq 0} \frac{t^n}{n!}\, e^{-c'n} = c \exp\{-t(1 - e^{-c'})\},$$

so that

(1.31) $$g(x,y) \leq \frac{1}{\lambda_y} \int_0^\infty (R_t 1_E)(x)\, dt \leq \frac{c}{\lambda_y} \frac{1}{1 - e^{-c'}} \leq c'' < \infty,$$

whence (1.27). □

1.3 Some potential theory

In this section we introduce some natural objects from potential theory such as the equilibrium measure, the equilibrium potential, and the capacity of a subset of E. We also provide two variational characterizations for the capacity. We then describe the orthogonal complement under the Dirichlet form of the space of functions vanishing on a subset of K. This also naturally leads us to the notion of trace form (and network reduction).

The Green function gives rise to the **potential operators**

$$Qf(x) = \sum_{y \in E} g(x,y)\, f(y) \lambda_y, \text{ for } f : E \to \mathbb{R} \text{ (a function)}, \tag{1.32}$$

the potential of the function f, and

$$G\nu(x) = \sum_{y \in E} g(x,y)\, \nu_y, \text{ for } \nu\colon\ E \to \mathbb{R} \text{ (a measure)}, \tag{1.33}$$

the potential of the measure ν. We also write the duality bracket (between functions and measures on E):

$$\langle \nu, f \rangle = \sum_x \nu_x\, f(x) \ \text{ for } f\colon\ E \to \mathbb{R},\ \nu\colon\ E \to \mathbb{R}. \tag{1.34}$$

In the next proposition we collect several useful properties of the Green function and Dirichlet form.

Proposition 1.3.

(1.35) $E(\nu, \mu) \overset{\text{def}}{=} \langle \nu, G\mu \rangle = \sum_{x,y \in E} \nu_x\, g(x,y)\, \mu_y$, *for* $\nu, \mu\colon\ E \to \mathbb{R}$ *defines a positive definite, symmetric bilinear form.*

(1.36) $Q = (I - P)^{-1}$ *(see* (1.19), (1.32) *for notation).*

(1.37) $G = (-L)^{-1}$, *where* $Lf(x) = \sum_{y \in E} c_{x,y} f(y) - \lambda_x f(x)$, *for* $f\colon\ E \to \mathbb{R}$.

(1.38) $\mathcal{E}(G\nu, f) = \langle \nu, f \rangle$, *for* $\nu\colon\ E \to \mathbb{R}$ *and* $f\colon\ E \to \mathbb{R}$.

(1.39) $\exists \rho > 0$, *such that* $\mathcal{E}(f,f) \geq \rho \|f\|^2_{L^2(d\lambda)}$, *for all* $f\colon\ E \to \mathbb{R}$.

(1.40) $G\kappa = 1$ *(and the killing measure* κ *is also called equilibrium measure of* E*).*

Proof.

- (1.35):

One can give a direct proof based on (1.23) - (1.26), but we will instead derive (1.35) with the help of (1.37) - (1.39). The bilinear form in (1.35) is symmetric by (1.27). Moreover, for $\nu\colon\ E \to \mathbb{R}$,

$$0 \leq \mathcal{E}(G\nu, G\nu) \overset{(1.38)}{=} \langle \nu, G\nu \rangle = E(\nu,\nu) \text{ (the } \textbf{energy of the measure } \boldsymbol{\nu}\text{)}.$$

By (1.39), $0 = \mathcal{E}(G\nu, G\nu) \Longrightarrow G\nu = 0$, and by (1.37) it follows that $\nu = (-L)\,G\nu = 0$. This proves (1.35) (assuming (1.37) - (1.39)).

• (1.36):

By (1.29):

$$\begin{aligned}\int_0^\infty \sum_{n\geq 0} e^{-t}\,\frac{t^n}{n!}\,|P^n f(x)|dt &\leq c\int_0^\infty e^{-t}\sum_{n\geq 0}\frac{t^n}{n!}\,e^{-c'n}dt\,\|f\|_\infty \\ &\overset{(1.30)}{=} c\|f\|_\infty\int_0^\infty e^{-t(1-e^{-c'})}dt < \infty.\end{aligned}$$

By Lebesgue's domination theorem, keeping in mind (1.18), (1.26),

$$\begin{aligned}Qf(x) &\overset{(1.32)}{=} \int_0^\infty R_t\,f(x)dt \overset{(1.18)}{=} \int_0^\infty \sum_{n\geq 0} e^{-t}\,\frac{t^n}{n!}\,P^n f(x)dt = \sum_{n\geq 0}\int_0^\infty e^{-t}\,\frac{t^n}{n!}\,dt\;P^n\,f(x) \\ &= \sum_{n\geq 0} P^n\,f(x) \overset{(1.29)}{=} (I-P)^{-1}f(x),\;\text{(1 is not in the spectrum of } P \text{ by (1.29)).}\end{aligned}$$

This proves (1.36).

• (1.37):

Note that in view of (1.19)

(1.41) $$\begin{aligned}-L = \lambda(I-P)\;\;&\text{(composition of } (I-P) \text{ and the multiplication by } \lambda_\cdot,\\ &\text{i.e. } (\lambda f)(x) = \lambda_x\,f(x) \text{ for } f\colon\, E\to\mathbb{R}, \text{ and } x\in E).\end{aligned}$$

Hence $-L$ is invertible and

$$(-L)^{-1} = (I-P)^{-1}\,\lambda^{-1} \overset{(1.36)}{=} Q\,\lambda^{-1} \underset{(1.33)}{\overset{(1.32)}{=}} G.$$

This proves (1.37).

• (1.38):

By (1.10) we find that

(1.42) $$\begin{aligned}\mathcal{E}(f,g) &= \sum_{x,y\in E} f(x)\,g(y)\,\mathcal{E}(1_x,1_y) \overset{(1.10)}{=} \sum_{x\in E}\lambda_x\,f(x)\,g(x) - \sum_{x,y\in E} c_{x,y}\,f(x)\,g(y)\\ &= \langle f, -Lg\rangle \overset{(1.2)}{=} \langle -Lf, g\rangle.\end{aligned}$$

As a result

$$\mathcal{E}(G\nu, f) = \langle -LG\nu, f\rangle \overset{(1.37)}{=} \langle \nu, f\rangle,\;\text{whence (1.38).}$$

• (1.39):

Note that for $x\in E$, $f\colon\, E\to\mathbb{R}$

$$f(x) = \langle 1_x, f\rangle \overset{(1.38)}{=} \mathcal{E}(G1_x, f).$$

Now $\mathcal{E}(\cdot,\cdot)$ is a non-negative symmetric bilinear form. We can thus apply Cauchy-Schwarz's inequality to find that

$$\begin{aligned} f(x)^2 \le \mathcal{E}(G1_x, G1_x)\, \mathcal{E}(f,f) &\overset{(1.38)}{=} \langle 1_x, G1_x\rangle\, \mathcal{E}(f,f) \\ &= g(x,x)\, \mathcal{E}(f,f). \end{aligned}$$

As a result we find that

$$\|f\|^2_{L^2(d\lambda)} = \sum_{x\in E} f(x)^2\, \lambda_x \le \sum_{x\in E} g(x,x)\, \lambda_x\, \mathcal{E}(f,f), \tag{1.43}$$

and (1.39) follows with

$$\rho^{-1} = \sum_{x\in E} g(x,x)\, \lambda_x.$$

• (1.40):

By (1.39), $\mathcal{E}(\cdot,\cdot)$ is positive definite and by (1.9)

$$\mathcal{E}(1,f) \overset{(1.9)}{=} \sum_x \kappa_x\, f(x) = \langle \kappa, f\rangle \overset{(1.38)}{=} \mathcal{E}(G\kappa, f), \text{ for all } f\colon E\to\mathbb{R}.$$

It thus follows that $1 = G\kappa$, whence (1.40). □

Remark 1.4. Note that we have shown in (1.42) that for all $f,g\colon E \to \mathbb{R}$,

$$\mathcal{E}(f,g) = \langle -Lf, g\rangle = \langle f, -Lg\rangle. \tag{1.44}$$

Since $-L = \lambda(I-P)$, we also find, see (1.11) for notation,

$$\mathcal{E}(f,g) = ((I-P)f,g)_\lambda = (f,(I-P)g)_\lambda. \tag{1.44'}$$

□

As a next step we introduce some important random times for the continuous-time Markov chain X_t, $t\ge 0$. Given $K \subseteq E$, we define

$$\begin{aligned} H_K &= \inf\{t\ge 0;\, X_t\in K\}, \text{ the } \textbf{entrance time} \text{ in } K, \\ \widetilde{H}_K &= \inf\{t>0;\ X_t\in K \text{ and there exists } s\in(0,t) \text{ with } X_s\ne X_0\}, \\ &\qquad\qquad \text{the } \textbf{hitting time} \text{ of } K, \\ T_K &= \inf\{t\ge 0;\ X_t\notin K\}, \text{ the } \textbf{exit time} \text{ from } K, \\ L_K &= \sup\{t>0;\ X_t\in K\}, \text{ the } \textbf{time of last visit} \text{ to } K \\ &\qquad\qquad (\text{with the convention } \sup\phi = 0,\ \inf\phi=\infty). \end{aligned} \tag{1.45}$$

$H_K, \widetilde{H}_K, T_K$ are stopping times for the canonical filtration $(\mathcal{F}_t)_{t\ge 0}$, on D_E (i.e. a $[0,\infty]$-valued map T on D_E, see above (1.17), such that $\{T\le t\}\in\mathcal{F}_t \overset{\text{def}}{=} \sigma(X_s, 0\le s\le t)$, for each $t\ge 0$). Of course L_K is in general not a stopping time.

Given $U\subseteq E$, the **transition density killed outside $\boldsymbol{U}$** is

$$r_{t,U}(x,y) = P_x[X_t = y, t< T_U]\,\frac{1}{\lambda_y} \le r_t(x,y), \text{ for } t\ge 0,\ x,y\in E, \tag{1.46}$$

and the **Green function killed outside $\boldsymbol{U}$** is

$$g_U(x,y) = \int_0^\infty r_{t,U}(x,y)dt \le g(x,y), \text{ for } x,y\in E. \tag{1.47}$$

Remark 1.5.

1) When U is a connected (non-empty) subgraph of the graph in (1.3), $r_{t,U}(x,y)$, $t \geq 0$, $x, y \in U$, and $g_U(x,y)$, $x, y \in U$, simply correspond to the transition density and the Green function in (1.22), (1.26), when one chooses on U

- the weights $c_{x,y}$, $x, y \in U$ (i.e. restriction to $U \times U$ of the weights on E),
- the killing measure $\widetilde{\kappa}_x = \kappa_x + \sum_{y \in E \setminus U} c_{x,y}$, $x \in U$.

2) When U is not connected the above remark applies to each connected component of U, and $r_{t,U}(x,y)$ and $g_U(x,y)$ vanish when x, y belong to different connected components of U. □

Proposition 1.6. $(U \subseteq E, A = E \setminus U)$

$$g_U(x,y) = g_U(y,x), \text{ for } x, y \in E. \tag{1.48}$$

$$g(x,y) = g_U(x,y) + E_x[H_A < \infty,\, g(X_{H_A}, y)], \text{ for } x, y \in E. \tag{1.49}$$

$$E_x[H_A < \infty, g(X_{H_A}, y)] = E_y[H_A < \infty, g(X_{H_A}, x)], \text{ for } x, y \in E \tag{1.50}$$

(Hunt's switching identity).

Proof.

• (1.48):

This is a direct consequence of the above remark and (1.27).

• (1.49):

$$\begin{aligned}
g(x,y) &\overset{(1.26)}{=} E_x\Big[\int_0^\infty 1\{X_t = y\}dt\Big]\,\frac{1}{\lambda_y} = E_x\Big[\int_0^\infty 1\{X_t = y, t < T_U\}dt\Big]\,\frac{1}{\lambda_y} \\
&+ E_x\Big[\int_{T_U}^\infty 1\{X_t = y\}dt,\, T_U < \infty\Big]\,\frac{1}{\lambda_y} \underset{(1.46),(1.47)}{\overset{\text{Fubini}}{=}} g_U(x,y) \\
&+ E_x\Big[T_U < \infty, \Big(\int_0^\infty 1\{X_t = y\}dt\Big)\circ \theta_{T_U}\Big]\,\frac{1}{\lambda_y} \overset{\text{strong Markov}}{=} g_U(x,y) \\
&+ E_x\Big[T_U < \infty,\, E_{X_{T_U}}\Big[\int_0^\infty 1\{X_t = y\}dt\Big]\Big]\,\frac{1}{\lambda_y} \underset{(1.26)}{\overset{T_U = H_A}{=}} g_U(x,y) \\
&+ E_x[H_A < \infty,\, g(X_{H_A}, y)].
\end{aligned}$$

This proves (1.49).

• (1.50):

This follows from (1.48), (1.49) and the fact that $g(\cdot,\cdot)$ is symmetric, cf. (1.27). □

Example:

Consider $x_0 \in E$. By (1.49) we find that for $x \in E$ (with $A = \{x_0\}$, $U = E \setminus \{x_0\}$):

$$g(x, x_0) = 0 + P_x[H_{x_0} < \infty]\, g(x_0, x_0),$$

writing H_{x_0} for $H_{\{x_0\}}$, so that

$$(1.51) \qquad P_x[H_{x_0} < \infty] = \frac{g(x,x_0)}{g(x_0,x_0)}\,, \quad \text{for } x \in E.$$

A second application of (1.49) now yields (with $U = E \backslash \{x_0\}$)

$$(1.52) \qquad g_U(x,y) = g(x,y) - \frac{g(x,x_0)g(x_0,y)}{g(x_0,x_0)}\,, \quad \text{for } x,y \in E.$$

□

Given $A \subseteq E$, we introduce the **equilibrium measure of** $\boldsymbol{A}$:

$$(1.53) \qquad e_A(x) = P_x[\widetilde{H}_A = \infty]\, 1_A(x)\, \lambda_x, \; x \in E.$$

Its total mass is called the **capacity of** $\boldsymbol{A}$ (or the **conductance** of A):

$$(1.54) \qquad \mathrm{cap}(A) = \sum_{x \in A} P_x[\widetilde{H}_A = \infty]\, \lambda_x.$$

Remark 1.7. As we will see below in the case of $A = E$ the terminology in (1.53) is consistent with the terminology in (1.40). There is an interpretation of the weights $(c_{x,y})$ and the killing measures (κ_x) on E as an **electric network** grounding E at the cemetery point Δ, which is implicit in the use of the above terms, see for instance Doyle-Snell [3]. □

Before turning to the next proposition, we simply recall that given $A \subseteq E$, by our convention in (1.45)

$$\{H_A < \infty\} = \{L_A > 0\} = \text{ the set of trajectories that enter } A.$$

Also given a measure ν on E, we write

$$(1.55) \qquad P_\nu = \sum_{x \in E} \nu_x\, P_x \text{ and } E_\nu \text{ for the } P_\nu\text{-integral (or “expectation”)}.$$

Proposition 1.8. $(A \subseteq E)$

$$(1.56) \qquad P_x[L_A > 0, X_{L_A^-} = y] = g(x,y)\, e_A(y), \; \textit{for } x,y \in E,$$

($X_{L_A^-}$ is the position of $X_{\cdot}$ at the last visit to A, when $L_A > 0$).

$$(1.57) \qquad h_A(x) \overset{\text{def}}{=} P_x[H_A < \infty] = P_x[L_A > 0] = G\, e_A(x), \; \textit{for } x \in E$$

*(the **equilibrium potential of A**).*

When $A \neq \phi$,

(1.58) *e_A is the unique measure ν supported on A such that $G\nu = 1$ on A.*

Let $A \subseteq B \subseteq E$ then under P_{e_B} the entrance “distribution” in A and the last exit “distribution” of A coincide with e_A:

$$(1.59) \qquad P_{e_B}[H_A < \infty, X_{H_A} = y] = P_{e_B}[L_A > 0, X_{L_A^-} = y] = e_A(y), \; \textit{for } y \in E.$$

In particular when $B = E$,

(1.60) *under P_κ, the entrance distribution in A and the exit distribution of A coincide with e_A.*

Proof.

• (1.56):

Both members vanish when $y \notin A$. We thus assume $y \in A$. Using the discrete-time Markov chain Z_n, $n \geq 0$ (see (1.15)), we can write:

$$P_x[L_A > 0, X_{L_A^-} = y] = P_x\Big[\bigcup_{n\geq 0} \{Z_n = y, \text{ and for all } k > n, Z_k \notin A\}\Big] \overset{\text{pairwise disjoint}}{=}$$
$$\sum_{n\geq 0} P_x[Z_n = y, \text{ and for all } \; k > n, \; Z_k \notin A] \overset{\text{Markov property}}{=}$$
$$\sum_{n\geq 0} P_x[Z_n = y]\, P_y[\text{for all } \; k > 0, \; Z_k \notin A] \underset{(1.45)}{\overset{\text{Fubini}}{=}}$$
$$E_x\Big[\sum_{n\geq 0} 1\{Z_n = y\}\Big]\; P_y[\widetilde{H}_A = \infty] = E_x\Big[\int_0^\infty 1\{X_t = y\} dt\Big]\, P_y[\widetilde{H}_A = \infty] \underset{(1.53)}{\overset{(1.26)}{=}} g(x,y)\, e_A(y).$$

This proves (1.56).

• (1.57):

Summing (1.56) over $y \in A$, we obtain

$$P_x[H_A < \infty] = P_x[L_A > 0] = \sum_{y\in A} g(x,y)\, e_A(y) \overset{(1.33)}{=} Ge_A(x), \text{ whence } (1.57).$$

• (1.58):

Note that e_A is supported on A and $Ge_A = 1$ on A by (1.57). If ν is another such measure and $\mu = \nu - e_A$,

$$\langle \mu, G\mu \rangle = 0$$

because $G\mu = 0$ on A, and μ is supported on A. By (1.35) it follows that $\mu = 0$, whence (1.58).

• (1.59), (1.60):

By (1.50) (Hunt's switching identity): for $y \in E$,

$$E_{e_B}[H_A < \infty, g(X_{H_A}, y)] = E_y[H_A < \infty, (Ge_B)(X_{H_A})] \underset{A\subseteq B}{\overset{(1.58)}{=}} P_y[H_A < \infty].$$

Denoting by μ the entrance distribution of X in A under P_{e_B}:

$$\mu_x = P_{e_B}[H_A < \infty, X_{H_A} = x], \; x \in E,$$

we see by the above identity and (1.57) that $G\mu(y) = Ge_A(y)$, for all $y \in E$, and by applying L to both sides, $\mu = e_A$. As for the last exit distribution of $X_{\boldsymbol{.}}$ from A under P_{e_B}, integrating over e_B in (1.56), we find:

$$P_{e_B}[L_A > 0, X_{L_A^-} = y] = \sum_{x\in E} e_B(x)\, g(x,y)\, e_A(y) \underset{A\subseteq B}{\overset{(1.57)}{=}} e_A(y), \text{ for } y \in E.$$

This completes the proof of (1.59). In the special case $B = E$, we know by (1.40), (1.58) that $e_B = \kappa$ and (1.60) follows. □

We now provide **two variational problems** for the **capacity**, where the **equilibrium measure** and the **equilibrium potential** appear. These characterizations are, of course, strongly flavored by the previously mentioned analogy with electric networks (we refer to Remark 1.7).

Proposition 1.9. *($A \subseteq E$)*

$$\mathrm{cap}(A) = (\inf\{E(\nu,\nu);\ \nu \text{ probability supported on } A\})^{-1} \tag{1.61}$$

and when $A \neq \phi$, the infimum is uniquely attained at $\overline{e}_A = e_A/\mathrm{cap}(A)$, the normalized equilibrium measure of A.

$$\mathrm{cap}(A) = \inf\{\mathcal{E}(f,f);\ f \geq 1 \text{ on } A\}, \tag{1.62}$$

and the infimum is uniquely attained at h_A, the equilibrium potential of A.

Proof.

• (1.61):

When $A = \phi$, both members of (1.61) vanish and there is nothing to prove. We thus assume $A \neq \phi$ and consider a probability measure ν supported on A. By (1.35), we have

$$\begin{aligned} 0 \leq E(\nu,\nu) &= E(\nu - \overline{e}_A + \overline{e}_A,\ \nu - \overline{e}_A + \overline{e}_A) \\ &= E(\overline{e}_A, \overline{e}_A) + 2E(\nu - \overline{e}_A, \overline{e}_A) + E(\nu - \overline{e}_A, \nu - \overline{e}_A). \end{aligned}$$

The last term is non-negative, by (1.35) it only vanishes when $\nu = \overline{e}_A$, and

$$E(\nu - \overline{e}_A, \overline{e}_A) = \sum_{x \in E} \big(\nu_x - \overline{e}_A(x)\big) \underbrace{\Big(\sum_{y \in E} g(x,y)\,\overline{e}_A(y)\Big)}_{\overset{(1.58)}{=} \frac{1}{\mathrm{cap}(A)} \text{ on } A} = \frac{1-1}{\mathrm{cap}(A)} = 0\,.$$

We thus find that $E(\nu,\nu)$ becomes (uniquely) minimal at

$$E(\overline{e}_A, \overline{e}_A) = \frac{1}{\mathrm{cap}(A)^2} \sum_{x,y \in E} e_A(x)\, g(x,y)\, e_A(y) \overset{(1.58)}{=} \frac{1}{\mathrm{cap}(A)^2}\, e_A(A) = \frac{1}{\mathrm{cap}(A)}.$$

This proves (1.61).

• (1.62):

We consider $f\colon E \to \mathbb{R}$ such that $f \geq 1_A$, and $h_A = Ge_A$, so that

$$h_A(x) = P_x[H_A < \infty] = 1, \text{ for } x \in A.$$

We have

$$\begin{aligned} \mathcal{E}(f,f) &= \mathcal{E}(f - h_A + h_A, f - h_A + h_A) \\ &= \mathcal{E}(h_A, h_A) + 2\mathcal{E}(f - h_A, h_A) + \mathcal{E}(f - h_A, f - h_A). \end{aligned}$$

Again, the last term is non-negative and only vanishes when $f = h_A$, see (1.39). Moreover, we have

$$\mathcal{E}(f - h_A, h_A) = \mathcal{E}(f - h_A, Ge_A) \overset{(1.38)}{=} \langle e_A, f - h_A \rangle \geq 0,$$
$$\text{since } h_A = 1 \text{ on } A,\ f \geq 1 \text{ on } A, \text{ and } e_A \text{ is supported on } A.$$

So the right-hand side of (1.62) equals

$$\mathcal{E}(h_A, h_A) = \mathcal{E}(Ge_A, h_A) \overset{(1.38)}{=} \langle e_A, h_A \rangle = e_A(A) = \text{cap}(A).$$

This proves (1.62). □

Orthogonal decomposition, trace Dirichlet form:

We consider $U \subseteq E$ and set $K = E \backslash U$. Our aim is to describe the orthogonal complement relative to the Dirichlet form $\mathcal{E}(\cdot, \cdot)$ of the space of functions supported in U:

$$\mathcal{F}_U = \{\varphi : E \to \mathbb{R};\ \varphi(x) = 0, \text{ for all } x \in K\}. \tag{1.63}$$

To this end we introduce the space of functions harmonic in U:

$$\mathcal{H}_U = \{h : E \to \mathbb{R};\ Ph(x) = h(x), \text{ for all } x \in U\}, \tag{1.64}$$

as well as the space of potentials of (signed) measures supported on K:

$$\mathcal{G}_K = \{f : E \to \mathbb{R};\ f = G\nu, \text{ for some } \nu \text{ supported on } K\}. \tag{1.65}$$

Recall that $\mathcal{E}(\cdot, \cdot)$ is a positive definite quadratic form on the space $\mathcal{F}$ of functions from E to $\mathbb{R}$ (see above (1.11)).

Proposition 1.10. *(orthogonal decomposition)*

$$\mathcal{H}_U = \mathcal{G}_K. \tag{1.66}$$

$$\mathcal{F} = \mathcal{F}_U \oplus \mathcal{H}_U, \text{ where } \mathcal{F}_U \text{ and } \mathcal{H}_U \text{ are orthogonal, relative to } \mathcal{E}(\cdot, \cdot). \tag{1.67}$$

Proof.

• (1.66):

We first show that $\mathcal{H}_U \subseteq \mathcal{G}_K$. Indeed when $h \in \mathcal{H}_U$, $h \overset{(1.37)}{=} G(-L)\, h = G\nu$ where $\nu = -Lh$ is supported on K by (1.64) and (1.41). Hence $\mathcal{H}_U \subseteq \mathcal{G}_K$.

To prove the reverse inclusion we consider ν supported on K. Set $h = G\nu$. By (1.37) we know that $Lh = LG\nu = -\nu$, so that Lh vanishes on U. It follows from (1.41) that $h \in \mathcal{H}_U$, and (1.66) is proved. Incidentally note that choosing $A = K$ in (1.49), we can multiply both sides of (1.49) by ν_y and sum over y. The first term in the right-hand side vanishes and we then see that $h = G\nu$ satisfies

$$h(x) = E_x[H_K < \infty, h(X_{H_K})], \text{ for } x \in E. \tag{1.68}$$

• (1.67):

We first note that when $\varphi \in \mathcal{F}_U$ and ν is supported on K,

$$\mathcal{E}(G\nu, \varphi) \overset{(1.38)}{=} \langle \nu, \varphi \rangle = 0.$$

So the spaces $\mathcal{F}_U$ and $\mathcal{H}_U$ are orthogonal under $\mathcal{E}(\cdot,\cdot)$. In addition, given f from $E \to \mathbb{R}$, we can define

$$h(x) = E_x[H_K < \infty, f(X_{H_K})], \text{ for } x \in E, \tag{1.69}$$

and note that

$$h(x) = f(x), \text{ when } x \in K,$$

and that by the same argument as above,

$$h \text{ is harmonic in } U.$$

If we now define $\varphi = f - h$, we see that φ vanishes on K, and hence

$$f = \varphi + h, \text{ with } \varphi \in \mathcal{F}_U \text{ and } h \in \mathcal{H}_U, \tag{1.70}$$

is the orthogonal decomposition of f. This proves (1.67). □

As we now explain the restriction of the Dirichlet form to the space $\mathcal{H}_U = \mathcal{G}_K$, see (1.64) - (1.66), gives rise to a new Dirichlet form on the space of functions from K to $\mathbb{R}$, the so-called trace form.

Given f: $K \to \mathbb{R}$, we also write f for the function on E that agrees with f on K and vanishes on U, when no confusion arises. Note that

$$\widetilde{f}(x) = E_x[H_K < \infty, f(X_{H_K})], \text{ for } x \in E, \tag{1.71}$$

is the unique function on E, harmonic in U, that agrees with f on K, cf. (1.67). Indeed the decomposition (1.67) applied to the case of the function equal to f on K, and to 0 on U, shows the existence of a function in $\mathcal{H}_U$ equal to f on K. By (1.68) and (1.66), it is necessarily equal to $\widetilde{f}$.

We then define for f: $K \to \mathbb{R}$, the **trace form**

$$\begin{aligned} \mathcal{E}^*(f,f) &= \mathcal{E}(\widetilde{f},\widetilde{f}) \\ &\overset{(1.69),(1.70)}{=} \inf\{\mathcal{E}(g,g);\ g\colon E \to \mathbb{R} \text{ coincides with } f \text{ on } K\}, \end{aligned} \tag{1.72}$$

where we used in the second line the fact that when g coincides with f on K, then $g = \varphi + \widetilde{f}$, with $\varphi \in \mathcal{F}_U$, and hence $\mathcal{E}(g,g) \geq \mathcal{E}(\widetilde{f},\widetilde{f})$ due to (1.67). We naturally extend this definition for f, g: $K \to \mathbb{R}$, by setting

$$\mathcal{E}^*(f,g) = \mathcal{E}(\widetilde{f},\widetilde{g}). \tag{1.73}$$

It is plain that $\mathcal{E}^*$ is a symmetric bilinear form on the space of functions from K to $\mathbb{R}$. As we now explain $\mathcal{E}^*$ does indeed correspond to a Dirichlet form on K induced by some (uniquely defined in view of (1.10)) non-negative weights and killing measure.

Proposition 1.11. $(K \neq \phi)$

The quantities defined by

$$\begin{aligned} c^*_{x,y} &= \lambda_x\, P_x[\widetilde{H}_K < \infty, X_{\widetilde{H}_K} = y], \textit{ for } x \neq y \textit{ in } K, \\ &= 0, \textit{ for } x = y \textit{ in } K, \end{aligned} \tag{1.74}$$

$$\kappa^*_x = \lambda_x\, P_x[\widetilde{H}_K = \infty], \textit{ for } x \in K, \tag{1.75}$$

$$\lambda^*_x = \lambda_x(1 - P_x[\widetilde{H}_K < \infty,\, X_{\widetilde{H}_K} = x]), \textit{ for } x \in K, \tag{1.76}$$

satisfy (1.2) - (1.5), (1.7), with E replaced by K (in particular $c^*_{x,y} = c^*_{y,x}$). The corresponding Dirichlet form coincides with $\mathcal{E}^*$, i.e.

$$\mathcal{E}^*(f,f) = \frac{1}{2} \sum_{x,y \in K} c^*_{x,y}\big(f(y) - f(x)\big)^2 + \sum_{x \in K} \kappa^*_x f^2(x), \text{ for } f\colon K \to \mathbb{R}. \tag{1.77}$$

The corresponding Green function $g^*(x,y)$, x,y in K, satisfies

$$g^*(x,y) = g(x,y), \text{ for } x,y \in K. \tag{1.78}$$

Proof. We first prove that

$$\begin{array}{l}\text{the quantities in (1.74) - (1.76) satisfy (1.2) - (1.5), (1.7),}\\ \text{with } E \text{ replaced by } K.\end{array} \tag{1.79}$$

To this end we note that for $x \neq y$ in K,

$$\begin{aligned} -\mathcal{E}^*(1_x, 1_y) &\overset{(1.73)}{=} -\mathcal{E}(\widetilde{1}_x, \widetilde{1}_y) \overset{(1.67)}{=} -\mathcal{E}(1_x, \widetilde{1}_y) \overset{(1.44)}{=} L\widetilde{1}_y(x) \\ &\overset{\widetilde{1}_y(x)=0}{=} \lambda_x \sum_{z \in E} p_{x,z}\, \widetilde{1}_y(z) \underset{\text{Markov}}{\overset{(1.71)}{=}} \lambda_x\, P_x[\widetilde{H}_K < \infty,\, X_{\widetilde{H}_K} = y] \\ &= c^*_{x,y}. \end{aligned} \tag{1.80}$$

By a similar calculation we also find that for $x \in K$,

$$\begin{aligned} \mathcal{E}^*(1_x, 1_x) &= -\, L\, \widetilde{1}_x(x) = \lambda_x\Big(1 - \sum_{z \in E} p_{x,z}\, \widetilde{1}_x(z)\Big) \\ &= \lambda_x\big(1 - P_x[\widetilde{H}_K < \infty, X_{\widetilde{H}_K} = x]\big) \\ &= \lambda^*_x. \end{aligned} \tag{1.81}$$

We further see that

$$\begin{aligned} \sum_{y \in K} c^*_{x,y} + \kappa^*_x &\overset{(1.74),(1.75)}{=} \lambda_x(1 - P_x[\widetilde{H}_K < \infty, X_{\widetilde{H}_K} = x]) \\ &\overset{(1.76)}{=} \lambda^*_x. \end{aligned} \tag{1.82}$$

From (1.80) we deduce the symmetry of $c^*_{x,y}$. These are non-negative weights on K. Moreover when x_0, y_0 are in K, we can find a nearest neighbor path in E from x_0 to y_0 and looking at the successive visits of K by this path, taking (1.74) into account, we see that K endowed with the edges $\{x,y\}$ for which $c^*_{x,y} > 0$, is a connected graph.

Further we know that for all x in E, P_x-a.s., the continuous-time chain on E reaches the cemetery state Δ after a finite time. As a result $P_y[\widetilde{H}_K = \infty] > 0$, for at least one y in K, since otherwise the chain starting from any x in K would a.s. never reach Δ. By (1.75) we thus see that κ^* does not vanish everywhere on K. In addition (1.7) holds by (1.82). We have thus proved (1.79).

• (1.77):
Expanding the square in the first sum in the right-hand side of (1.77), we see using the symmetry of $c^*_{x,y}$, (1.82), and the second line of (1.74), that the right-hand side of (1.77) equals

$$\begin{aligned} &\sum_{x \in K} \lambda^*_x\, f^2(x) - \sum_{x \neq y \text{ in } K} c^*_{x,y}\, f(x)\, f(y) \overset{(1.80),(1.81)}{=} \\ &\sum_{x \in K} \mathcal{E}^*(1_x, 1_x)\, f^2(x) + \sum_{x \neq y \text{ in } K} \mathcal{E}^*(1_x, 1_y)\, f(x)\, f(y) = \mathcal{E}^*(f,f), \end{aligned}$$

and this proves (1.77).

• (1.78):

Consider $x \in K$ and ψ_x the restriction to K of $g(x,\cdot)$. By (1.66) we see that $g(x,\cdot) = G1_x(\cdot) = \widetilde{\psi}_x(\cdot)$, and therefore for any $y \in K$ we have

$$\mathcal{E}^*(\psi_x, 1_y) \overset{(1.73)}{=} \mathcal{E}(G1_x, \widetilde{1}_y) \overset{(1.66),(1.67)}{=} \mathcal{E}(G1_x, 1_y) \overset{(1.38)}{=} 1_{\{x=y\}}$$
$$\overset{(1.38)}{=} \mathcal{E}^*(\psi_x^*, 1_y), \text{ if } \psi_x^*(\cdot) = g^*(x,\cdot).$$

It follows that $\psi_x = \psi_x^*$ for any x in K, and this proves (1.78). □

Remark 1.12.

1) The trace form, with its expressions (1.72), (1.77), is intimately related to the notion of network reduction, or electrical network equivalence, see [1], p. 56.

2)When $K \subseteq K' \subseteq E$ are non-empty subsets of E, the trace form on K, of the trace form on K' of $\mathcal{E}$, coincides with the trace form on K of $\mathcal{E}$. Indeed this follows for instance by (1.78) and the fact that the Green function determines the Dirichlet form, see (1.37), (1.10). This feature is referred to as the "**tower property**" of **traces**. □

1.4 Feynman-Kac formula

Given a function $V\colon E \to \mathbb{R}$, we can also view V as a multiplication operator:

$$(Vf)(x) = V(x)\, f(x), \text{ for } f\colon E \to \mathbb{R}. \tag{1.83}$$

In this short section we recall a celebrated probabilistic representation formula for the operator $e^{t(P-I+V)}$, when $t \geq 0$. We recall the convention stated above (1.18).

Theorem 1.13. *(Feynman-Kac formula)*

For $V, f : E \to \mathbb{R}$, $t \geq 0$, one has

$$E_x\Big[f(X_t)\, \exp\Big\{\int_0^t V(X_s)ds\Big\}\Big] = \big(e^{t(P-I+V)}f\big)(x), \text{ for } x \in E, \tag{1.84}$$

(the case $V = 0$ corresponds to (1.18)).

Proof. We denote by $S_t\, f(x)$ the left-hand side of (1.84). By the Markov property we see that for $t, s \geq 0$,

$$\begin{aligned} S_{t+s}\, f(x) &= E_x\Big[f(X_{t+s})\, \exp\Big\{\int_0^{t+s} V(X_u)du\Big\}\Big] \\ &= E_x\Big[\exp\Big\{\int_0^t V(X_u)du\Big\}\, \exp\Big\{\int_0^s V(X_u)du\Big\} \circ \theta_t\, f(X_s) \circ \theta_t\Big] \\ &= E_x\Big[\exp\Big\{\int_0^t V(X_u)du\Big\}\, E_{X_t}\Big[\exp\Big\{\int_0^s V(X_u)du\Big\}\, f(X_s)\Big]\Big] \\ &= E_x\Big[\exp\Big\{\int_0^t V(X_u)du\Big\}\, S_s\, f(X_t)\Big] = S_t(S_s f)(x) = (S_t S_s) f(x). \end{aligned}$$

In other words S_t, $t \ge 0$ has the semi-group property

$$S_{t+s} = S_t S_s, \text{ for } t, s \ge 0. \tag{1.85}$$

Moreover, observe that

$$\begin{aligned}\frac{1}{t}\,(S_t\, f - f)(x) &= \frac{1}{t}\, E_x\Big[f(X_t)\, \exp\Big\{\int_0^t V(X_s)ds\Big\} - f(X_0)\Big] \\ &= \frac{1}{t}\, E_x[f(X_t) - f(X_0)] + E_x\Big[f(X_t)\, \frac{1}{t}\, \int_0^t V(X_s)\, e^{\int_0^s V(X_u)du} ds\Big],\end{aligned}$$

and as $t \to 0$,

$$\frac{1}{t}\, E_x[f(X_t) - f(X_0)] \to (P - I)\, f(x), \text{ by (1.18)},$$

whereas by dominated convergence

$$E_x\Big[f(X_t)\, \frac{1}{t}\, \int_0^t V(X_s)\, e^{\int_0^s V(X_u)du} ds\Big] \to E_x[f(X_0)\, V(X_0)] = V f(x).$$

So we see that

$$\frac{1}{t}\,(S_t\, f - f)(x) \underset{t\to 0}{\longrightarrow} (P - I + V)\, f(x). \tag{1.86}$$

Then considering $S_{t+h}\, f(x) - S_t\, f(x) = (S_h - I)\, S_t\, f(x)$, with $h > 0$ small, as well as (when $t > 0$ and $0 < h < t$)

$$S_{t-h}\, f(x) - S_t\, f(x) = -(S_h - I)\, S_{t-h}\, f(x),$$

one sees that (using in the second case that $\sup_{u \le t} |S_u\, f(x)| \le e^{t\|V\|_\infty} \|f\|_\infty$) the function $t \ge 0 \to S_t\, f(x)$ is continuous.

Now dividing by h and letting $h \to 0$, we find that

(1.87) $t \ge 0 \to S_t\, f(x)$ is continuously differentiable with derivative:
$(P - I + V)\, S_t\, f(x)$.

It now follows that the function

$$s \in [0, t] \to F(s) = e^{(t-s)(P-I+V)} S_s\, f(x)$$

is continuously differentiable on $[0, t]$, with derivative

$$F'(s) = -e^{(t-s)(P-I+V)}(P - I + V)\, S_s\, f(x) + e^{(t-s)(P-I+V)}(P - I + V)\, S_s\, f(x) = 0.$$

We thus find that $F(0) = F(t)$ so that $e^{t(P-I+V)} f(x) = S_t\, f(x)$. This proves (1.84). □

1.5 Local times

In this short section we define the local time of the Markov chain X_t, $t \geq 0$, and discuss some of its basic properties.

The local time of $X_{\bf .}$ at site $x \in E$, and time $t \geq 0$, is defined as

(1.88) $$L_t^x = \int_0^t 1\{X_s = x\}\, ds\, \frac{1}{\lambda_x}\,.$$

Note that the normalization is different from (0.2) (we have not yet introduced $\overline{X}_t$, $t \geq 0$). We extend (1.88) to the case $x = \Delta$ (cemetery point) with the convention

(1.89) $$\lambda_\Delta = 1,\ L_t^\Delta = \int_0^t 1\{X_s = \Delta\} ds, \ \text{ for } t \geq 0.$$

By direct inspection of (1.88) we see that for $x \in E$, $t \in [0,\infty) \to L_t^x \in [0,\infty)$ is a continuous non-decreasing function with a finite limit L_∞^x (because $X_t = \Delta$ for t large enough). We record in the next proposition a few simple properties of the local time.

Proposition 1.14.

(1.90) $$E_x[L_\infty^y] = g(x,y), \ \textit{ for } x,y \in E.$$

(1.91) $$E_x[L_{T_U}^y] = g_U(x,y), \ \textit{ for } x,y \in E,\, U \subseteq E.$$

(1.92) $$\sum_{x \in E \cup \{\Delta\}} V(x)\, L_t^x = \int_0^t \frac{V}{\lambda}\,(X_s) ds, \ \textit{ for } t \geq 0,\, V: E \cup \{\Delta\} \to \mathbb{R}.$$

(1.93) $$L_t^x \circ \theta_s + L_s^x = L_{t+s}^x, \ \textit{for } x \in E,\ s \geq 0 \ \ \textit{(additive function property)}.$$

Proof.

- (1.90):

$$E_x[L_\infty^y] = E_x\Big[\int_0^\infty 1\{X_t = y\}\, \frac{dt}{\lambda_y}\Big] \overset{(1.26)}{=} g(x,y).$$

- (1.91):

Analogous argument to (1.90), cf. (1.45), (1.47), and Remark 1.5.

- (1.92):

$$\sum_{x \in E \cup \{\Delta\}} V(x)\, L_t^x = \sum_{x \in E \cup \{\Delta\}} V(x) \int_0^t 1\{X_s = x\}\, \frac{ds}{\lambda_x} =$$
$$\int_0^t \sum_{x \in E \cup \{\Delta\}} \frac{V}{\lambda}\,(x)\, 1\{X_s = x\}\, ds = \int_0^t \frac{V}{\lambda}\,(X_s) ds\,.$$

- (1.93):

Note that $\int_0^{t+s} V(X_u) du = (\int_0^t V(X_u) du) \circ \theta_s + \int_0^s V(X_u) du$, and apply this identity with $V(\cdot) = \frac{1}{\lambda_x}\, 1_x(\cdot)$. □

1.6 The Markov chain $\overline{X}_.$ (with variable jump rate)

Using time change, we construct in this section the Markov chain $\overline{X}_.$ with same discrete skeleton Z_n, $n \ge 0$, as $X_.$, but with variable jump rate λ_x, $x \in E \cup \{\Delta\}$. We describe the transition semi-group attached to $\overline{X}_.$, relate the local times for $\overline{X}_.$ and $X_.$, and briefly discuss the Feynman-Kac formula for $\overline{X}_.$. As a last topic, we explain how the trace process of $\overline{X}_.$ on a subset K of E is related to the trace Dirichlet form introduced in Section 1.4.

We define

$$L_t = \sum_{x \in E \cup \{\Delta\}} L_t^x = \int_0^t \lambda_{X_s}^{-1} \, ds, \; t \ge 0, \tag{1.94}$$

so that $t \in \mathbb{R}_+ \to L_t \in \mathbb{R}_+$ is a continuous, strictly increasing, piecewise differentiable function, tending to ∞. In particular it is an increasing bijection of $\mathbb{R}_+$, and using the formula for the derivative of the inverse one can write for the inverse function of $L_.$:

$$\tau_u = \inf\{t \ge 0; \; L_t \ge u\} = \int_0^u \lambda_{X_{\tau_v}} \, dv = \int_0^u \lambda_{\overline{X}_v} \, dv, \tag{1.95}$$

where we have introduced the time changed process (with values in $E \cup \{\Delta\}$)

$$\overline{X}_u \overset{\text{def}}{=} X_{\tau_u}, \text{ for } u \ge 0, \tag{1.96}$$

(the path of $\overline{X}_.$ thus belongs to D_E, cf. above (1.17)).

We also introduce the local times of $\overline{X}_.$ (note that the normalization is different from (1.88), but in agreement with (0.2)):

$$\overline{L}_u^x \overset{\text{def}}{=} \int_0^u 1\{\overline{X}_v = x\} \, dv, \;\; \text{for } u \ge 0, \, x \in E \cup \{\Delta\}. \tag{1.97}$$

Proposition 1.15. *$\overline{X}_u$, $u \ge 0$, is a Markov chain with cemetery state Δ and sub-Markovian transition semi-group on E:*

$$\overline{R}_t \, f(x) \overset{\text{def}}{=} E_x[f(\overline{X}_t)] = e^{tL} f(x), \; \textit{for } t \ge 0, \, x \in E, \; f: E \to \mathbb{R}, \tag{1.98}$$

(i.e. $\overline{X}_.$ has the jump rate λ_x in x and jumps according to $p_{x,y}$ in (1.12), (1.14)).

Moreover one has the identities:

$$X_t = \overline{X}_{L_t}, \;\; \textit{for } t \ge 0 \; (\textit{“time } t \textit{ for } X_. \textit{ is time } L_t \textit{ for } \overline{X}_.\textit{”}), \tag{1.99}$$

and

$$L_t^x = \overline{L}_{L_t}^x, \;\; \textit{for } x \in E \cup \{\Delta\}, \; t \ge 0, \tag{1.100}$$

$$L_\infty^x = \overline{L}_\infty^x, \;\; \textit{for } x \in E. \tag{1.101}$$

Proof.

• Markov property of $\overline{X}_{\textstyle .}$: (sketch)

Note that τ_u, $u \geq 0$, are $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$-stopping times and using the fact that L_t, $t \geq 0$, satisfies the additive functional property

$$L_{t+s} = L_s + L_t \circ \theta_s, \quad \text{for } t, s \geq 0,$$

we see that $L_{\tau_u \circ \theta_{\tau_v} + \tau_v} = u + v$, and taking inverses

$$\tau_{u+v} = \tau_u \circ \theta_{\tau_v} + \tau_v, \tag{1.102}$$

and

$$\overline{X}_{u+v} \underset{(1.102)}{\overset{(1.96)}{=}} X_{\tau_u \circ \theta_{\tau_v} + \tau_v} = X_{\tau_u} \circ \theta_u = \overline{X}_u \circ \theta_{\tau_v}.$$

(Note incidentally that $\overline{\theta}_u = \theta_{\tau_u}$, for $u \geq 0$, satisfies the semi-flow property $\overline{\theta}_{u+v} = \overline{\theta}_u \circ \overline{\theta}_v$, for $u, v \geq 0$, and, in this notation, the above equality reads $\overline{X}_{u+v} = \overline{X}_u \circ \overline{\theta}_v$.)

It now follows that for $u, v \geq 0$, $B \in \mathcal{F}_{\tau_v}$, one has for $f : E \cup \{\Delta\} \to \mathbb{R}$,

$$\begin{aligned} &E_x[f(\overline{X}_{u+v})\, 1_B] = E_x[f(X_{\tau_u} \circ \theta_{\tau_v})\, 1_B] \overset{\text{strong Markov for } X_{\textstyle .}}{=} \\ &E_x[E_{X_{\tau_v}}[f(X_{\tau_u})]\, 1_B] = E_x[E_{\overline{X}_v}[f(\overline{X}_u)]\, 1_B]\,. \end{aligned}$$

Since for $v' \leq v$, $\overline{X}_{v'} = X_{\tau_{v'}}$ are (see for instance Proposition 2.18, p. 9 of [12]) $\mathcal{F}_{\tau_v}$-measurable, this proves the Markov property.

• (1.98):

From the Markov property one deduces that $\overline{R}_t$, $t \geq 0$, is a sub-Markovian semi-group.

Now for f: $E \to \mathbb{R}$, $x \in E$, $u > 0$, one has

$$\frac{1}{u}\,(\overline{R}_u\, f - f)(x) = \frac{1}{u}\, E_x[f(\overline{X}_u) - f(\overline{X}_0)] = \frac{1}{u}\, E_x[f(X_{\tau_u}) - f(X_0)].$$

By (1.95) we see that $\tau_u \leq cu$, for $u \geq 0$. We also know that the probability that $X_{\textstyle .}$ jumps at least twice in $[0, t]$ is $o(t)$ as $t \to 0$. So as $u \to 0$,

$$\begin{aligned} &\frac{1}{u}\; E_x[f(X_{\tau_u}) - f(X_0)] = \\ &\frac{1}{u}\, E_x\big[f(X_{\tau_u}) - f(X_0),\ X_{\textstyle .} \text{ has exactly one jump in } [0, \tau_u]\big] + o(1) = \\ &E_x[f(Z_1) - f(X_0)]\, \frac{1}{u}\, P_x[L_{T_1} \leq u] + o'(1), \quad \text{with } T_1 \text{ the first jump of } X_{\textstyle .}, \\ &\overset{(1.94)}{=} (Pf - f)(x)\, \frac{1}{u}\, P_x\Big[\frac{T_1}{\lambda_x} \leq u\Big] + o'(1) \underset{u \to 0}{\longrightarrow} \lambda_x (Pf - f)(x) \overset{(1.37)}{=} Lf(x). \end{aligned}$$

So we have shown that:

$$\frac{1}{u}\,(\overline{R}_u\, f - f)(x) \underset{u \to 0}{\longrightarrow} Lf(x)\,. \tag{1.103}$$

Just as below (1.86) one now shows the corresponding statement to (1.87):

(1.104) $\quad u \geq 0 \to \overline{R}_u f(x)$ is continuously differentiable with derivative $L\overline{R}_u f(x)$.

One then concludes in the same fashion as below (1.87), that $e^{uL} f(x) = \overline{R}_u f(x)$, and this proves (1.98).

• (1.99):

By (1.96), $\overline{X}_{L_t} = X_{\tau_{L_t}} = X_t$, for $t \geq 0$, whence (1.99).

• (1.100):

$$\frac{d}{dt}\,\overline{L}^x_{L_t} = \frac{d\overline{L}^x_u}{du}\Big|_{u=L_t} \times \frac{dL_t}{dt} \overset{(1.97)}{\underset{(1.94)}{=}} 1\{\overline{X}_{L_t} = x\}\,\frac{1}{\lambda_{X_t}} \overset{(1.96)}{=} 1\{X_t = x\}\,\frac{1}{\lambda_x} = \frac{dL^x_t}{dt}\,,$$

except when t is a jump time of $X_.$, and integrating we find (1.100).

• (1.101):

Letting $t \to \infty$ in (1.100) yields $\overline{L}^x_\infty = L^x_\infty$, that is (1.101). □

One then has the Feynman-Kac formula for $\overline{X}_.$.

Theorem 1.16. *(Feynman-Kac formula for $\overline{X}_.$)*

For $V, f : E \to \mathbb{R}$, $u \geq 0$, one has

(1.105)
$$E_x\Big[f(\overline{X}_u)\,\exp\Big\{\int_0^u V(\overline{X}_v)dv\Big\}\Big] = e^{u(L+V)} f(x), \text{ for } x \in E.$$

Proof. The proof is similar to that of (1.84). One simply uses (1.98) in place of (1.18). □

Remark 1.17. As a closing remark for Chapter 1, we briefly sketch a link between the Markov chain obtained as the trace of $\overline{X}_.$ on a non-empty subset K of E and the trace form $\mathcal{E}^*$, cf. (1.72) and Proposition 1.11. To this end we introduce

(1.106)
$$\overline{L}^K_u = \sum_{x \in K \cup \{\Delta\}} \overline{L}^x_u = \int_0^u 1\big\{\overline{X}_v \in K \cup \{\Delta\}\big\}\,dv, \ \text{ for } u \geq 0,$$

which is a continuous non-decreasing function of u tending to infinity, and its right-continuous inverse,

(1.107)
$$\overline{\tau}^K_v = \inf\{u \geq 0;\, \overline{L}^K_u > v\}, \ \text{ for } v \geq 0.$$

The trace process of $\overline{X}_.$ on K is defined as

(1.108)
$$\overline{X}^K_v = \overline{X}_{\overline{\tau}^K_v}, \ \text{ for } v \geq 0$$

(intuitively at time v, $\overline{X}^K_.$ is at the location where $\overline{X}_.$ sits once $\overline{L}^K_.$ accumulates $v + \varepsilon$ units of time, with $\varepsilon \to 0$). With similar arguments as in the case of $\overline{X}_.$ (see the proof of Proposition 1.15, in particular, using the strong Markov property of $\overline{X}_.$ and the fact that, in the notation from below (1.102), $\overline{X}^K_{u+v} = \overline{X}^K_u \circ \overline{\theta}_{\overline{\tau}^K_v}$), one can show that under P_x, $x \in K \cup \{\Delta\}$, $\overline{X}^K_v$, $v \geq 0$, is a Markov chain on K with cemetery state Δ.

One can further show that its corresponding sub-Markovian transition semi-group on K has the form

$$\overline{R}_t^K f(x) = E_x[f(\overline{X}_t^K)] = e^{t\,L^*} f(x), \text{ for } x \in K,\ f\colon K \to \mathbb{R}, \tag{1.109}$$

where in the notation of (1.74), (1.76),

$$L^* f(x) = \sum_{y \in K} c^*_{x,y}\, f(y) - \lambda^*_x\, f(x), \text{ for } x \in K,\ f\colon K \to \mathbb{R}. \tag{1.110}$$

To see this last point one notes that if $\mathcal{L}$ stands for the generator of $\overline{X}_{\cdot}^K$, the inverse of $-\mathcal{L}$ has the $K \times K$ matrix:

$$\begin{aligned} E_x\Big[\int_0^\infty 1\{\overline{X}_v^K = y\}\, dv\Big] &= E_x\Big[\int_0^\infty 1\{\overline{X}_u = y\}\, du\Big] \\ &\overset{(1.101),(1.90)}{=} g(x,y) \overset{(1.78)}{=} g^*(x,y), \ \text{ for } x,y \in K. \end{aligned} \tag{1.111}$$

A similar identity holds for the continuous-time chain on K with variable jump rate $\lambda^*_{\cdot}$ attached to the weights $c^*_{x,y}$ and the killing measure $\kappa^*_{\cdot}$. Its generator is L^* (by Proposition 1.15), and we thus find that $\mathcal{L} = L^*$. □

2 Isomorphism theorems

In this chapter we will discuss the isomorphism theorems of Dynkin and Eisenbaum mentioned in the introduction, cf. (0.4) - (0.7), as well as some of the so-called generalized Ray-Knight theorems, in the terminology of Marcus-Rosen [19]. We still need to introduce some objects such as the Gaussian free field and the measures on paths entering the Dynkin isomorphism theorem. We keep the same set-up and notation as in Chapter 1.

2.1 The Gaussian free field

In this section we define the Gaussian free field. We also describe the conditional law of the field given its values on a given subset K of E, as well as the law of its restriction to K. Interestingly this brings into play the orthogonal decomposition under the Dirichlet form and the notion of trace form discussed in Section 1.4.

As we now see we can use $g(x,y)$, $x,y \in E$, as the covariance function of a centered Gaussian field indexed by E. An important step to this effect is (1.35).

We endow the canonical space $\mathbb{R}^E$ of functions on E with the canonical product σ-algebra and with the canonical coordinates

(2.1) $$\varphi_x : f \in \mathbb{R}^E \to \varphi_x(f) = f(x), \;\; x \in E.$$

Proposition 2.1. *There exists a unique probability P^G on $\mathbb{R}^E$, under which*

(2.2) $$\begin{array}{l} (\varphi_x)_{x \in E} \text{ is a centered Gaussian field with covariance} \\ E^G[\varphi_x \, \varphi_y] = g(x,y), \text{ for } x,y \in E. \end{array}$$

Proof.

Uniqueness

Under such a P^G, for any ν: $E \to \mathbb{R}$, $\langle \nu, \varphi \rangle = \sum_{x \in E} \nu_x \, \varphi_x$ is a centered Gaussian variable with variance

$$\sum_{x,y} \nu_x \, \nu_y \, E^G[\varphi_x \, \varphi_y] = \sum_{x,y \in E} \nu_x \, \nu_y \, g(x,y) = E(\nu,\nu).$$

As a result

(2.3) $$E^G[e^{i\langle \nu, \varphi \rangle}] = e^{-\frac{1}{2}\, E(\nu,\nu)}, \text{ for any } \nu : E \to \mathbb{R}.$$

This specifies the characteristic function of P^G, and hence P^G is unique.

Existence

We give both an abstract and a concrete construction of the law P^G.

Abstract construction:

We choose ν_ℓ, $1 \le \ell \le |E|$ an orthonormal basis for $E(\cdot,\cdot)$, cf. (1.35), of the space of measures, and consider the dual basis f_i, $1 \le i \le |E|$, of functions so: $\langle \nu_\ell, f_i \rangle = \delta_{\ell,i}$, for $1 \le i$, $\ell \le |E|$. If ξ_i, $i \ge 1$ are i.i.d. $N(0,1)$ variables on some auxiliary space $(\Omega, \mathcal{A}, P)$, we define the random function

(2.4) $$\psi(\cdot,\omega) = \sum_{1 \le i \le |E|} \xi_i(\omega) \, f_i(\cdot).$$

For any $x \in E$, $1_x = \sum_{\ell=1}^{|E|} E(1_x, \nu_\ell)\, \nu_\ell$, so that

$$\psi(x,\omega) = \langle 1_x, \psi(\cdot,\omega)\rangle = \sum_{1 \le \ell, i \le |E|} E(1_x, \nu_\ell)\, \xi_i(\omega) \langle \nu_\ell, f_i\rangle = \sum_{\ell=1}^{|E|} E(1_x, \nu_\ell)\, \xi_\ell.$$

It now follows that for $x, y \in E$

$$\begin{aligned} E^P[\psi(x,\omega)\, \psi(y,\omega)] &= \sum_{1 \le \ell, \ell' \le |E|} E(1_x, \nu_\ell)\, E(1_y, \nu_{\ell'})\, E^P[\xi_\ell\, \xi_{\ell'}] = \sum_{1 \le \ell \le |E|} E(1_x, \nu_\ell)\, E(1_y, \nu_\ell) \\ &\overset{\text{Parseval}}{=} E(1_x, 1_y) = g(x,y). \end{aligned}$$

So the law of $\psi(\cdot,\omega)$ on $\mathbb{R}^E$ satisfies (2.2).

Concrete construction:

The matrix $g(x,y)$, $x, y \in E$, has inverse $\langle -L1_x, 1_y\rangle$, $x, y \in E$, cf. (1.37), and hence under the probability

$$P^G = \frac{1}{(2\pi)^{\frac{|E|}{2}} \sqrt{\det G}}\ \exp\Big\{ -\tfrac{1}{2}\, \mathcal{E}(\varphi,\varphi)\Big\} \prod_{x \in E} d\varphi_x \tag{2.5}$$

(using that $\sum_{x,y \in E} \varphi_x\, \varphi_y \langle -L1_x, 1_y\rangle = \langle -L\varphi, \varphi\rangle \overset{(1.44)}{=} \mathcal{E}(\varphi,\varphi)$), $(\varphi_x)_{x \in E}$ is a centered Gaussian vector with covariance $g(x,y)$, $x, y \in E$, i.e. (2.2) holds. □

Remark 2.2. In the above abstract construction the dual basis f_i, $1 \le i \le |E|$, of ν_ℓ, $1 \le \ell \le |E|$, is simply given by

$$f_i = G\nu_i,\ 1 \le i \le |E|. \tag{2.6}$$

Indeed $\langle \nu_\ell, f_i\rangle = \langle \nu_\ell, G\nu_i\rangle = E(\nu_\ell, \nu_i) = \delta_{\ell,i}$. Note that f_i, $1 \le i \le |E|$, is an orthonormal basis under $\mathcal{E}(\cdot,\cdot)$:

$$\mathcal{E}(f_i, f_j) = \mathcal{E}(G\nu_i, G\nu_j) \overset{(1.38)}{=} \langle \nu_i, G\nu_j\rangle = E(\nu_i, \nu_j) = \delta_{i,j}.$$

□

Conditional expectations:

We consider $K \subseteq E$, $U = E \backslash K$, and want to describe the conditional law under P^G of $(\varphi_x)_{x \in U}$ given $(\varphi_x)_{x \in K}$, as well as the law of $(\varphi_x)_{x \in K}$. The orthogonal decomposition in Proposition 1.10 together with the description of the trace form in Proposition 1.11 will be useful for this purpose. We write $P^{G,U}$ for the law on $\mathbb{R}^E$ of the centered Gaussian field with covariance

$$E^{G,U}[\varphi_x\, \varphi_y] = g_U(x,y),\ \text{for } x, y \in E, \tag{2.7}$$

(so $\varphi_x = 0$, $P^{G,U}$-a.s., when $x \in K$).

Proposition 2.3. *$(K \ne \phi)$*

For $x \in E$, define on $\mathbb{R}^E$ the $\sigma(\varphi_y, y \in K)$-measurable

$$\begin{aligned} h_x &= E_x[H_K < \infty,\ \varphi_{X_{H_K}}] \\ &= \sum_{y \in K} P_x[H_K < \infty,\ X_{H_K} = y]\, \varphi_y \quad \textit{(so } h_x = \varphi_x, \textit{ for } x \in K). \end{aligned} \tag{2.8}$$

Then we can write

(2.9) $$\varphi_x = \psi_x + h_x, \text{ for } x \in E \quad (\text{so } \psi_x = 0, \text{ for } x \in K).$$

Under P^G,

(2.10) $$(\psi_x)_{x\in E} \text{ is independent from } \sigma(\varphi_y, y \in K),$$

and

(2.11) $$(\psi_x)_{x\in E} \text{ is distributed as } (\varphi_x)_{x\in E} \text{ under } P^{G,U}.$$

In addition, in the notation of (1.77),

(2.12) $$(\varphi_x)_{x\in K} \text{ has the law } \frac{1}{(2\pi)^{\frac{|K|}{2}}\sqrt{\det_{K\times K} G}}\ \exp\Big\{-\frac{1}{2}\,\mathcal{E}^*(\varphi,\varphi)\Big\}\prod_{x\in K} d\varphi_x,$$

where $\det_{K\times K} G$ *denotes the determinant of the* $K\times K$*-matrix obtained by restricting* $g(\cdot,\cdot)$ *to* $K\times K$.

Proof.

• (2.10):

For any $x \in E$, ψ_x belongs to the linear space generated by the centered jointly Gaussian collection φ_z, $z \in E$. In addition when $x \in E$ and $y \in K$, we find that by (2.8), (2.9) and (2.2),

$$\begin{aligned} E^G[\psi_x\,\varphi_y] &= E^G[\varphi_x\,\varphi_y] - E^G[h_x\,\varphi_y] \\ &= g(x,y) - \sum_{z\in K} P_x[H_K < \infty, X_{H_K} = z]\, g(z,y) \\ &= g(x,y) - E_x[H_K<\infty, g(X_{H_K},y)] \overset{(1.49)}{=} 0. \end{aligned}$$

The claim (2.10) now follows.

• (2.11):

Since $\psi_x = 0$, for $x \in K$, and $P^{G,U}$-a.s., $\varphi_x = 0$, for $x\in K$, we only need to focus on the law of $(\psi_x)_{x\in U}$ under P^G. When F is a bounded measurable function on $\mathbb{R}^U$, we find that by (2.5), setting c as the inverse of $(2\pi)^{\frac{|E|}{2}}\sqrt{\det G}$, we have

$$E^G\big[F\big((\psi_x)_{x\in U}\big)\big] = c\int_{\mathbb{R}^E} F\big((\varphi_x - h_x)_{x\in U}\big)\ \exp\Big\{-\frac{1}{2}\,\mathcal{E}(\varphi,\varphi)\Big\}\prod_{x\in E} d\varphi_x.$$

Using Proposition 1.10 and (1.71), (1.72), we see that for φ in $\mathbb{R}^E$,

$$\mathcal{E}(\varphi,\varphi) = \mathcal{E}(\varphi-h,\varphi-h) + \mathcal{E}^*(\varphi_{|K},\varphi_{|K}),$$

where $\varphi_{|K}$ denotes the restriction to K of $\varphi\in\mathbb{R}^E$. We then make a change of variables in the above integral. We set $\varphi'_x = \varphi_x - h_x$, for $x\in U$, and $\varphi'_x = \varphi_x$, for $x \in K$, and note that the Jacobian of this transformation equals 1, so that $\prod_{x\in E} d\varphi'_x = \prod_{x\in E} d\varphi_x$. We thus see that for all F as above:

$$E^G[F\big((\psi_x)_{x\in U}\big)] = \tag{2.13}$$
$$c\int_{\mathbb{R}^E} F\big((\varphi_x)_{x\in U}\big)\ \exp\Big\{-\frac{1}{2}\,\mathcal{E}(\varphi_U,\varphi_U) - \frac{1}{2}\,\mathcal{E}^*(\varphi_{|K},\varphi_{|K})\Big\}\prod_{x\in E} d\varphi_x,$$

where we have set $\varphi_U(x) = 1_U(x)\varphi_x$. Integrating over the variables φ_x, $x\in K$, we find that for a suitable constant c',

$$= c'\int_{\mathbb{R}^U} F\big((\varphi_x)_{x\in U}\big)\ \exp\Big\{-\frac{1}{2}\,\mathcal{E}(\varphi_U,\varphi_U)\Big\}\prod_{x\in U} d\varphi_x$$

and using Remark 1.5 and (2.5)

$$= E^{G,U}\big[F\big((\varphi_x)_{x\in U}\big)\big].$$

This proves (2.11).

• (2.12):

A simple modification of the above calculation replacing $F\big((\psi_x)_{x\in U}\big)$ by $H\big((\varphi_x)_{x\in K}\big)$, with H a bounded measurable functions on $\mathbb{R}^K$, yields (2.12). □

Remark 2.4. As a result of Proposition 2.3, when x is a given point of U, under P^G, conditionally on the variables φ_y, $y\in K$,

(2.14) φ_x is distributed as a Gaussian variable with mean $E_x[H_K<\infty, \varphi_{X_{H_K}}]$, and variance $g_U(x,x)$.

Note that by Proposition 1.9 (and Remark 1.5), for $x\in U$,

(2.15) $g_U(x,x)$ is the inverse of the minimum energy of a function taking the value 1 on x and 0 on K

(this provides an interpretation of the conditional variance as an effective resistance between x and $K\cup\{\Delta\}$). □

2.2 The measures $P_{x,y}$

In this section we introduce a further ingredient of the Dynkin isomorphism theorem, namely the kind of measures on paths that appear in (0.4), which live on paths in E with finite duration that go from x to y. We provide several descriptions of these measures, and derive an identity for the Laplace transform of the local time of the path, which prepares the ground for the proof of the Dynkin isomorphism theorem in the next section.

We introduce the space of E-valued trajectories with duration $t\ge 0$:

(2.16) Γ_t = the space of right-continuous functions $[0,t]\to E$, with finitely many jumps, left-continuous at t.

We still denote by X_s, $0\le s\le t$, the canonical coordinates, and by convention we set $X_s=\Delta$ (cemetery point) if $s>t$.

We then define the space of E-valued trajectories with finite duration as

$$\Gamma = \bigcup_{t>0} \Gamma_t \,. \tag{2.17}$$

For $\gamma \in \Gamma$, we denote the duration of γ by

$$\zeta(\gamma) = \text{the unique } t > 0 \text{ such that } \gamma \in \Gamma_t. \tag{2.18}$$

The σ-algebra we choose on Γ is simply obtained by "transport". We use the bijection Φ: $\Gamma_1 \times (0,\infty) \to \Gamma$:

$$(w,t) \in \Gamma_1 \times (0,\infty) \xrightarrow{\Phi} \gamma(\cdot) = w\Big(\frac{\cdot}{t}\Big) \in \Gamma,$$

where we endow $\Gamma_1 \times (0,\infty)$ with the canonical product σ-algebra (and Γ_1 is endowed with the σ-algebra generated by the maps X_s, $0 \le s \le 1$, from Γ_1 into E). We thus take the image by Φ of the σ-algebra on $\Gamma_1 \times (0,\infty)$, and obtain the σ-algebra on Γ. We define for $x, y \in E$, $t > 0$, the measure

$$\begin{aligned} &P^t_{x,y} \text{ the image on } \Gamma_t \text{ of } 1\{X_t = y\}\, \frac{P_x}{\lambda_y}, \\ &\text{under the map: } (X_s)_{0\le s\le t} \text{ from } D_E \cap \{X_t = y\} \text{ into } \Gamma_t (\subseteq \Gamma). \end{aligned} \tag{2.19}$$

Note that the total mass of $P^t_{x,y}$ is

$$P^t_{x,y}[\Gamma_t] = \frac{1}{\lambda_y}\, P_x[X_t = y] \overset{(1.22)}{=} r_t(x,y). \tag{2.20}$$

We then define the finite measure $P_{x,y}$ on Γ via:

$$P_{x,y}[B] = \int_0^\infty P^t_{x,y}\,[B]\, dt, \tag{2.21}$$

for any measurable subset B of Γ (noting that $t > 0 \to P^t_{x,y}$ defines a finite measure kernel from $(0,\infty)$ to Γ). The total mass of $P_{x,y}$ is

$$P_{x,y}[\Gamma] = \int_0^\infty P^t_{x,y}[\Gamma]\, dt \overset{(2.20)}{=} \int_0^\infty r_t(x,y)\, dt \overset{(1.26)}{=} g(x,y). \tag{2.22}$$

The next proposition describes some relations between the measures $P_{x,y}$ and P_x. In particular it provides an interpretation of $P_{x,y}$ as a non-normalized h-transform of P_x, with $h(\cdot) = g(\cdot, y)$, see for instance Section 3.9 of [19]. The Remark 2.6 below gives yet an other description of $P_{x,y}$.

Proposition 2.5. $(x, y \in E)$

For $0 < t_1 < \cdots < t_n$, $x_1, \dots, x_n \in E$, *one has*

$$\begin{aligned} &P_{x,y}[X_{t_1} = x_1, \dots, X_{t_n} = x_n] = P_x[X_{t_1} = x_1, \dots, X_{t_n} = x_n]\, g(x_n, y) = \\ &r_{t_1}(x,x_1)\, r_{t_2 - t_1}(x_1, x_2) \dots r_{t_n - t_{n-1}}(x_{n-1}, x_n)\, g(x_n, y)\, \lambda_{x_1} \dots \lambda_{x_n}. \end{aligned} \tag{2.23}$$

If $K \subseteq E$ and H_K is defined as in (1.45), for $B \in \sigma(X_{H_K \wedge s}, s \geq 0)$ and ζ as in (2.18),

$$P_{x,y}[B, H_K \leq \zeta] = E_x[B \cap \{H_K < \infty\},\, g(X_{H_K}, y)]\,. \tag{2.24}$$

Proof.

• (2.23):

$$P_{x,y}[X_{t_1} = x_1, \dots, X_{t_n} = x_n] \overset{(2.21)}{=} \int_0^\infty P^t_{x,y}[X_{t_1} = x_1, \dots, X_{t_n} = x_n]\,, dt$$
$$= \int_{t_n}^\infty P_x[X_{t_1} = x_1, \dots, X_{t_n} = x_n,\ X_t = y]\,\frac{dt}{\lambda_y} \overset{\text{Markov property at time } t_n}{=}$$
$$\int_{t_n}^\infty E_x[X_{t_1} = x_1, \dots, X_{t_n} = x_n,\ r_{t-t_n}(x_n, y)]\,dt$$
$$\overset{(1.26)}{=} P_x[X_{t_1} = x_1, \dots, X_{t_n} = x_n]\,g(x_n, y),$$

and the second equality of (2.23) follows from the Markov property.

• (2.24):

$$P_{x,y}[B, H_K \leq \zeta] = \int_0^\infty P^t_{x,y}[B, H_K \leq \zeta]\,dt \overset{(2.18),(2.19)}{=}$$
$$\int_0^\infty E_x[B, H_K \leq t,\ X_t = y]\,\frac{dt}{\lambda_y} \overset{\text{strong Markov property}}{=}$$
$$\int_0^\infty E_x[B, H_K \leq t,\ r_{t-H_K}(X_{H_K}, y)]\,dt \underset{(1.26)}{\overset{\text{Fubini}}{=}} E_x[B, H_K < \infty,\ g(X_{H_K}, y)].$$

□

Remark 2.6. Given $\gamma \in \Gamma$, we can introduce $N(\gamma) \geq 0$, the number of jumps of γ strictly before $\zeta(\gamma)$, the duration of γ, and when $N(\gamma) = n \geq 1$, we can consider $0 < T_1(\gamma) < \dots < T_n(\gamma) < \zeta(\gamma)$ the successive jump times of X_s, $0 \leq s \leq \zeta(\gamma)$.

As we now explain, for $n \geq 1$, $t_i > 0$, $1 \leq i \leq n$, $t > 0$, and $x_1, \dots, x_n \in E$, one has the following formula complementing Proposition 2.5:

$$\begin{aligned}&P_{x,y}[N = n, X_{T_1} = x_1, \dots, X_{T_n} = x_n,\\ &\qquad T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n, \zeta \in t + dt] =\\ &\frac{c_{x,x_1}\, c_{x_1,x_2} \dots c_{x_{n-1},y}}{\lambda_x\, \lambda_{x_1} \dots \lambda_{x_{n-1}}\, \lambda_y}\, \delta_{x_n,y}\ 1\{0 < t_1 < t_2 < \dots < t_n < t\}\, e^{-t}\, dt_1 \dots dt_n\, dt,\end{aligned} \tag{2.25}$$

where the precise meaning of (2.25) is obtained by considering some subsets $A_1, \dots, A_n$, A of $(0, \infty)$, replacing "$T_1 \in t_1 + dt_1$",…,"$T_n \in t_n + dt_n$", "$\zeta \in t + dt$" by $\{T_1 \in A_1\}, \dots, \{T_n \in A_n\}$, $\{\zeta \in A\}$, in the left-hand side of (2.25), and in the right-hand side multiplying by $1_{A_1}(t_1) \dots 1_{A_n}(t_n)\, 1_A(t)$, and integrating the expression over the variables $t_1, \dots, t_n, t$.

To find (2.25), we note that for $t > 0$,

$$P^t_{x,y}[N = n,\ X_{T_1} = x_1, \dots, X_{T_n} = x_n,\ T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n] \overset{(2.19)}{=}$$

$$P_x[X_. \text{ has } n \text{ jumps in } [0,t],\ X_{T_1} = x_1, \dots, X_{T_n} = x_n,$$
$$T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n]\, \delta_{x_n,y}\, \lambda_y^{-1} =$$

$$P_x[Z_1 = x_1, Z_2 = x_2, \dots, Z_n = x_n]\, P_x[T_n < t < T_{n+1},$$
$$T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n]\, \delta_{x_n,y}\, \lambda_y^{-1} =$$

$$\frac{c_{x,x_1}\, c_{x_1,x_2} \dots c_{x_{n-1},y}}{\lambda_x\, \lambda_{x_1} \dots \lambda_{x_{n-1}}\, \lambda_y}\ 1\{0 < t_1 < \dots < t_n < t\}\, e^{-t}\, dt_1 \dots dt_n\, \delta_{x_n,y},$$

where we made use of (1.12), (1.15), and Remark 1.1. Multiplying both sides by dt readily yields (2.25), in view of (2.18), (2.21).

Similarly, we see that for $n = 0$, we have

$$P_{x,y}[N = 0,\ \zeta \in t + dt] = \delta_{x,y}\, e^{-t}\, dt. \tag{2.26}$$

We record an interesting consequence of (2.25), (2.26). We denote by $\overset{\vee}{\gamma} \in \Gamma$ the "time reversal" of $\gamma \in \Gamma$, i.e. $\overset{\vee}{\gamma}$ is the element of Γ such that $\zeta(\overset{\vee}{\gamma}) = \zeta(\gamma)$, $\overset{\vee}{\gamma}(0) = \gamma(\zeta)$, $\overset{\vee}{\gamma}(\zeta) = \gamma(0)$, and $\overset{\vee}{\gamma}(s) = \lim_{\varepsilon \downarrow 0} \gamma(\zeta - s - \varepsilon)$, for $0 < s < \zeta(\gamma)$.

Observe that on $\{N = n\}$ one can reconstruct γ from $T_1, \dots, T_n, \zeta$, and $X_{T_1}, \dots, X_{T_n}$. It is then a straightforward consequence of (2.25), (2.26) that

$$P_{y,x} \text{ is the image of } P_{x,y} \text{ under the map } \gamma \to \overset{\vee}{\gamma}. \tag{2.27}$$

We will later see some analogous formulas to (2.25), (2.26) for rooted loops in Proposition 3.1, see also (3.40). □

We will now provide some formulas for moments of $\int_0^\infty V(X_s)ds$ and L^z_∞ under the measure $P_{x,y}$.

Proposition 2.7. $(x, y \in E)$

For $V\colon E \to \mathbb{R}$ and $n \geq 0$, one has, in the notation of (1.32) and (1.83),

$$E_{x,y}\Big[\Big(\int_0^\infty V(X_s)ds\Big)^n\Big] = n!\ \big((QV)^n g^y\big)(x),\ \text{with } g^y(\cdot) = (G1_y)(\cdot) = g(\cdot, y). \tag{2.28}$$

For $x_1, x_2, \dots, x_n \in E$, one has

$$E_{x,y}\Big[\prod_{i=1}^n L^{x_i}_\infty\Big] = \sum_{\sigma \in \mathcal{S}_n} g(x, x_{\sigma(1)})\, g(x_{\sigma(1)}, x_{\sigma(2)}) \dots g(x_{\sigma(n)}, y), \tag{2.29}$$

with $\mathcal{S}_n$ the set of permutations of $\{1, \dots, n\}$.

When $\|\, G|V|\, \|_\infty < 1$, one has

$$E_{x,y}\Big[\exp\Big\{\sum_{z \in E} V(z)\, L^z_\infty\Big\}\Big] = \big((I - GV)^{-1} g^y\big)(x) = \big((I - GV)^{-1} G1_y\big)(x). \tag{2.30}$$

Proof. We begin with a slightly more general calculation and consider

$V_1, \dots, V_n : E \to \mathbb{R}$ and

$$E_{x,y}\Big[\prod_{i=1}^n \int_0^\infty V_i(X_s) ds\Big] = E_{x,y}\Big[\int_{\mathbb{R}_+^n} V_1(X_{s_1}) \dots V_n(X_{s_n})\, ds_1 \dots ds_n\Big]$$

and decomposing over the various orthants

$$= \sum_{\sigma \in \mathcal{S}_n} \int_{0 < s_{\sigma(1)} < \dots < s_{\sigma(n)} < \infty} E_{x,y}[V_1(X_{s_1}) \dots V_n(X_{s_n})]\, ds_1 \dots ds_n$$

$$\overset{(2.23)}{=} \sum_{\sigma \in \mathcal{S}_n} \int_{0 < s_{\sigma(1)} < \dots < s_{\sigma(n)}} \sum_{x_1, \dots, x_n} r_{s_{\sigma(1)}}(x, x_1)\, V_{\sigma(1)}(x_1)\, \lambda_{x_1}\, r_{s_{\sigma(2)} - s_{\sigma(1)}}(x_1, x_2)\, V_{\sigma(2)}(x_2)\, \lambda_{x_2}$$
$$\dots r_{s_{\sigma(n)} - s_{\sigma(n-1)}}(x_{n-1}, x_n)\, V_{\sigma(n)}(x_n)\, g(x_n, y)\, \lambda_{x_n}\, ds_{\sigma(1)} \dots ds_{\sigma(n)}$$

integrating over $ds_{\sigma(n)}$ and summing over x_n

$$\overset{(1.32)}{=} \sum_{\sigma \in \mathcal{S}_n} \int_{0 < s_{\sigma(1)} < \dots < s_{\sigma(n-1)} < \infty} \sum_{x_1, \dots, x_{n-1}} r_{s_{\sigma(1)}}(x, x_1)\, V_{\sigma(1)}(x_1)\, \lambda_{x_1} \dots \lambda_{x_{n-1}} (QV_{\sigma(n)} g^y)(x_{n-1})$$
$$ds_{\sigma(1)} \dots ds_{\sigma(n-1)}$$

and by induction

$$= \sum_{\sigma \in \mathcal{S}_n} (QV_{\sigma(1)}\, QV_{\sigma(2)} \dots QV_{\sigma(n)} g^y)(x).$$

In other words, we have:

$$E_{x,y}\Big[\prod_{i=1}^n \int_0^\infty V_i(X_s)\, ds\Big] = \sum_{\sigma \in \mathcal{S}_n} (QV_{\sigma(1)} QV_{\sigma(2)} \dots QV_{\sigma(n)} g^y)(x). \tag{2.31}$$

• (2.28):

We choose $V_i = V$, $1 \le i \le n$, and (2.31) yields (2.28).

• (2.29):

We chose $V_i = \frac{1}{\lambda_{x_i}} 1_{x_i}$, $1 \le i \le n$, and note that $QV_i = g^{x_i}$, so that (2.31) yields

$$E_{x,y}\Big[\prod_{i=1}^n L_\infty^{x_i}\Big] = \sum_{\sigma \in \mathcal{S}_n} g(x, x_{\sigma(1)})\, g(x_{\sigma(1)}, x_{\sigma(2)}) \dots g(x_{\sigma(n)}, y),$$

i.e., (2.29) is proved.

• (2.30):

$$E_{x,y}\Big[\exp\Big\{\sum_{z \in E} V(z)\, L_\infty^z\Big\}\Big] \overset{(1.92)}{=} E_{x,y}\Big[\sum_{n \ge 0} \frac{1}{n!} \Big(\int_0^\infty \frac{V}{\lambda}\, (X_s)\, ds\Big)^n\Big]. \tag{2.32}$$

The calculation below shows we can apply dominated convergence:

$$\sum_{n \ge 0} \frac{1}{n!}\, E_{x,y}\Big[\Big(\int_0^\infty \frac{|V|}{\lambda}\, (X_s)\, ds\Big)^n\Big] \overset{(2.28)}{=} \sum_{n \ge 0} \Big(\Big(Q\, \frac{|V|}{\lambda}\Big)^n g^y\Big)(x)$$
$$\overset{(1.33)}{=} \sum_{n \ge 0} ((G|V|)^n g^y)(x) < \infty,$$

since $\|G|V|\,\|_\infty < 1$.

So the left hand-side of (2.32) equals

$$\sum_{n\ge 0} \frac{1}{n!}\, E_{x,y}\Big[\Big(\int_0^\infty \frac{V}{\lambda}\,(X_s)\,ds\Big)^n\Big] \overset{(2.28)}{\underset{(1.33)}{=}} \sum_{n\ge 0} \big((GV)^n g^y\big)(x)$$

and since $\|GV\|_{L^\infty(E)\to L^\infty(E)} < 1$, we find that

$$\begin{aligned}&\sum_{n\ge 0}(GV)^n = (I-GV)^{-1}, \text{ so that}\\ &E_{x,y}\Big[\exp\Big\{\sum_{x\in E} V(x)\,L_\infty^x\Big\}\Big] = \big((I-GV)^{-1}g^y\big)(x),\end{aligned}$$

i.e. (2.30) holds. □

2.3 Isomorphism theorems

This section is devoted to the isomorphism theorems of Dynkin and Eisenbaum, which explore the nature of the relations between occupation time and free field. We refer to Marcus-Rosen [19] for a discussion of these theorems under very general assumptions.

We first state and prove the Dynkin isomorphism theorem, which shows that L_∞^z, $z \in E$, under $P_{x,y}$, has similar features, in a suitable sense, to $\frac{1}{2}\varphi_z^2$, $z \in E$, under P^G.

Theorem 2.8. ***(Dynkin isomorphism theorem)***

For any $x, y \in E$, and bounded measurable F on $\mathbb{R}^E$, one has

$$E_{x,y}\otimes E^G\Big[F\Big(\big(L_\infty^z + \frac{1}{2}\,\varphi_z^2\big)_{z\in E}\Big)\Big] = E^G\Big[\varphi_x\,\varphi_y\,F\Big(\big(\frac{1}{2}\,\varphi_z^2\big)_{z\in E}\Big)\Big]. \tag{2.33}$$

In other words:

$\big(L_\infty^z + \frac{1}{2}\,\varphi_z^2\big)_{z\in E}$ *under $P_{x,y}\otimes P^G$, has the same "law" as*

$\big(\frac{1}{2}\,\varphi_z^2\big)_{z\in E}$ *under $\varphi_x\,\varphi_y\,P^G$ ($\leftarrow$ signed measure when $x \neq y$!).*

Proof. We will show that for small $V\colon E \to \mathbb{R}$,

$$E_{x,y}\otimes E^G\Big[\exp\Big\{\sum_{z\in E} V(z)\Big(L_\infty^z + \frac{1}{2}\,\varphi_z^2\Big)\Big\}\Big] = E^G\Big[\varphi_x\,\varphi_y\,\exp\Big\{\sum_{z\in E} V(z)\,\frac{1}{2}\,\varphi_z^2\Big\}\Big] \tag{2.34}$$

(i.e. both integrals are well-defined and equal).

Let us first explain how the claim (2.33) then follows.

For any $V\colon E\to\mathbb{R}$, it follows by application of (2.34) to $u_0 V$ and $-u_0 V$ for small $u_0 > 0$, that $\cosh(u_0 \sum_z V(z)(L_\infty^z + \frac{1}{2}\,\varphi_z^2))$ is integrable for $P_{x,y}\otimes P^G$ and $\cosh(u_0\sum_z V(z)\frac{1}{2}\,\varphi_z^2))$ is integrable under $|\varphi_x\,\varphi_y|\,P^G$, and hence the functions

$$u \in (-u_0,u_0) + i\,\mathbb{R}(\supseteq \mathbb{C}) \longrightarrow E_{x,y}\otimes E^G\Big[\exp\Big\{u\sum_{z\in E} V(z)\Big(L_\infty^z + \frac{1}{2}\,\varphi_z^2\Big)\Big\}\Big]$$

and

$$u \in (-u_0,u_0) + i\,\mathbb{R} \longrightarrow E^G\Big[\varphi_x\,\varphi_y\,\exp\Big\{u\sum_{z\in E} V(z)\,\frac{1}{2}\,\varphi_z^2\Big\}\Big]$$

are analytic. And by (2.34) they are equal for u small and real. Hence they agree in $(-u_0, u_0) + i\,\mathbb{R}$, and in particular for the choice $u = i$, that is for any $V: E \to \mathbb{R}$,

$$E_{x,y} \otimes E^G\Big[\exp\Big\{i \sum_{z\in E} V(z)\Big(L^z_\infty + \frac{1}{2}\,\varphi_z^2\Big)\Big\}\Big] = E^G\Big[\varphi_x\,\varphi_y\,\exp\Big\{i \sum_{z\in E} V(z)\,\frac{1}{2}\,\varphi_z^2\Big\}\Big].$$

This means that the characteristic function of the law of $L^z_\infty + \frac{1}{2}\,\varphi_z^2$, $z \in E$, under $P_{x,y} \otimes P^G$ equals the characteristic function of the law (i.e. image measure) of $\frac{1}{2}\,\varphi_z^2$, $z \in E$, under $\varphi_x\,\varphi_y\,P^G$. It follows that these laws are equal (in particular the law of $\frac{1}{2}\,\varphi_z^2$, $z \in E$, under the signed measure $\varphi_x\,\varphi_y\,P^G$, is a positive measure!), and (2.33) is proved.

We now turn to the proof of (2.34).

We already know by (2.30) that when V is small

$$\tag{2.35} E_{x,y}\Big[\exp\Big\{\sum_{z\in E} V(z)\,L^z_\infty\Big\}\Big] = \big((I - GV)^{-1}\,G1_y\big)(x).$$

Moreover, by (2.5), we see that for small V, the random variable $\exp\{\frac{1}{2}\sum_{z\in E} V(z)\,\varphi_z^2\}$ is P^G-integrable, because when V is small, thanks to (1.39), $\mathcal{E}_V(\varphi,\varphi) \overset{\text{def}}{=} \mathcal{E}(\varphi,\varphi) - \sum_{z\in E} V(z)\,\varphi_z^2$ is a positive definite quadratic form. In addition, for any φ: $E \to \mathbb{R}$:

$$\tag{2.36} \mathcal{E}_V(\varphi,\varphi) \overset{(1.44)}{=} \langle -L\varphi,\varphi\rangle - \langle V\varphi,\varphi\rangle = \langle (-L-V)\,\varphi,\varphi\rangle.$$

Thus if we define the probability measure on $\mathbb{R}^E$:

$$\tag{2.37} P^{G,V} = \frac{1}{(2\pi)^{\frac{|E|}{2}}\sqrt{\det G}\,E^G\big[\exp\big\{\frac{1}{2}\sum_{z\in E} V(z)\,\varphi_z^2\big\}\big]}\;\exp\Big\{-\frac{1}{2}\,\mathcal{E}_V(\varphi,\varphi)\Big\}\prod_{x\in E} d\varphi_x,$$

then, the field φ_z, $z \in E$, is a centered Gaussian field under $P^{G,V}$, with covariance matrix $\langle (-L-V)^{-1}1_z, 1_{z'}\rangle$, $z, z' \in E$. Note that

$$\tag{2.38} (-L-V)^{-1} \overset{(1.37)}{=} (-L(I-GV))^{-1} = (I-GV)^{-1}(-L)^{-1} = (I-GV)^{-1}\,G.$$

As a result, we see that when V is small:

$$E^{G,V}[\varphi_x\,\varphi_y] = \big((I-GV)^{-1}\,G1_y\big)(x) \overset{(2.35)}{=} E_{x,y}\Big[\exp\Big\{\sum_{z\in E} V(z)\,L^z_\infty\Big\}\Big].$$

Multiplying these equalities by the P^G-expectation in the denominator of the normalizing constant of $P^{G,V}$ in (2.37) yields (2.34), and this concludes the proof of (2.33). □

The fact that the measures $P_{x,y}$ and not simply the measures P_x appear in the Dynkin isomorphism theorem makes its use somewhat cumbersome in a number of applications. We now discuss the Eisenbaum isomorphism theorem, which does not make use of the measures $P_{x,y}$, but instead directly involves the measures P_x.

As a preparation, we record the statements corresponding to (2.28) - (2.30), when $P_{x,y}$ is replaced by P_x.

Proposition 2.9. *For $V\colon E \to \mathbb{R}$, and $n \geq 0$, one has*

$$E_x\Big[\Big(\int_0^\infty V(X_s)\,ds\Big)^n\Big] = n!\,\big((QV)^n 1_E\big)(x), \text{ for } x \in E. \tag{2.39}$$

For $x, x_1, \dots, x_n \in E$, one has

$$E_x\Big[\prod_{i=1}^n L_\infty^{x_i}\Big] = \sum_{\sigma\in\mathcal{S}_n} g(x, x_{\sigma(1)})\, g(x_{\sigma(1)}, x_{\sigma(2)}) \dots g(x_{\sigma(n-1)}, x_{\sigma(n)}) \quad \text{(Kac's moment formula)}. \tag{2.40}$$

When $\|G|V|\,\|_\infty < 1$, one has

$$E_x\Big[\exp\Big\{\sum_{z\in E} V(z)\,L_\infty^z\Big\}\Big] = \big((I - GV)^{-1} 1_E\big)(x), \text{ for } x \in E. \tag{2.41}$$

Proof. The proofs are straightforward modifications of the proofs of (2.28), (2.29), (2.30), replacing (2.23) by the identity

$$\begin{aligned} &P_x[X_{t_1} = x_1, \dots, X_{t_n} = x_n] = \\ &r_{t_1}(x, x_1)\, r_{t_2 - t_1}(x_1, x_2) \dots r_{t_n - t_{n-1}}(x_{n-1}, x_n)\, \lambda_{x_1} \dots \lambda_{x_n}, \\ &\text{for } 0 < t_1 < \dots < t_n \text{ and } x, x_1, \dots, x_n \in E. \end{aligned} \tag{2.42}$$

□

We are now ready to state and prove

Theorem 2.10. ***(Eisenbaum isomorphism theorem)***

For any $x \in E$, $s \neq 0$, and bounded measurable F on $\mathbb{R}^E$, one has

$$E_x \otimes E^G\Big[F\Big(\Big(L_\infty^z + \frac{(\varphi_z + s)^2}{2}\Big)_{z\in E}\Big)\Big] = E^G\Big[\Big(1 + \frac{\varphi_x}{s}\Big)\, F\Big(\Big(\frac{(\varphi_z + s)^2}{2}\Big)_{z\in E}\Big)\Big], \tag{2.43}$$

in other words:

$\big(L_\infty^z + \frac{1}{2}\,(\varphi_z + s)^2\big)_{z\in E}$ *under $P_x \otimes P^G$, has the same "law" as*

$\big(\frac{1}{2}\,(\varphi_z + s)^2\big)_{z\in E}$ *under $\big(1 + \frac{\varphi_x}{s}\big)\, P^G$ (← signed measure!).*

Proof. The same arguments we employed below (2.34), show that it suffices to prove that for small $V\colon E \to \mathbb{R}$,

$$\begin{aligned} &E_x \otimes E^G\Big[\exp\Big\{\sum_{z\in E} V(z)\Big(L_\infty^z + \frac{(\varphi_z + s)^2}{2}\Big)\Big\}\Big] = \\ &E^G\Big[\Big(1 + \frac{\varphi_x}{s}\Big)\, \exp\Big\{\sum_{z\in E} V(z) \frac{(\varphi_z + s)^2}{2}\Big\}\Big] \end{aligned} \tag{2.44}$$

(i.e. both integrals are well-defined and equal).

We already know by (2.41) that for small V,

$$E_x\Big[\exp\Big\{\sum_{z\in E} V(z)\,L_\infty^z\Big\}\Big] = \big((I - GV)^{-1} 1_E\big)(x). \tag{2.45}$$

We further note that when V is small $\exp\{\frac{1}{2}\sum_{z\in E} V(z)(\varphi_z+s)^2\}$ is P^G is integrable (see above (2.36)), and

$$\begin{aligned}
&\frac{E^G\big[\big(1+\frac{\varphi_x}{s}\big)\ \exp\big\{\frac{1}{2}\sum\limits_{z\in E} V(z)(\varphi_z+s)^2\big\}\big]}{E^G\big[\exp\big\{\sum\limits_{z\in E} V(z)\frac{(\varphi_z+s)^2}{2}\big\}\big]} = 1 + \frac{E^G\big[\varphi_x\, e^{\frac{1}{2}\sum_{z\in E} V(z)(\varphi_z+s)^2}\big]}{s\, E^G\big[e^{\frac{1}{2}\sum_{z\in E} V(z)(\varphi_z+s)^2}\big]} = \\
&1+\frac{1}{s}\ \frac{E^{G,V}\big[\varphi_x \exp\big\{s\sum\limits_{z\in E} V(z)\varphi_z\big\}\big]}{E^{G,V}\big[\exp\big\{s\sum\limits_{z\in E} V(z)\varphi_z\big\}\big]}\,,
\end{aligned} \tag{2.46}$$

where $P^{G,V}$ is defined in (2.37).

We are going to use the next

Lemma 2.11. *If (X,Y) is a two-dimensional centered Gaussian vector, then for $s \neq 0$,*

$$\frac{E[X\ \exp\{sY\}]}{s\, E[\exp\{sY\}]} = E[XY]. \tag{2.47}$$

Proof. For t, s in $\mathbb{R}$, we have

$$\begin{aligned}
E[\exp\{tX+sY\}] &= \exp\Big\{\frac{1}{2}\ E[(tX+sY)^2]\Big\} \\
&= \exp\Big\{\frac{1}{2}\ \big(t^2\, E[X^2] + 2ts\, E[XY] + s^2 E[Y^2]\big)\Big\}.
\end{aligned}$$

By differentiating in t and setting $t=0$, we find

$$\begin{aligned}
E[X\exp\{sY\}] &= s\, E[XY]\exp\Big\{\frac{s^2}{2}\ E[Y^2]\Big\} \\
&= s\, E[XY] E[\exp\{sY\}].
\end{aligned}$$

This proves (2.47). □

We apply (2.47) with $X=\varphi_x$ and $Y=\sum_{z\in E} V(z)\,\varphi_z$. We find that the last expression in (2.46) equals

$$\begin{aligned}
1+E^{G,V}\Big[\varphi_x \sum_{z\in E} V(z)\,\varphi_z\Big] &= 1 + \sum_{z\in E}\ E^{G,V}[\varphi_x\,\varphi_z]\,V(z) \\
&\overset{(2.38)}{=} 1+\big((I-GV)^{-1}\,GV\big)(x).
\end{aligned} \tag{2.48}$$

Note that $(I-GV)^{-1} = (I-GV)^{-1}(I-GV+GV) = I+(I-GV)^{-1}GV$, so the expressions in (2.45) and (2.48) are equal. The claim (2.43) follows. □

2.4 Generalized Ray-Knight theorems

We will now discuss some so-called "generalized Ray-Knight theorems" (we will explain the terminology below, see also [8] and [19] for a presentation of these results in a general framework). These results are closely linked to the isomorphism theorems of Dynkin and Eisenbaum, see also [6] and [19].

We begin with a direct application of the isomorphism theorem of Eisenbaum. We consider $x_0 \in E$, and assume that κ_x vanishes for $x \neq x_0$. We set

$$U = E\backslash\{x_0\}, \tag{2.49}$$

(so $T_U \overset{P_x\text{-a.s.}}{=} H_{x_0}$, for all $x \in E$, by our assumptions on κ), and denote by $P^{G,U}$ the law on $\mathbb{R}^E$ of the centered Gaussian free field with covariance

$$E^{G,U}[\varphi_x\,\varphi_y] = g_U(x,y), \text{ for } x,y \in E. \tag{2.50}$$

Theorem 2.12. ***(Generalized first Ray-Knight theorem)***

For any $x \in E$, and $s \neq 0$,

$$\begin{array}{l} \Big(L^z_{H_{x_0}} + \frac{1}{2}\,(\varphi_z + s)^2\Big)_{z\in E} \text{ under } P_x \otimes P^{G,U}, \text{ has the same "law" as} \\ \Big(\frac{1}{2}\,(\varphi_z+s)^2\Big)_{z\in E} \text{ under } (1 + \frac{\varphi_x}{s})\,P^{G,U}. \end{array} \tag{2.51}$$

Proof. This is the direct application of (2.43) to the case where U replaces E, $g_U(\cdot,\cdot)$ replaces $g(\cdot,\cdot)$ and $L^z_{H_{x_0}}$ now plays the role of L^z_∞, with the help of Remark 1.5 1) and 2). □

Remark 2.13.

1) The above terminology stems from the fact that the corresponding statement in the case of Brownian motion when $x_0 = 0 < x$ in $\mathbb{R}$, can be shown to be equivalent, see Marcus-Rosen [19], p. 367, to the statement

$$\begin{array}{l} \big(L^z_{H_0} + (B^2_{z-x} + \widetilde{B}^2_{z-x})\,1_{\{z\ge x\}}\big)_{z\ge 0} \text{ has the same law as} \\ (B^2_z + \widetilde{B}^2_z)_{z\ge 0}, \end{array} \tag{2.52}$$

where $(L^z_t, z \in \mathbb{R}, t \ge 0)$, $(B_z, z \ge 0)$, $(\widetilde{B}_z, z \ge 0)$ are independent and respectively distributed as the local time process of Brownian motion starting at x and two independent copies of Brownian motion starting at 0. Note incidentally that in the present situation $g_U(z,z') = 2\; z \wedge z'$, for $z, z' \ge 0$, so that $(\varphi_z, z \ge 0)$ under $P^{G,\{0\}^c=U}$ is distributed as $\sqrt{2}\,B_z, z \ge 0$.

The statement (2.52) can be shown to be equivalent to the more classical statement of the first Ray-Knight theorem (see Marcus and Rosen [19], p. 52):

(2.53) $L^z_{H_0}$, for $z \in [0,x]$, under P_x (the law of Brownian motion starting from x), has the law of a two-dimensional squared Bessel process starting at 0, and then proceeds from x as a zero-dimensional squared Bessel process (i.e. conditional on $(L^z_{H_0})_{0\le z\le x}$, $(L^{x+y}_{H_0})_{y\ge 0}$ is distributed as a zero-dimensional squared Bessel process),

which stems from the work of Ray [21] and Knight [13]. We also recall that (see Revuz-Yor [23], chapter XI) the law of the δ-dimensional squared Bessel process starting from x, with $\delta \ge 0$, and $x \ge 0$, is that of the solution of the stochastic differential equation:

$$Z_t = x + 2\int_0^t \sqrt{Z_s}\,dB_s + \delta t,\; t \ge 0, \tag{2.54}$$

where B_x, $s \geq 0$, is a Brownian motion. This law is commonly denoted by $BESQ^\delta(x)$. It satisfies the important relation, see Revuz-Yor [23], p. 410:

(2.55) If $Z_.$ and $Z'_.$ are independent, respectively $BESQ^\delta(x)$ and $BESQ^{\delta'}(x')$-distributed, then $Z_. + Z'_.$ is $BESQ^{\delta+\delta'}(x+x')$-distributed, for $\delta, \delta', x, x' \geq 0$.

2) The type of argument which enables to go from (2.52) to (2.53) uses (2.55) (when $\delta = \delta' = 1$, noting that for $a \geq 0$, $BESQ^1(a)$ is the law of the square of Brownian motion starting from $\sqrt{a}$), as well as the following property:

(2.56) If X, Y, Z are random n-vectors with non-negative components such that X, Z are independent, and Y, Z are independent, and $X + Z \overset{\text{law}}{=} Y + Z$, then $X \overset{\text{law}}{=} Y$.

(*Proof.* Let $\mathcal{L}_X(t) = E[e^{-\sum_i^n t_i X_i}]$, for $t \in \mathbb{R}_+^n$, be the Laplace transform of X, and $\mathcal{L}_Y, \mathcal{L}_Z$ be defined analogously, then we see, by independence, that $\mathcal{L}_X(t)\,\mathcal{L}_Z(t) = \mathcal{L}_{X+Z}(t) = \mathcal{L}_{Y+Z}(t) = \mathcal{L}_Y(t)\,\mathcal{L}_Z(t)$. Now $\mathcal{L}_Z(t) > 0$, and simplifying we see that $\mathcal{L}_X(t) = \mathcal{L}_Y(t)$, for all $t \in \mathbb{R}_+^n$, so that $X \overset{\text{law}}{=} Y$.)

It is important to realize that the assumption that **the components of** X, Y, Z are **non-negative is important** for the validity of (2.56)! One can find in Feller [9], p. 506, an example of random variables X, Y, Z such that X, Z are independent, Y, Z are independent, and $X+Z \overset{\text{law}}{=} Y+Z$, but X has not the same law as Y! (These examples come from the fact that there are characteristic functions of distributions, which are supported in $[-1,1]$, and there are distinct distributions which can have the same restriction of their respective characteristic function to $[-1,1]$. Thus, if two distinct characteristic functions agree in $[-1,1]$, their products by a characteristic function supported in $[-1,1]$ will be equal. Interpreting products of characteristic functions in terms of characteristic functions of sums of independent variables, one obtains the desired examples.)

3) Simple random walk in continuous time on $\mathbb{Z}$ also satisfies a corresponding Ray-Knight theorem (or (2.51)). Actually, when $x_0 = 0 < x$, an integer, and when one chooses $c_{z,z+1} = \frac{1}{2}$ for all z (for the weights), then

(2.57) $$g_U(z,z') = 2\; z \wedge z', \text{ for all } z, z' \in \mathbb{N}, \text{ when } U = \mathbb{Z}\backslash\{0\}.$$

Indeed, when $z \geq z' \geq 1$, by the strong Markov property, letting H_0 denote the entrance time of $Z_n, n \geq 0$, in $\{0\}$,

$$g_U(z,z') = g_U(z',z') = E_{z'}\Big[\sum_{k=0}^{H_0} 1\{Z_k = z'\}\Big],$$

and the number of visits to z' before hitting 0 under $P_{z'}$ is geometric with success parameter $\frac{1}{2z'}$ (and hence has expectation $2z'$). So $g_U(\cdot,\cdot)$ corresponds to the restriction of the Brownian Green function $g_{\mathbb{R}\backslash\{0\}}(\cdot,\cdot)$ to $\mathbb{N} \times \mathbb{N}$! It now follows from (2.51) (in this slightly more general context) that

(2.58) $(L^z_{H_0})_{z\in\mathbb{N}}$ under P_x, for $x \geq 1$ integer, has the same law as the restriction to $z \in \mathbb{N}$ of the Brownian local times $L^z_{H_0}$, $z \geq 0$, under Wiener measure starting from x (see (2.53)).

□

As a preparation for the proof of the generalized second Ray-Knight theorem we record the following fact concerning Gaussian vectors.

Proposition 2.14. *Let $n \geq 1$, $\psi = (\psi_1, \dots, \psi_n)$ be a centered Gaussian vector with invertible covariance matrix A, and $V = \mathrm{diag}(v_1, \dots, v_n)$ be a deterministic diagonal matrix, such that $A^{-1} - V$ is positive definite. Then for any real numbers b_i, $1 \leq i \leq n$,*

$$(2.59)\qquad \begin{aligned} &E\Big[\exp\Big\{\sum_{i=1}^n v_i\,\frac{(\psi_i+b_i)^2}{2}\Big\}\Big] = \\ &\frac{1}{\sqrt{\det(I-AV)}}\ \exp\Big\{\tfrac{1}{2}\Big(\sum_{1\leq i\leq n} v_i\,b_i^2 + \sum_{1\leq i,j\leq n} v_i\,b_i\,\widetilde{A}_{ij}\,v_j\,b_j\Big)\Big\}\end{aligned}$$

where $\widetilde{A} = (A^{-1}-V)^{-1} = (I-AV)^{-1}A$.

Moreover, when $\widetilde{\psi}$ is an independent copy of ψ, and $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}$ are real numbers such that

$$(2.60)\qquad \alpha^2 + \widetilde{\alpha}^2 = \beta^2 + \widetilde{\beta}^2,$$

then for any b_i, $1 \leq i \leq n$,

$$(2.61)\qquad \big((\psi_i+\alpha b_i)^2 + (\widetilde{\psi}_i + \widetilde{\alpha}\,b_i)^2\big)_{1\leq i\leq n} \overset{\text{law}}{=} \big((\psi_i+\beta b_i)^2 + (\widetilde{\psi}_i+\widetilde{\beta} b_i)^2\big)_{1\leq i\leq n}.$$

Proof.

• (2.59):

Note that by assumption $A^{-1}-V$ is positive definite, hence invertible, and $A^{-1}-V = A^{-1}(I-AV)$, so that $I-AV$ is invertible and $\widetilde{A} = (A^{-1}-V)^{-1} = (I-AV)^{-1}A$. Moreover,

$$E\Big[\exp\Big\{\sum_{i=1}^n v_i\,\frac{(\psi_i+b_i)^2}{2}\Big\}\Big] = E\Big[\exp\Big\{\sum_{i=1}^n v_i\,b_i\,\psi_i + \frac{1}{2}\sum_{i=1}^n v_i\,\psi_i^2\Big\}\Big]\,e^{\frac{1}{2}\sum_{i=1}^n v_i\,b_i^2}$$

and using that $A^{-1}-V = \widetilde{A}^{-1}$,

$$\begin{aligned} &E\Big[\exp\Big\{\sum_{i=1}^n v_i\,b_i\,\psi_i + \frac{1}{2}\sum_{i=1}^n v_i\,\psi_i^2\Big\}\Big] = \\ &\frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det A}}\int_{\mathbb{R}^n}\exp\Big\{\sum_{i=1}^n v_i\,b_i\,x_i - \frac{1}{2}\sum_{i,j=1}^n x_i\,\widetilde{A}^{-1}_{ij}\,x_j\Big\}\,dx_1\dots dx_n = \\ &\Big(\frac{\det\widetilde{A}}{\det A}\Big)^{\frac{1}{2}}\,\widetilde{E}\Big[\exp\Big\{\sum_{i=1}^n v_i\,b_i\,\psi_i\Big\}\Big]\end{aligned}$$

where under $\widetilde{P}$, $(\psi_1, \dots, \psi_n)$ is a centered Gaussian vector with covariance matrix $\widetilde{A}$, and $\widetilde{E}$ denotes the $\widetilde{P}$-expectation. Since $\det\widetilde{A} = \det A/\det(I-AV)$, the last expression equals

$$\frac{1}{\sqrt{\det(I-AV)}}\cdot\exp\Big\{\frac{1}{2}\sum_{i,j=1}^n v_i\,b_i\,\widetilde{A}_{ij}\,v_j\,b_j\Big\}.$$

This proves (2.59).

• (2.61):

By (2.59), we see that for small v_i, $1 \le i \le n$, one has

$$
\begin{aligned}
(2.62) \quad & E\Big[\exp\Big\{\sum_{i=1}^n v_i \Big[\frac{(\psi_i+\alpha b_i)^2}{2} + \frac{(\widetilde{\psi}_i+\widetilde{\alpha} b_i)^2}{2}\Big]\Big\}\Big] = \\
& \frac{1}{\det(I-AV)} \exp\Big\{\frac{1}{2}\Big(\sum_{i=1}^n v_i\, b_i^2(\alpha^2+\widetilde{\alpha}^2) + \frac{1}{2}\sum_{1\le i,j\le n} v_i\, b_i\, \widetilde{A}_{ij}\, v_j\, b_j(\alpha^2+\widetilde{\alpha}^2)\Big)\Big\}
\end{aligned}
$$

and we obtain the same expression if we now replace α by β and $\widetilde{\alpha}$ by $\widetilde{\beta}$, thanks to (2.60).

The same argument as below (2.34) then shows that the random vectors on the left-hand side and on the right-hand of (2.61) have the same characteristic function. The claim (2.61) now follows. □

We continue with some preparation for the generalized second Ray-Knight theorem.

For the next proposition, once again we assume that the killing measure κ vanishes everywhere except at a point:

(2.63) there is $x_0 \in E$, such that $\kappa_{x_0} = \lambda > 0$, and $\kappa_x = 0$, for $x \ne x_0$.

We write $U = E\backslash\{x_0\}$, and recall that

$$
\begin{aligned}
(2.64) \quad g_U(x,y) &\overset{(1.52)}{=} g(x,y) - \frac{g(x,x_0)\, g(x_0,y)}{g(x_0,x_0)} \\
&\overset{(1.91)}{=} E_x[L^y_{H_{x_0}}], \quad \text{for } x,y \in E.
\end{aligned}
$$

We introduce (in the same fashion as in (2.50))

(2.65) $P^{G,U}$, the probability on $\mathbb{R}^E$ under which $\varphi_x, x \in E$, is a centered Gaussian field with covariance $E^{G,U}[\varphi_x\, \varphi_y] = g_U(x,y)$, for $x,y \in E$.

(2.66) Y, an exponential random variable with parameter λ under Q.

Proposition 2.15. *(under (2.63) - (2.66))*

$$
\begin{aligned}
(2.67) \quad & \Big(L^x_\infty + \frac{1}{2}\,\varphi_x^2\Big)_{x\in E} \ \textit{under } P_{x_0}\otimes P^{G,U}, \textit{ has same law as} \\
& \Big(\frac{1}{2}\,(\varphi_x + \sqrt{2Y})^2\Big)_{x\in E} \ \textit{under } P^{G,U}\otimes Q.
\end{aligned}
$$

Proof. Consider Z, Z' independent centered Gaussian variables with variance λ^{-1}, independent from φ_x, $x \in E$, and φ'_x, $x \in E$, two independent copies $P^{G,U}$-distributed.

By (2.61), we find that

$$
\begin{aligned}
\big((\varphi_x+Z)^2 + (\varphi'_x+Z')^2\big)_{x\in E} &\overset{\text{law}}{=} \big(\varphi_x^2 + (\varphi'_x + \sqrt{Z^2+Z'^2})^2\big)_{x\in E} \\
&\overset{\text{law}}{=} \big(\varphi_x^2 + (\varphi'_x+\sqrt{2Y})^2\big)_{x\in E},
\end{aligned}
$$

where Y is an exponential random variable with parameter λ, independent of $(\varphi_x)_{x\in E}$ and $(\varphi'_x)_{x\in E}$, and we used that $Z^2 + Z'^2 \overset{\text{law}}{=} 2Y$ (use polar coordinates).

Thus if we show that

$$\Big(L^x_\infty + \frac{1}{2}\,\varphi_x^2 + \frac{1}{2}\,(\varphi'_x)^2\Big)_{x\in E} \overset{\text{law}}{=} \Big(\frac{1}{2}\,(\varphi_x + Z)^2 + \frac{1}{2}\,(\varphi'_x + Z')^2\Big)_{x\in E}, \tag{2.68}$$

it will follow that

$$\Big(L^x_\infty + \frac{1}{2}\,\varphi_x^2 + \frac{1}{2}\,(\varphi'_x)^2\Big)_{x\in E} \overset{\text{law}}{=} \Big(\frac{1}{2}\,\varphi_x^2 + \frac{1}{2}\,(\varphi'_x + \sqrt{2Y})^2\Big)_{x\in E},$$

and, "simplifying" on both sides, i.e. applying (2.56), we will conclude that

$$\Big(L^x_\infty + \frac{1}{2}\,(\varphi'_x)^2\Big)_{x\in E} \overset{\text{law}}{=} \Big(\frac{1}{2}\,(\varphi'_x + \sqrt{2Y})^2\Big)_{x\in E},$$

i.e. (2.67) will be proved.

It remains to prove (2.68). By the same argument as below (2.34), it suffices to prove that for small $V\colon E \to \mathbb{R}$,

$$\begin{aligned} &E_{x_0} \otimes E^{G,U} \otimes E^{G,U}\Big[\exp\Big\{\sum_{x\in E} V(x)\Big(L^x_\infty + \frac{1}{2}\,\varphi_x^2 + \frac{1}{2}\,(\varphi'_x)^2\Big)\Big\}\Big] = \\ &E^{G,U} \otimes E^{G,U} \otimes E^{Z,Z'}\Big[\exp\Big\{\sum_{x\in E} V(x)\Big(\frac{1}{2}\,(\varphi_x + Z)^2 + \frac{1}{2}\,(\varphi'_x + Z')^2\Big)\Big\}\Big], \end{aligned} \tag{2.69}$$

where $E^{Z,Z'}$ denotes the expectation relative to the probability governing Z, Z'. Observe that (writing E in place of $E^{G,U} \otimes E^{Z,Z'}$),

$$E[(\varphi_x + Z)(\varphi_y + Z)] = E[\varphi_x\,\varphi_y] + E[Z^2] = g_U(x,y) + \lambda^{-1},$$

i.e.

(2.70) $\varphi_x + Z$, $x \in E$ is a centered Gaussian field with covariance $g_U(x,y) + \lambda^{-1}$, $x, y \in E$.

Lemma 2.16. *(under (2.63), with $U = E\backslash\{x_0\}$)*

$$g(x,y) = g_U(x,y) + \lambda^{-1},\ \textit{for } x,y \in E. \tag{2.71}$$

Proof. We have by (2.64)

$$g(x,y) = g_U(x,y) + \frac{g(x,x_0)\,g(x_0,y)}{g(x_0,x_0)},\ \text{for } x,y \in E.$$

Since $\kappa = 0$, for $x \neq x_0$, we see that $P_x[H_{x_0} < \infty] = 1$, for any $x \in E$, and by (1.51) $g(x,x_0) = g(x_0,x_0) = g(x_0,x)$, for all $x \in E$. Hence

$$g(x,y) = g_U(x,y) + g(x_0,x_0),\ \text{for } x,y \in E. \tag{2.72}$$

To compute $g(x_0,x_0)$, we use the fact that

$$g(x_0,x_0) \overset{(1.26)}{=} \frac{1}{\lambda_{x_0}}\,E_{x_0}\Big[\sum_{n\geq 0} 1\{Z_n = x_0\}\Big]. \tag{2.73}$$

Under P_{x_0}, $\sum_{n\geq 0} 1\{Z_n = x_0\}$, the total number of visits to x_0, is a geometric random variable with success parameter $\frac{\kappa_{x_0}}{\lambda_{x_0}} = \frac{\lambda}{\lambda_{x_0}}$ (i.e. the probability to jump to the cemetery point). As a result we obtain

$$E_{x_0}\Big[\sum_{n\geq 0} 1\{Z_n = x_0\}\Big] = \Big(\frac{\lambda}{\lambda_{x_0}}\Big)^{-1} = \frac{\lambda_{x_0}}{\lambda},$$

and

$$g(x_0, x_0) = \frac{1}{\lambda_{x_0}} \, \frac{\lambda_{x_0}}{\lambda} = \lambda^{-1}.$$

With (2.72) we now find (2.71). □

By (2.41), we see that for small V

$$E_{x_0}\Big[\exp\Big\{\sum_{x\in E} V(x)\, L^x_\infty\Big\}\Big] = \big((I-GV)^{-1} 1_E\big)(x_0), \tag{2.74}$$

and by (2.59), with $b_i \equiv 0$, and A playing the role of G or G_U, where G_U stands for the matrix with components $g_U(x,y)$, $x, y \in E$, we have for small V:

$$\frac{E^{G,U} \otimes E^{G,U} \otimes E^{Z,Z'}\big[e^{\sum\limits_{x\in E} V(x)(\frac{1}{2}(\varphi_x+Z)^2+\frac{1}{2}(\varphi'_x+Z')^2)}\big]}{E^{G,U} \otimes E^{G,U}\big[e^{\sum\limits_{x\in E} V(x)(\frac{\varphi_x^2}{2}+\frac{(\varphi'_x)^2}{2})}\big]} = \frac{\det(I-G_U V)}{\det(I-GV)}\,. \tag{2.75}$$

We will now see that the expressions in (2.74) and (2.75) are equal. We observe that by Cramer's rule for the inverse of a matrix

$$\big((I-GV)^{-1} 1_E\big)(x_0) = \frac{\det(M)}{\det(I-GV)},$$

where M is the matrix obtained by replacing the x_0-column in $I - GV$ with 1 everywhere. Subtracting the x_0-row from all other rows of M, we obtain a matrix $\widetilde{M}$ with coefficients $\widetilde{M}_{x,y}$, which for $x, y \neq x_0$ equal

$$\begin{aligned} \widetilde{M}_{x,y} &= \delta_{x,y} - g(x,y)\, V(y) + g(x_0,y)\, V(y) \\ &\overset{(2.71)}{=} \delta_{x,y} - g_U(x,y)\, V(y), \end{aligned}$$

and the x_0-column of $\widetilde{M}$ vanishes everywhere except at row x_0:

$$\widetilde{M}_{x,x_0} = \delta_{x,x_0}.$$

Clearly $\det M = \det \widetilde{M}$, and if we develop the determinant of $\widetilde{M}$ along the x_0-column, we find that $\det M = \det \widetilde{M} = \det\big((I - G_U\, V)_{|U\times U}\big) = \det(I - G_U\, V)$.

We have thus proved that the expressions in (2.74) and (2.75) are equal. This concludes the proof of (2.69) and hence of (2.67). □

We will later give another proof of the above proposition based instead on the Dynkin isomorphism theorem. For the time being we proceed with the **generalized second Ray-Knight theorem**.

We now consider the **situation where the killing measure vanishes** on E, i.e. (1.1) - (1.3) hold but **instead of our usual set-up**,

$$\kappa_x = 0, \text{ for all } x \in E. \tag{2.76}$$

We denote by X_t^0, $t \geq 0$, the canonical process on the space D_E^0 of right-continuous E-valued trajectories with finitely many jumps on finite intervals and infinitely many jumps, and by P_x^0, $x \in E$, the law of the walk with jump rate 1, and Markovian transition probability $p_{x,y}^0 = \frac{c_{x,y}}{\lambda_x^0}$, for $x, y \in E$, with $\lambda_x^0 = \sum_{y \in E} c_{x,y}$, for $x \in E$.

The local time of the walk is defined by

$$\ell_t^x = \int_0^t 1\{X_s^0 = x\}\, ds\, \frac{1}{\lambda_x^0}\,, \text{ for } x \in E, t \geq 0. \tag{2.77}$$

The map $t \geq 0 \to \ell_t^x \geq 0$, is continuous non-decreasing, $\ell_0^x = 0$, P_z^0-a.s., $\ell_\infty^x = \infty$, for any $x, z \in E$, since we are now in a recurrent situation.

We now consider a special point

$$x_0 \in E, \tag{2.78}$$

and keep the notation $U = E \backslash \{x_0\}$.

Note that the law of $X^0_{t \wedge H_{x_0}}$, $t \geq 0$, under P_x^0 agrees with that of $X_{t \wedge H_{x_0}}$, $t \geq 0$, under P_x, when we instead pick the killing measure κ with the unique non-vanishing point x_0 of (2.78), as in (2.63). In particular, the killed Green function $g_U^0(\cdot,\cdot)$ (attached to the walk X_t^0, $t \geq 0$) coincides with $g_U(\cdot,\cdot)$, and

$$g_U^0(x,y) = g_U(x,y) = E_x^0[\ell^y_{H_{x_0}}], \text{ for } x, y \in E. \tag{2.79}$$

We now introduce the right-continuous inverse of $t \to \ell_t^{x_0}$:

$$\sigma_u = \inf\{t \geq 0;\, \ell_t^{x_0} > u\}, \text{ for } u \geq 0. \tag{2.80}$$

Theorem 2.17. *(***Generalized second Ray-Knight theorem***)*

Keeping the notation of (2.65), for any $u > 0$,

$$\begin{aligned} &\Big(\ell^x_{\sigma_u} + \frac{1}{2}\,\varphi_x^2\Big)_{x \in E} \;\; \textit{under } P_{x_0}^0 \otimes P^{G,U}, \textit{ has the same law as} \\ &\Big(\frac{1}{2}\,(\varphi_x + \sqrt{2u})^2\Big)_{x \in E} \;\; \textit{under } P^{G,U}, \end{aligned} \tag{2.81}$$

(we will later explain the origin of the above terminology, see Remark 2.19).

Proof. Consider as in (2.66) an exponential variable Y with parameter $\lambda > 0$ (see (2.63)), under some auxiliary probability Q. First assume that we can show that

$$\begin{aligned} &\big(\ell^x_{\sigma_Y}\big)_{x \in E} \;\; \text{under } P_{x_0}^0 \otimes Q, \text{ has the same law as} \\ &\big(L^x_\infty\big)_{x \in E} \;\; \text{under } P_{x_0}, \end{aligned} \tag{2.82}$$

and let us explain how (2.81) follows.

By (2.67) and (2.82), we then find that under $P^0_{x_0} \otimes P^{G,U} \otimes Q$,

$$\Big(\ell^x_{\sigma_Y} + \frac{1}{2}\,\varphi^2_x\Big)_{x\in E} \text{ has the same law as } \Big(\frac{1}{2}\,(\varphi_x + \sqrt{2Y})^2\Big)_{x\in E}. \tag{2.83}$$

As a result for any $V\colon\, E \to \mathbb{R}_+$:

$$\begin{array}{l} \displaystyle\int_0^\infty E^0_{x_0} \otimes E^{G,U}\Big[\exp\Big\{-\sum_{x\in E} V(x)\Big(\ell^x_{\sigma_u} + \frac{1}{2}\,\varphi^2_x\Big)\Big\}\Big]\, e^{-\lambda u} du = \\ \displaystyle\int_0^\infty E^{G,U}\Big[\exp\Big\{-\sum_{x\in E} V(x)\,\frac{1}{2}\,(\varphi_x + \sqrt{2u})^2\Big\}\Big]\, e^{-\lambda u} du, \end{array} \tag{2.84}$$

(indeed, multiplying both members by λ yields on the left-hand side the $P^0_{x_0} \otimes P^{G,U} \otimes Q$-expectation of $\exp\{-\sum_{x\in E} V(x)(\ell^x_{\sigma_Y} + \frac{1}{2}\,\varphi^2_x)\}$, and on the right-hand side the corresponding expectation of $\exp\{-\sum_{x\in E} V(x)\frac{1}{2}\,(\varphi_x + \sqrt{2Y})^2\}$).

Now

$$u \ge 0 \to E^0_{x_0} \otimes E^{G,U}\Big[\exp\Big\{-\sum_{x\in E} V(x)\Big(\ell^x_{\sigma_u} + \frac{1}{2}\,\varphi^2_x\Big)\Big\}\Big] \ge 0,$$

is a bounded right-continuous (because $u \to \sigma_u$ is right-continuous, $t \ge 0 \to \ell^x_t \ge 0$ is continuous, and we use dominated convergence) function. Similarly, $u \ge 0 \to E^{G,U}[\exp\{-\sum_{x\in E} V(x)\frac{1}{2}\,(\varphi_x + \sqrt{2u})^2\}] \ge 0$, is a bounded continuous function. By (2.84) their Laplace transforms are equal and they are hence equal. But this implies that for any $u > 0$, the Laplace transform of the law of $\big(\ell^x_{\sigma_u} + \frac{1}{2}\,\varphi^2_x\big)_{x\in E}$ under $P^0_{x_0} \otimes P^{G,U}$, is equal to the Laplace transform of the law of $\big(\frac{1}{2}\,(\varphi_x + \sqrt{2u})^2\big)_{x\in E}$ under $P^{G,U}$, and the claim (2.81) follows.

So there remains to prove (2.82).

For this purpose it is convenient to introduce the time-changed process defined similarly as in (1.96):

$$\begin{aligned} &\overline{X}^0_u = X^0_{\tau^0_u}, \text{ for } u \ge 0, \text{ where} \\ &\tau^0_u = \inf\{t \ge 0;\ \ell^0_t \ge u\} = \int_0^u \lambda^0_{\overline{X}_v}\, dv, \text{ with } \ell^0_t = \sum_{x\in E} \ell^x_t \\ &\quad = \int_0^t 1/\lambda^0_{X^0_s}\, ds, (t \ge 0 \to \ell^0_t \ge 0 \text{ is an increasing bijection of } \mathbb{R}_+). \end{aligned} \tag{2.85}$$

If we now define, cf. (1.97),

$$\overline{\ell}^x_u = \int_0^u 1\{\overline{X}^0_v = x\}\, dv, \text{ for } u \ge 0,\, x \in E, \tag{2.86}$$

then as in (1.99), (1.100) we see that

$$X^0_t = \overline{X}^0_{\ell^0_t},\ t \ge 0, \text{ and } \ell^x_t = \overline{\ell}^x_{\ell^0_t}, \text{ for } x \in E,\, t \ge 0. \tag{2.87}$$

Now, corresponding to (2.80), we can introduce

$$\overline{\sigma}_v = \inf\{u \geq 0;\ \overline{\ell}_u^{x_0} > v\} \underset{(2.80)}{\overset{(2.87)}{=}} \ell^0_{\sigma_v}, \ \text{ for } v > 0, \tag{2.88}$$

so that

$$\ell^x_{\sigma_Y} \overset{(2.87)}{=} \overline{\ell}^x_{\ell^0_{\sigma_Y}} \overset{(2.88)}{=} \overline{\ell}^x_{\overline{\sigma}_Y}, \ \text{ for any } x \in E. \tag{2.89}$$

The key to the identity in law (2.82) will come from the next representation of the law of $\overline{X}_{\cdot}$ under P_x.

Lemma 2.18. $(x \in E)$

(2.90)
$$\begin{aligned} Z_u &\overset{\text{def}}{=} \overline{X}^0_u, \ \textit{for } u < \overline{\sigma}_Y, \\ &\overset{\text{def}}{=} \Delta, \ \textit{for } u \geq \overline{\sigma}_Y, \end{aligned}$$
has the same law (on D_E) under $P^0_x \otimes Q$ as $\overline{X}_u$, $u \geq 0$, under P_x.

Proof. We consider $0 = u_0 < u_1 \cdots < u_n$ and $f_0, f_1, \dots, f_n$: $E \to \mathbb{R}$. Then

$$\begin{aligned} &E^0_x \otimes E^Q[f_0(Z_{u_0})\, f_1(Z_{u_1}) \dots f_n(Z_{u_n})] \overset{(2.90)}{=} \\ &E^0_x \otimes E^Q[f_0(\overline{X}^0_{u_0})\, f_1(\overline{X}^0_{u_1}) \dots f_n(\overline{X}^0_{u_n})\, 1\{u_n < \overline{\sigma}_Y\}]. \end{aligned} \tag{2.91}$$

Note that by (2.88)

$$\{\overline{\ell}^{x_0}_{u_n} < Y\} \subseteq \{u_n < \overline{\sigma}_Y\} \subseteq \{\overline{\ell}^{x_0}_{u_n} \leq Y\}$$

and the Q-probability of both events on the right-hand side and left-hand side is equal to $\exp\{-\lambda \overline{\ell}^{x_0}_{u_n}\}$. So integrating over Q in the second line of (2.91) we see that the expression on the first line equals

$$E^0_x[f_0(\overline{X}^0_{u_0})\, f_1(\overline{X}^0_{u_1}) \dots f_n(\overline{X}^0_{u_n})\, e^{-\lambda \overline{\ell}^{x_0}_u}]. \tag{2.92}$$

By the corresponding statement to (1.98), we know that $\overline{X}^0_u$, $u \geq 0$, is a Markov chain with Markovian transition semi-group

$$\begin{aligned} \overline{R}^0_t f(x) &= E^0_x[f(\overline{X}^0_t)] = e^{tL^0} f(x), \ t \geq 0, \ \text{ where} \\ L^0 f(x) &= \sum_{y \in E} c_{x,y}\, f(y) - \lambda^0_x\, f(x), \ \text{ for } f\!: E \to \mathbb{R}. \end{aligned}$$

The application of the Markov property to (2.92) at times $u_{n-1}, \dots, u_0$ shows that the expression in (2.92) equals

$$(f_0\, \overline{S}^0_{u_1}\, f_1\, \overline{S}^0_{u_2 - u_1}\, f_2 \dots f_{n-1}\, \overline{S}^0_{u_n - u_{n-1}} f_n)(x),$$

where

$$\overline{S}^0_u\, f(x) \overset{\text{def}}{=} E^0_x[f(\overline{X}^0_u)\, e^{-\lambda \overline{\ell}^{x_0}_u}], \text{ for } f\!: E \to \mathbb{R},\, x \in E,\, u \geq 0.$$

The corresponding version of the Feynman-Kac formula (1.105), see (2.86), shows that

$$\overline{S}^0_u\, f(x) = e^{u(L^0 - \lambda 1_{\{x_0\}})} f(x) = e^{uL}\, f(x),$$

because $\kappa_{x_0} = \lambda = \lambda_{x_0} - \lambda^0_{x_0}$, cf. (2.63). We have thus found that

$$E^0_x \otimes E^Q[f_0(Z_{u_0})\, f_1(Z_{u_1}) \dots f_n(Z_{u_n})] = (f_0\, e^{u_1 L}\, f_1\, e^{(u_2-u_1)L}\, f_2 \dots f_{n-1}\, e^{(u_n-u_{n-1})L}\, f_n)(x)$$

and by (1.98) we see that this is equal to $E_x[f_0(\overline{X}_{u_0})\, f_1(\overline{X}_{u_1}) \dots f_n(\overline{X}_{u_n})]$. From this and the fact that $Z_{\textstyle .}$ and $\overline{X}_{\textstyle .}$ remain in Δ once they reach Δ, one easily deduces that the finite dimensional marginals of $Z_{\textstyle .}$ and $\overline{X}_{\textstyle .}$ coincide, whence (2.90). This concludes the proof of the lemma. □

We can now conclude the proof of (2.82).

For all $x \in E$, we have $\ell^x_{\sigma_Y} \overset{(2.89)}{=} \overline{\ell}^x_{\overline{\sigma}_Y} = \int_0^\infty 1\{Z_u = x\} du$, by the definition of Z. So under $P^0_{x_0} \otimes Q$

$$\begin{aligned}(\ell^x_{\sigma_Y})_{x\in E} &= \Big(\int_0^\infty 1\{Z_u = x\}\, du\Big)_{x\in E} \underset{(2.90)}{\overset{\text{law}}{=}} \Big(\int_0^\infty 1\{\overline{X}_u = x\}\, du\Big)_{x\in E} \\ &\overset{(1.97)}{=} (\overline{L}^x_\infty)_{x\in E} \overset{(1.101)}{=} (L^x_\infty)_{x\in E}, \text{ under } P_{x_0},\end{aligned} \tag{2.93}$$

and we have completed the proof of (2.82) and hence of (2.81). □

Remark 2.19.

1) The “generalized second Ray-Knight theorem” was originally proved in [8]. The terminology stems from the fact that in the case of Brownian motion when $x_0 = 0$, the statement corresponding to (2.81) yields that, see Marcus-Rosen [19], p. 53, for any $u > 0$,

$$\big(L^x_{\sigma_u} + B^2_x\big)_{x\ge 0} \text{ has the same law as } \big((B_x + \sqrt{u})^2\big)_{x\ge 0}, \tag{2.94}$$

when $(L^z_t, z \in \mathbb{R}, t \ge 0)$ and $(B_x, x \ge 0)$ are independent and respectively distributed as the local time process of a Brownian motion starting at 0, and a Brownian motion starting at 0, and we have set

$$\sigma_u = \inf\{t \ge 0;\ L^0_t > u\}.$$

This statement, using arguments described earlier, is equivalent to the more traditional formulation:

(2.95) Under Wiener measure starting at 0, $(L^x_{\sigma_u})_{x\ge 0}$ has the same law as a zero-dimensional squared Bessel process starting at u, (i.e. $BESQ^0(u)$ in the notation below (2.54)).

2) The same argument that we used below (2.57), shows that one also has a similar identity for the random walk on $\mathbb{Z}$, when $c_{z,z+1} = \frac{1}{2}$, for all z. Namely for $u > 0$,

(2.96) $(\ell^x_{\sigma_u})_{x\in\mathbb{N}}$ under P_0 has same law as the restriction to integer times $x \in \mathbb{N}$ of $L^x_{\sigma_u}$ under Wiener measure in (2.95).

3) In the case of random interlacements on a transient weighted graph, one can establish an identity in law in the spirit of the generalized second Ray-Knight theorem, see [28]. It relates the field of occupation times of random interlacements at level u to the Gaussian free field on the transient weighted graph, cf. (4.86). □

Complement: a proof of (2.67) based on the Dynkin isomorphism theorem

We now provide a second proof of Proposition 2.15, which makes direct use of the Dynkin isomorphism theorem. We recall the notation (2.63) - (2.66).

Second proof of (2.67):

By the Dynkin isomorphism theorem, cf. (2.33), we see that

(2.97) $\Big(L^x_\infty + \frac{1}{2}\,\varphi^2_x\Big)_{x\in E}$ under $\overline{P}_{x_0,x_0} \otimes P^G$, has the same law as $\Big(\frac{1}{2}\,\varphi^2_x\Big)_{x\in E}$ under μ_0,

where we have introduced the probabilities

$$(2.98)\qquad \begin{aligned} \overline{P}_{x_0,x_0} &= \frac{1}{g(x_0,x_0)}\; P_{x_0,x_0} \qquad (\text{on } \Gamma),\\ \mu_0 &= \frac{1}{g(x_0,x_0)}\; \varphi^2_{x_0}\; P^G \quad (\text{on } \mathbb{R}^E)\,. \end{aligned}$$

The next observation is that if we consider the process with trajectories in D_E,

$$X_{t^+}(\gamma) = \lim_{\varepsilon\downarrow 0} X_{t+\varepsilon}(\gamma), \text{ for } \gamma\in\Gamma,$$

(the only time where $X_t(\gamma) \neq X_{t^+}(\gamma)$ is when $t = \zeta(\gamma)$, the duration of γ, see below (2.16)), we have the following:

(2.99) the law on D_E of X_{t^+}, $t\ge 0$, under $\overline{P}_{x_0,x_0}$ is equal to P_{x_0}.

Indeed $g(\cdot,x_0) = g(x_0,x_0)$, by (2.71), and looking at (2.23), we see that the finite dimensional distributions of X_t, $t \ge 0$, under $\overline{P}_{x_0,x_0}$, which coincide with those of X_{t^+}, $t\ge 0$, under $\overline{P}_{x_0,x_0}$, since $\overline{P}_{x_0,x_0}[\zeta = t] = 0$, for any t, are equal to the finite dimensional distributions of X_t, $t\ge 0$ (i.e. the canonical process on D_E), under P_{x_0}. The claim (2.99) follows.

As a result of (2.99) and of the formula $L^x_\infty = \frac{1}{\lambda_x}\int_0^\infty 1\{X_s = x\}\,ds$, $x\in E$, we see that

(2.100) L^x_∞, $x\in E$, under $\overline{P}_{x_0,x_0}$, has the same law as L^x_∞, $x\in E$, under P_{x_0}.

As for the right-hand side of (2.97), we use the following

Lemma 2.20.

(2.101) $\Big(\frac{1}{2}\,\varphi^2_x\Big)_{x\in E}$ *under* μ_0, *has the same law as* $\Big(\frac{1}{2}\,(\psi_x + X)^2\Big)_{x\in E}$,

where ψ_x, $x\in E$, *is* $P^{G,U}$*-distributed and independent from* X, *which is distributed as* $(Z^2 + Z'^2 + \widetilde{Z}^2)^{\frac{1}{2}}$, *where* $Z, Z', \widetilde{Z}$ *are independent centered Gaussian variables with variance* λ^{-1}.

Proof. Note that for any $x,y\in E$:

$$\begin{aligned} &E^G[(\varphi_x - \varphi_{x_0})\,\varphi_{x_0}] = g(x,x_0) - g(x_0,x_0) \overset{(2.71)}{=} 0, \text{ and}\\ &E^G[(\varphi_x - \varphi_{x_0})\,(\varphi_y - \varphi_{x_0})] \overset{(2.71)}{=} g_U(x,y). \end{aligned}$$

As a result (this is also a special case of Proposition 2.3), we find that:

$$\begin{array}{l}\varphi_{x_0} \text{ and } (\varphi_x - \varphi_{x_0})_{x\in E} \text{ are independent under } P^G, \\ \text{and } (\varphi_x - \varphi_{x_0})_{x\in E} \text{ is } P^{G,U}\text{-distributed.}\end{array} \tag{2.102}$$

As a consequence, under μ_0,

$$\Big(\frac{1}{2}\,\varphi_x^2\Big)_{x\in E} = \Big(\frac{1}{2}\,(\varphi_x - \varphi_{x_0} + \varphi_{x_0})^2\Big)_{x\in E} \overset{\text{law}}{=} \Big(\frac{1}{2}\,(\psi_x + |\varphi_{x_0}|)^2\Big)_{x\in E}$$

where ψ and $|\varphi_{x_0}|$ are independent in the last expression, with ψ having distribution $P^{G,U}$, and $|\varphi_{x_0}|$ being under the law μ_0. The claim (2.101) will thus follow once we see that

$$\text{under } \mu_0,\ |\varphi_{x_0}| \text{ has the distribution of the variable } X. \tag{2.103}$$

To this end we note that for $f\colon \mathbb{R}_+ \to \mathbb{R}$, bounded measurable

$$\begin{aligned} E^{\mu_0}[f(|\varphi_{x_0}|)] &\overset{(2.98),(2.71)}{=} \lambda E^G[\varphi_{x_0}^2\, f(|\varphi_{x_0}|)] = \frac{\lambda^{\frac{3}{2}}}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} t^2\, f(|t|)\, e^{-\frac{\lambda t^2}{2}} dt \\ &= \frac{2\lambda^{\frac{3}{2}}}{(2\pi)^{\frac{1}{2}}} \int_0^\infty r^2\, f(r)\, e^{-\frac{\lambda r^2}{2}} dr, \end{aligned}$$

and that using polar coordinates in $\mathbb{R}^3$,

$$\begin{aligned} E[f(X)] &= \frac{\lambda^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(|z|)\, e^{-\frac{\lambda|z|^2}{2}} dz = \frac{\lambda^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} 4\pi \int_0^\infty r^2\, f(r)\, e^{-\frac{\lambda r^2}{2}} dr \\ &= E^{\mu_0}[f(|\varphi_{x_0}|)], \ \text{ whence the claim (2.101).} \end{aligned}$$

□

By (2.97), (2.100), (2.101) we can conclude, with L_∞^x, $x \in E$, now considered under P_{x_0} (and $(\psi_x + Z)_{x\in E}$, having the law P^G, see (2.70), (2.71)), that:

$$\Big(L_\infty^x + \frac{1}{2}\,(\psi_x + Z)^2\Big)_{x\in E} \overset{\text{law}}{=} \Big(\frac{1}{2}\,(\psi_x + X)^2\Big)_{x\in E}.$$

Adding to both sides an independent copy $\frac{1}{2}\,(\psi_x')^2$, $x \in E$, of $\frac{1}{2}\,\psi_x^2$, $x \in E$, we see that

$$\begin{aligned} \Big(L_\infty^x + \frac{1}{2}\,(\psi_x + Z)^2 + \frac{1}{2}\,(\psi_x')^2\Big)_{x\in E} &\overset{\text{law}}{=} \Big(\frac{1}{2}\,(\psi_x + X)^2 + \frac{1}{2}\,(\psi_x')^2\Big)_{x\in E} \\ &\underset{(2.61)}{\overset{\text{law}}{=}} \Big(\frac{1}{2}\,\Big(\psi_x + \sqrt{Z^2 + \widetilde{Z}^2}\Big)^2 + \frac{1}{2}\,(\psi_x' + Z')^2\Big)_{x\in E} \\ &\overset{\text{law}}{=} \Big(\frac{1}{2}\,(\psi_x + \sqrt{2Y})^2 + \frac{1}{2}\,(\psi_x' + Z')\Big)_{x\in E}, \end{aligned} \tag{2.104}$$

where Y, as in (2.66), is an independent exponential variable with parameter λ. Of course, $\frac{1}{2}\,(\psi_x + Z)^2$ has the same law as $\frac{1}{2}\,(\psi_x' + Z')^2$, $x \in E$, and simplifying on both sides of (2.104), i.e. applying (2.56), we obtain:

$$\Big(L_\infty^x + \frac{1}{2}\,(\psi_x')^2\Big)_{x\in E} \overset{\text{law}}{=} \Big(\frac{1}{2}\,(\psi_x + \sqrt{2Y})^2\Big)_{x\in E}. \tag{2.105}$$

This is simply a reformulation of (2.67). □

3 The Markovian loop

In this chapter we will introduce the measure describing the Markovian loop and study some of its properties. For this purpose it will be convenient to first discuss the rooted (or based) loops as well as pointed loops. The Markovian loops come up as unrooted loops, which live in the space of rooted loops modulo time-shift. We refer to Le Jan [18] for an extensive discussion of Markovian loops and their properties.

3.1 Rooted loops and the measure μ_r on rooted loops

This section is devoted to the introduction of the general set-up for loops, the construction of the σ-finite measure μ_r governing rooted loops, and the discussion of some of its basic properties.

We first introduce the space of rooted loops of duration $t > 0$:

(3.1)
$$\begin{aligned} L_{r,t} = &\text{ the space of right-continuous functions } [0,t] \to E, \\ &\text{ with finitely many jumps and same value in } 0 \text{ and } t. \end{aligned}$$

We denote by X_s, $0 \le s \le t$, the canonical coordinates, and we extend periodically the function $s \in [0,t] \to X_s(\gamma)$, for any $\gamma \in L_{r,t}$, so that $X_s(\gamma)$ is well-defined for any $s \in \mathbb{R}$. Let us underline that the spaces $L_{r,t}$ are pairwise disjoint, as t varies over $(0,\infty)$.

We then define the space of rooted loops via the formula

(3.2)
$$L_r = \bigcup_{t>0} L_{r,t},$$

and for $\gamma \in L_r$, a rooted loop, we denote the duration of γ by

(3.3)
$$\zeta(\gamma) = \text{the unique } t > 0 \text{ such that } \gamma \in L_{r,t}.$$

We define a σ-algebra on L_r in a similar fashion as below (2.18), i.e. we identify L_r with $L_{r,1} \times (0,\infty)$ via the map $(w,t) \in L_{r,1} \times (0,\infty) \to \gamma(\cdot) = w(\frac{\cdot}{t}) \in L_r$, and endow $L_{r,1} \times (0,\infty)$ with the canonical product σ-algebra (where $L_{r,1}$ is endowed with the σ-algebra generated by the maps X_s, $0 \le s \le 1$, from $L_{r,1}$ into E).

It will be convenient to parametrize rooted loops with the help of the random variables, which we now introduce.

The discrete duration of the rooted loop γ is

(3.4)
$$\begin{aligned} N(\gamma) = &\,(\text{the total number of jumps of } 0 \le s \le t \to X_s) \vee 1 \\ &\text{if } t = \zeta(\gamma) > 0 \text{ stands for the duration of } \gamma. \end{aligned}$$

When $N(\gamma) = n > 1$,

(3.5)
$$0 < T_1(\gamma) < \dots < T_{n-1}(\gamma) < T_n(\gamma) \le \zeta(\gamma)$$

are the successive jump times of the rooted loop γ, and

(3.6)
$$Z_0(\gamma) = \gamma(0),\ Z_1(\gamma) = \gamma(T_1), \dots, Z_{n-1}(\gamma) = \gamma(T_{n-1}),\ Z_n(\gamma) = \gamma(T_n) = Z_0(\gamma)$$

are the successive positions of the rooted loop.

We also extend the definition of $Z_p(\gamma)$ to all $p \in \mathbb{Z}$ by periodicity (so that when $N(\gamma) = n$, $Z_n(\gamma) = Z_0(\gamma)$, $Z_{n+1}(\gamma) = Z_1(\gamma), \dots$).

In the case where $N(\gamma) = 1$, the rooted loop γ does not move away from its initial position $Z_0(\gamma)$, and has duration $\zeta(\gamma)$; we will call γ a **trivial loop**. The case where

$$N(\gamma) = n > 1, \text{ and } T_n(\gamma) = \zeta(\gamma), \tag{3.7}$$

corresponds to the situation where the rooted loop has a jump at time $\zeta(\gamma)$ (which taking into account the periodicity of the function $s \to X_s(\gamma)$, "corresponds to time 0"); we will call γ such that either $N(\gamma) = 1$, or $N(\gamma) = n > 1$ and $T_n(\gamma) = \zeta(\gamma)$, a **pointed loop**.

With the above variables we can of course reconstruct $\gamma \in L_r$ since

$$\begin{aligned}
&\text{for } N(\gamma) = 1, && \gamma(s) = Z_0, \quad \text{for } 0 \le s \le \zeta(\gamma),\\
&\text{for } N(\gamma) = n > 1, && \gamma(s) = Z_0, \quad \text{for } 0 \le s < T_1(\gamma),\\
& && = Z_k, \quad \text{for } T_k(\gamma) \le s < T_{k+1}(\gamma), \text{ for } 1 \le k \le n-1,\\
& && = Z_0, \quad \text{for } T_n(\gamma) \le s \le \zeta(\gamma).
\end{aligned} \tag{3.8}$$

We also define the shift $\theta_v \colon L_r \to L_r$, when $v \in \mathbb{R}$, via:

$$\begin{array}{l}\theta_v(\gamma) \in L_r, \text{ for } \gamma \in L_r, \text{ is the rooted loop } \gamma' \text{ such that}\\ \zeta(\gamma') = \zeta(\gamma), \text{ and } X_s(\gamma') = X_{s+v}(\gamma), \text{ for all } s \in \mathbb{R}.\end{array} \tag{3.9}$$

The measure μ_r on rooted loops:

For any $t > 0$ and $x \in E$, we have defined the finite measure $P^t_{x,x}$ in (2.19) as the image of $1\{X_t = x\} \frac{P_x}{\lambda_x}$ on Γ_t, under the map $(X_s)_{0 \le s \le t}$. The measure $P^t_{x,x}$ is in fact concentrated on $\Gamma_t \cap \{\gamma(0) = \gamma(t)\} \subseteq L_{r,t}$, cf. (3.1), and we can view $t > 0 \to P^t_{x,x}$ as a positive measure kernel from $(0,\infty)$ to L_r. We then introduce the measure on L_r:

$$\mu_r[B] = \sum_{x \in E} \int_0^\infty P^t_{x,x}(B)\, \lambda_x \,\frac{dt}{t}, \ \text{ for measurable subsets } B \text{ of } L^t_r. \tag{3.10}$$

Note that for $a > 0$, by (2.20), $\mu_r[\zeta \ge a] = \sum_{x\in E} \int_a^\infty r_t(x,x)\, \lambda_x \frac{dt}{t} \le \frac{1}{a} \sum_{x \in E} g(x,x)\, \lambda_x < \infty$, whereas $\mu_r[L_r] = \infty$. So we see that μ_r is a σ-finite measure. We now collect some useful properties of μ_r.

Proposition 3.1. *For $0 \le t_1 < \dots < t_k < t$, $x_1, \dots, x_k \in E$, one has*

$$\begin{aligned}
&\mu_r[X_{t_1} = x_1, \dots, X_{t_k} = x_k,\ \zeta \in t + dt] =\\
&r_{t_2 - t_1}(x_1, x_2)\, \lambda_{x_2} \dots r_{t_k - t_{k-1}}(x_{k-1}, x_k)\, \lambda_{x_k}\, r_{t_1 + t - t_k}(x_k, x_1)\, \lambda_{x_1} \frac{dt}{t}, \ \textit{when } k > 1,\\
&\qquad = r_t(x_1, x_1)\, \lambda_{x_1} \frac{dt}{t}, \ \textit{when } k = 1,
\end{aligned} \tag{3.11}$$

(see below for the precise meaning of this formula).

When $n > 1$, for $t_i > 0$, $1 \le i \le n$, $t > 0$, and $x_0, \dots, x_{n-1} \in E$, one has

$$(3.12)\qquad \begin{aligned} &\mu_r(N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1},\\ &T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n,\, \zeta \in t + dt) =\\ &p_{x_0,x_1}\, p_{x_1,x_2} \dots p_{x_{n-1},x_0}\, 1\{0 < t_1 < \dots < t_{n-1} < t_n < t\}\, \frac{e^{-t}}{t}\, dt_1\, dt_2 \dots dt_n\, dt \end{aligned}$$

where $p_{x,y}$ are defined in (1.12) (see below for the precise meaning of this formula).

When $n = 1$, for $x_0 \in E$, $t > 0$,

$$(3.13)\qquad \mu_r[N = 1, Z_0 = x_0,\, \zeta \in t + dt] = e^{-t}\, \frac{dt}{t}$$

(see below for the precise meaning of this formula).

Proof.

• (3.11):

The precise meaning of (3.11) is obtained by considering some measurable $A \subseteq (t_k, \infty)$, replacing "$\zeta \in t + dt$" on the left-hand side, by "$\zeta \in A$", and on the right-hand side multiplying by $1_A(t)$ and integrating the expression over t. When $k > 1$, we have for $t > t_k$, with the help of the Markov property (see also the proof of (2.23)):

$$\begin{aligned} &P^t_{x,x}[X_{t_1} = x_1, \dots, X_{t_k} = x_k]\, \lambda_x \overset{(2.19)}{=}\\ &r_{t_1}(x, x_1)\, \lambda_{x_1}\, r_{t_2 - t_1}(x_1, x_2)\, \lambda_{x_2} \dots r_{t_k - t_{k-1}}(x_{k-1}, x_k)\, \lambda_{x_k}\, r_{t - t_k}(x_k, x)\, \lambda_x \end{aligned}$$

Summing over x and applying the Chapman-Kolmogorov identity, cf. (1.24), we find that

$$(3.14)\qquad \begin{aligned} &\sum_{x \in E} P^t_{x,x}[X_{t_1} = x_1, \dots, X_{t_k} = x_k]\, \lambda_x =\\ &r_{t_2 - t_1}(x_1, x_2)\, \lambda_{x_2} \dots r_{t_k - t_{k-1}}(x_{k-1}, x_k)\, \lambda_{x_k}\, r_{t_1 + t - t_k}(x_k, x_1)\, \lambda_{x_1}, \end{aligned}$$

and (3.11) follows from the above formula and (3.10).

When $k = 1$, (3.14) is replaced for $t > t_1$ by

$$(3.15)\qquad \sum_{x \in E} P^t_{x,x}[X_{t_1} = x_1]\, \lambda_x = r_t\,(x_1, x_1)\, \lambda_{x_1}$$

and the last line of (3.11) follows similarly.

• (3.12):

We use a similar procedure as indicated at the beginning of the proof (3.11) to give a precise meaning to (3.12), see also Remark 2.6. We then observe that for $t > 0$, $x \in E$,

$$(3.16)\qquad \begin{aligned} &P^t_{x,x}[N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1},\, T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n]\, \lambda_x \overset{(2.19)}{=}\\ &P_{x_0}\,[X_\cdot \text{ has } n \text{ jumps in } [0,t],\, X_{T_1} = x_1, \dots, X_{T_{n-1}} = x_{n-1}, X_{T_n} = x_0,\\ &\quad T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n]\, \delta_{x,x_0} =\\ &P_{x_0}\,[Z_1 = x_1, Z_2 = x_2, \dots, Z_{n-1} = x_{n-1}, Z_n = x_0] \cdot P_{x_0}[T_n < t < T_{n+1},\\ &\quad T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n]\, \delta_{x,x_0} = p_{x_0,x_1}\, p_{x_1,x_2} \dots p_{x_{n-1},x_0}\\ &\quad 1\{0 < t_1 < t_2 < \dots < t_n < t\}\, e^{-t}\, dt_1\, dt_2 \dots dt_n\, \delta_{x,x_0}, \end{aligned}$$

where we have also denoted by T_k, $k \geq 1$, the successive jump times for the continuous Markov chain, see Remark 1.1.

Summing over $x \in E$ and multiplying by $\frac{dt}{t}$, in view of (3.10), (3.16) yields

$$\mu_r[N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}, T_1 \in t_1 + dt_1, T_n \in t_n + dt_n, \zeta \in t + dt] =$$
$$p_{x_0,x_1}\, p_{x_1,x_2} \dots p_{x_{n-1},x_0}\, 1\{0 < t_1 < t_2 < \dots < t_n < t\}\, \frac{e^{-t}}{t}\, dt_1\, dt_2 \dots dt_n\, dt,$$

i.e. we have proved (3.12).

• (3.13):
As above, we can write for $t > 0$ and $x \in E$,

$$P^t_{x,x}[N = 1, Z_0 = x_0]\, \lambda_{x_0} = P_{x_0}\big[X_\cdot \text{ has no jump in } [0,t]\big]\, \delta_{x,x_0} =$$
$$P_{x_0}[T_1 > t]\, \delta_{x,x_0} = e^{-t}\delta_{x,x_0}.$$

so that summing over x and multiplying by $\frac{dt}{t}$ yields

$$\mu_r[N = 1, Z_0 = x_0, \zeta \in t + dt] = \frac{e^{-t}}{t}\, dt,$$

i.e. we have proved (3.13). □

We continue the discussion of the properties of the measure μ_r on rooted (also called based) loops, which was introduced in (3.10). In particular we will see that μ_r is invariant under time-shift and, in a suitable sense, under time-reversal as well.

Proposition 3.2. *For $n > 1$, $x_0, \dots, x_{n-1} \in E$, one has*

(3.17) $$\mu_r[N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}] = \frac{1}{n}\, p_{x_0,x_1}\, p_{x_1,x_2} \dots p_{x_{n-1},x_0} = \frac{1}{n}\, \frac{c_{x_0,x_1}\, c_{x_1,x_2} \dots c_{x_{n-1},x_0}}{\lambda_{x_0}\, \lambda_{x_1} \dots \lambda_{x_{n-1}}},$$

(3.18) $$\mu_r[N = n] = \frac{1}{n}\, Tr(P^n) \qquad \textit{(recall notation from (1.19))}.$$

(3.19) $$\mu_r[N > 1] = -\log\big(\det(I - P)\big) < \infty.$$

(3.20) $$\mu_r[N > 1, (Z_{k+m})_{m\in\mathbb{Z}} \in \cdot] = \mu_r[N > 1, (Z_m)_{m\in\mathbb{Z}} \in \cdot],\ \textit{for any } k \in \mathbb{Z},$$
(stationarity property of the discrete loop).

(3.21) $\theta_v \circ \mu_r = \mu_r$, *for any $v \in \mathbb{R}$, in the notation of (3.9),*
(stationarity property of the continuous-time loop).

Proof.

• (3.17):

$$\begin{aligned}
&\mu_r[N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}] \\
&\overset{(3.12)}{=} p_{x_0,x_1} \dots p_{x_{n-1},x_0} \int_{0<t_1<\dots<t_n<t} \frac{e^{-t}}{t}\, dt_1 \dots dt_n\, dt \\
&= p_{x_0,x_1} \dots p_{x_{n-1},x_0} \int_0^\infty \frac{t^{n-1}}{n!}\, e^{-t}\, dt = \frac{1}{n}\, p_{x_0,x_1} \dots p_{x_{n-1},x_0} \\
&\overset{(1.12)}{=} \frac{1}{n}\, \frac{c_{x_0,x_1} \dots c_{x_{n-1},x_0}}{\lambda_{x_0} \dots \lambda_{x_{n-1}}}, \quad \text{whence (3.17).}
\end{aligned}$$

• (3.18):

$$\mu_r[N=n] = \sum_{x_0,\dots,x_{n-1}\in E} \mu_r[N=n, Z_0=x_0,\dots,Z_{n-1}=x_{n-1}]$$
$$\overset{(3.17)}{=} \frac{1}{n} \sum_{x_0\in E} \langle 1_{x_0}, P^n 1_{x_0}\rangle = \frac{1}{n}\, Tr(P^n),$$

whence (3.18).

• (3.19):

$$\mu_r[N>1] = \sum_{n=2}^{\infty} \frac{1}{n}\, Tr(P^n) = \sum_{n=1}^{\infty} \frac{1}{n}\, Tr(P^n), \text{ since } Tr(P)=0,$$
$$= Tr\Big(\sum_{n=1}^{\infty} \frac{1}{n}\, P^n\Big) \qquad \text{(the series is convergent by (1.29)).}$$

As already observed, by (1.29) all the eigenvalues of the self-adjoint operator P on $L^2(d\lambda)$, say $\gamma_1 \le \dots \le \gamma_{|E|}$, belong to $(-1,1)$. So we have the identity

$$\sum_{n=1}^{\infty} \frac{1}{n}\, P^n = -\log(I-P), \tag{3.22}$$

which can for instance be seen after diagonalization of P in some orthonormal basis of $L^2(d\lambda)$, and hence we find that

$$\mu_r[N>1] = Tr(-\log(I-P)) = \sum_{i=1}^{|E|} -\log(1-\gamma_i) = -\log \prod_{i=1}^{|E|} (1-\gamma_i)$$
$$= -\log(\det(I-P)),$$

and (3.19) is proved.

• (3.20):

We pick $n>1$, and first show that

$$\mu_r[N=n, (Z_{k+m})_{m\in\mathbb{Z}} \in \cdot\,] = \mu_r[N=n, (Z_m)_{m\in\mathbb{Z}} \in \cdot\,]. \tag{3.23}$$

On $\{N=n\}$, $m \to Z_m$ has period n (so $Z_{m+\ell n} = Z_m$, for all $m,\ell \in \mathbb{Z}$), see below (3.6). We can thus assume that $0<k<n$, and restrict m to $\{0,\dots,n-1\}$. But for $x_0,\dots,x_{n-1}\in E$,

$$\mu_r[N=n, (Z_m)_{0\le m<n} = (x_m)_{0\le m<n}] \overset{(3.17)}{=} \frac{1}{n}\, p_{x_0,x_1}\dots p_{x_{n-1},x_0},$$

whereas

$$\mu_r[N=n, (Z_{k+m})_{0\le m<n} = (x_m)_{0\le m<n}] =$$
$$\mu_r[N=n, Z_k=x_0, Z_{k+1}=x_1,\dots,Z_{n-1}=x_{n-1-k}, Z_0=x_{n-k},\dots,Z_{k-1}=x_{n-1}] \overset{(3.17)}{=}$$
$$\frac{1}{n}\, p_{x_{n-k},x_{n-k+1}}\dots p_{x_{n-2},x_{n-1}}\, p_{x_{n-1},x_0}\dots p_{x_{n-1-k},x_{n-k}} =$$
$$\frac{1}{n}\, p_{x_0,x_1}\dots p_{x_{n-1},x_0}, \text{ whence (3.23).}$$

Summing over $n>1$ in (3.23) yields (3.20).

• (3.21):

We use the following lemma:

Lemma 3.3. $(t > 0, v \in \mathbb{R})$

$$\theta_v \circ \Big(\sum_{x\in E} P^t_{x,x}\, \lambda_x \Big) = \sum_{x\in E} P^t_{x,x}\, \lambda_x. \tag{3.24}$$

Proof. The measure $\sum_{x\in E} P^t_{x,x}\, \lambda_x$ is concentrated on $L_{r,t}$, and for $\gamma \in L_{r,t}$, $\theta_{\ell t}(\gamma) = \gamma$, for any $\ell \in \mathbb{Z}$, due to the fact that $s \in \mathbb{R} \to X_s(\gamma) \in E$ has period t. We can thus assume that $0 < v < t$. The claim (3.24) will then follow once we show that for any $0 < t_1 < \dots < t_k = t - v < t_{k+1} < \dots < t_n = t$, one has for $x_1, \dots, x_n \in E$,

$$\begin{aligned} &\sum_{x\in E} P^t_{x,x}[X_{t_1} = x_1, \dots, X_{t_n} = x_n]\, \lambda_x = \\ &\sum_{x\in E} P^t_{x,x}\big[X_{v+t_1} = x_1, \dots, \underset{\substack{\uparrow \\ t_k = t-v}}{X_{v+t_k}} = x_k, X_{v+t_{k+1}-t} = x_{k+1}, \dots, X_v = x_n\big]\, \lambda_x. \end{aligned} \tag{3.25}$$

The expression on the left-hand side of (3.25), as in (3.14), is equal to

$$r_{t_2-t_1}(x_1,x_2)\, \lambda_{x_2} \dots r_{t_n-t_{n-1}}(x_{n-1},x_n)\, \lambda_{x_n}\, r_{t_1}(x_n,x_1)\, \lambda_{x_1} \quad \text{(recall that } t_n = t).$$

Note that $0 < v + t_{k+1} - t < \dots < v = v + t_n - t < v + t_1 < \dots < v + t_k = t$, and therefore using once again the calculation in (3.14), the expression on the right-hand side of (3.25) equals:

$$\begin{aligned} &r_{t_{k+2}-t_{k+1}}(x_{k+1},x_{k+2})\, \lambda_{x_{k+2}} \dots r_{t_n-t_{n-1}}(x_{n-1},x_n)\, \lambda_{x_n} \\ &r_{t_1}(x_n,x_1)\, \lambda_{x_1} \dots r_{t_{k+1}-t_k}(x_k,x_{k+1})\, \lambda_{x_{k+1}}. \end{aligned}$$

This shows that (3.25) holds and completes the proof of (3.24). □

We can now complete the proof of (3.21). When B is a measurable subset of L_r, we thus find that:

$$\mu_r[B] \overset{(3.10)}{=} \int_0^\infty \sum_{x\in E} P^t_{x,x}[B]\, \lambda_x\, \frac{dt}{t}\,,$$

and therefore

$$\begin{aligned} \theta_v \circ \mu_r[B] = \mu_r[\theta_v^{-1}(B)] &= \int_0^\infty \sum_{x\in E} P^t_{x,x}[\theta_v^{-1}(B)]\, \lambda_x\, \frac{dt}{t} \\ &\overset{(3.24)}{=} \int_0^\infty \sum_{x\in E} P^t_{x,x}[B]\, \lambda_x\, \frac{dt}{t} = \mu_r[B]. \end{aligned}$$

This completes the proof of (3.21). □

We will now highlight some invariance properties of the measure μ_r under time-reversal. For this purpose it is convenient to introduce the time-reversal map $\overset{\vee}{\theta}$: $L_r \to L_r$ via:

$$\begin{aligned} &\overset{\vee}{\theta}(\gamma) \in L_r, \text{ for } \gamma \in L_r, \text{ is the rooted loop } \gamma' \text{ such that} \\ &\zeta(\gamma') = \zeta(\gamma) \text{ and } X_s(\gamma') = \lim_{\varepsilon \downarrow 0} X_{-s-\varepsilon}(\gamma) = X_{(-s)-}(\gamma), \text{ for all } s \in \mathbb{R}. \end{aligned} \tag{3.26}$$

Proposition 3.4.

$$\mu_r[N > 1, (Z_{-m})_{m\in\mathbb{Z}} \in \cdot] = \mu_r[N > 1, (Z_m)_{m\in\mathbb{Z}} \in \cdot] \tag{3.27}$$
(time-reversal invariance of the discrete loop).

$$\overset{\vee}{\theta} \circ \mu_r = \mu_r \quad \textit{(time-reversal invariance of the continuous loop)}. \tag{3.28}$$

Proof.

• (3.27):

One could use (3.28), but it is instructive to give a direct proof. For $n > 1$, $x_0, \dots, x_{n-1} \in E$, one has by (3.17)

$$\mu_r[N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}] = \frac{1}{n}\, \frac{c_{x_0,x_1}\, c_{x_1,x_2} \dots c_{x_{n-1},x_0}}{\lambda_{x_0} \dots \lambda_{x_{n-1}}}\,,$$

whereas

$$\begin{aligned}
&\mu_r[N = n, Z_0 = x_0, Z_{-1} = x_1, \dots, Z_{-(n-1)} = x_{n-1}] = \\
&\mu_r[N = n, Z_0 = x_0, Z_1 = x_{n-1}, Z_2 = x_{n-2}, \dots, Z_{n-1} = x_1] \\
&\overset{(3.17)}{=} \frac{1}{n}\, \frac{c_{x_0,x_{n-1}}\, c_{x_{n-1},x_{n-2}} \dots c_{x_2,x_1}\, c_{x_1,x_0}}{\lambda_{x_0}\, \lambda_{x_{n-1}} \dots \lambda_{x_1}} = \mu_r[N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}].
\end{aligned}$$

Summing over $n > 1$ yields the claim (3.27).

• (3.28):

We first show that for $t > 0$,

$$\overset{\vee}{\theta} \circ \Big(\sum_{x\in E} P^t_{x,x}\, \lambda_x\Big) = \sum_{x\in E} P^t_{x,x}\, \lambda_x. \tag{3.29}$$

Indeed, for arbitrary $k \ge 1$, $0 < t_1 < \dots < t_k < t$, and $x_1, \dots, x_k \in E$, by (3.14) we have

$$\begin{aligned}
&\sum_{x\in E} P^t_{x,x}[X_{t_1} = x_1, \dots, X_{t_k} = x_k]\, \lambda_x = \\
&r_{t_2-t_1}(x_1, x_2) \dots r_{t_k-t_{k-1}}(x_{k-1}, x_k)\, r_{t_1+t-t_k}(x_k, x_1)\, \lambda_{x_1} \dots \lambda_{x_k},
\end{aligned}$$

whereas using (3.26) and the facts that $s \to X_s(\gamma)$ has period t, when $\gamma \in L_{r,t}$, and $P^t_{x,x}[v$ is a jump time of $s \to X_s(\gamma)] = 0$, for any $v \in \mathbb{R}$,

$$\begin{aligned}
&\overset{\vee}{\theta} \circ \Big(\sum_{x\in E} P^t_{x,x}\, \lambda_x\Big)[X_{t_1} = x_1, \dots, X_{t_k} = x_k] = \\
&(\text{with } t'_1 = t - t_k < t'_2 = t - t_{k-1} < \dots < t'_k = t - t_1) \\
&\sum_{x\in E} P^t_{x,x}[X_{t'_1} = x_k, X_{t'_2} = x_{k-1}, \dots, X_{t'_k} = x_1]\, \lambda_x \overset{(3.14)}{=} \\
&r_{t'_2-t'_1}(x_k, x_{k-1})\, r_{t'_3-t'_2}(x_{k-1}, x_{k-2}) \dots r_{t'_1+t-t'_k}(x_1, x_k)\, \lambda_{x_1} \dots \lambda_{x_k} =
\end{aligned}$$

(using the symmetry of $r_s(\cdot,\cdot)$, see (1.23), and the definition of the t'_i)

$$r_{t_k-t_{k-1}}(x_{k-1}, x_k)\, r_{t_{k-1}-t_{k-2}}(x_{k-2}, x_{k-1}) \dots r_{t_1+t-t_k}(x_k, x_1)\, \lambda_{x_1} \dots \lambda_{x_k},$$

and the claim (3.29) now follows.

As a result, for a measurable subset B of L_r, we find that

$$\overset{\vee}{\theta} \circ \mu_r[B] \overset{(3.10)}{=} \int_0^\infty \sum_{x \in E} P^t_{x,x}\big[\overset{\vee}{\theta}^{\,-1}(B)\big]\, \lambda_x \,\frac{dt}{t} \overset{(3.29)}{=} \int_0^\infty \sum_{x \in E} P^t_{x,x}[B]\, \lambda_x \,\frac{dt}{t} \overset{(3.10)}{=} \mu_r[B],$$

and this completes the proof of (3.28). □

3.2 Pointed loops and the measure μ_p on pointed loops

In this section we define the σ-finite measure μ_p governing pointed loops and relate it to the measure μ_r on rooted loops. As it turns out, the measures μ_r and μ_p agree on functions invariant under the shift $\theta_v, v \in \mathbb{R}$.

We introduce the measurable subspace of L_r of pointed loops, see below (3.7),

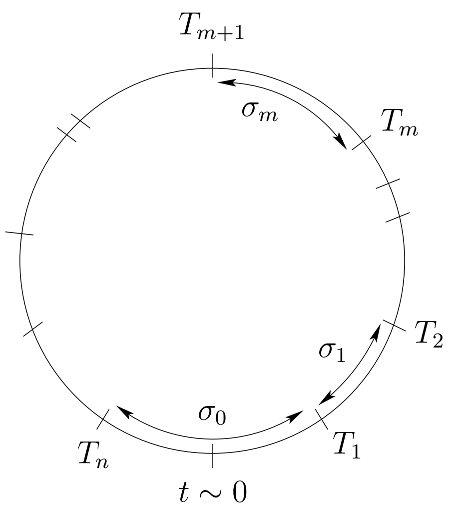
(3.30) $$\begin{aligned} L_p &= \{\gamma \in L_r;\ \gamma \text{ is trivial, or } N(\gamma) = n > 1, \text{ and } T_n(\gamma) = \zeta(\gamma)\} \\ &= \{\gamma \in L_r;\ \gamma \text{ is trivial, or } 0 \text{ is a jump time of } s \in \mathbb{R} \to X_s(\gamma)\}. \end{aligned}$$

It is convenient to introduce for $\gamma \in L_r$ with $N(\gamma) = n > 1$, the variables describing the successive durations between jumps of the loop γ (see also Figure 3.1):

(3.31) $$\begin{aligned} &\sigma_0(\gamma) = T_1(\gamma) + \zeta(\gamma) - T_n(\gamma), \\ &\sigma_1(\gamma) = T_2(\gamma) - T_1(\gamma), \dots, \sigma_{n-1}(\gamma) = T_n(\gamma) - T_{n-1}(\gamma), \end{aligned}$$

so that

(3.32) $$\sigma_0(\gamma) + \dots + \sigma_{n-1}(\gamma) = \zeta(\gamma).$$



the variables σ_ℓ and T_k in a rooted loop γ with $N(\gamma) = n$ and $\zeta(\gamma) = t$

Fig. 3.1

In the case where γ is a pointed loop with $N(\gamma) = n > 1$, the variables $Z_0, \dots, Z_{n-1}$ and $\sigma_0, \dots, \sigma_{n-1}$ enable to reconstruct γ, with the help of (3.8) and the identity $T_n(\gamma) = \zeta(\gamma)$. We now introduce a measure μ_p on L_p via:

$$(3.33)\quad \begin{cases} \mu_p[N=1, Z_0 = x_0, \zeta \in t + dt] = e^{-t}\,\dfrac{dt}{t}, \quad \text{for any } x_0 \in E,\ t > 0, \\ \mu_p[N = n, Z_0 = x_0, \dots Z_{n-1} = x_{n-1}, \\ \sigma_0 \in s_0 + ds_0, \dots, \sigma_{n-1} \in s_{n-1} + ds_{n-1}] = \\ \dfrac{1}{n}\, p_{x_0,x_1} \dots p_{x_{n-1},x_0}\, e^{-(s_0 + \dots + s_{n-1})} ds_0 \dots ds_{n-1}, \quad \text{for } n > 1,\ x_0, \dots, x_{n-1} \in E, \\ \qquad s_0, \dots, s_{n-1} > 0. \end{cases}$$

The meaning of this formla is the same as in (3.11) - (3.13). We will now relate the measure μ_r on rooted loops and μ_p on pointed loops.

When $\gamma \in L_r$ is such that $N(\gamma) = n > 1$, the loops $\theta_{T_m(\gamma)}(\gamma)$, for $1 \le m \le n$, are pointed. We will denote by θ_{T_m} the corresponding map from $L_r \cap \{N = n\} \ni \gamma \to \gamma' = \theta_{T_m(\gamma)}(\gamma) \in L_p \cap \{N = n\}$. Thus $\theta_{T_m} \circ (1\{N = n\}\,\mu_r)$ is a measure on $L_p \cap \{N = n\}$. As we will see below it corresponds to a type of size-biased modification of $1\{N = n\}\mu_p$.

Proposition 3.5. $(n > 1)$

For any $1 \le m \le n$, $x_0, \dots, x_{n-1} \in E$, $s_0, s_1, \dots, s_{n-1} > 0$,

$$(3.34)\quad \begin{aligned} &\theta_{T_m} \circ (1\{N = n\}\,\mu_r)\,[Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}, \\ &\sigma_0 \in s_0 + ds_0, \dots, \sigma_{n-1} \in s_{n-1} + ds_{n-1}] = \\ &p_{x_0,x_1} \dots p_{x_{n-1},x_0},\ \frac{s_{n-m}}{s_0 + \dots + s_{n-1}}\ e^{-(s_0 + \dots + s_{n-1})}\, ds_0\, ds_1 \dots ds_{n-1}. \end{aligned}$$

For any $1 \le m \le n$,

$$(3.35)\quad \theta_{T_m} \circ (1\{N = n\}\,\mu_r) = n\,\frac{\sigma_{n-m}}{\sigma_0 + \dots + \sigma_{n-1}}\ 1\{N = n\}\,\mu_p.$$

When $F\colon L_r \to \mathbb{R}_+$ *is a bounded measurable function such that* $F \circ \theta_v = F$, *for all* $v \in \mathbb{R}$, *then one has*

$$(3.36)\quad \int_{\{N=n\}} F\,d\mu_r = \int_{\{N=n\}} F\,d\mu_p \ \textit{(this equality holds as well when } n = 1\textit{)}.$$

Proof.

- (3.34):

We consider $h_0, \dots, h_{n-1}$ bounded measurable functions on $(0, \infty)$, and we extend the definition of σ_ℓ for $\ell \in \{0, \dots, n-1\}$ on $\{N = n\}$, to all $\ell \in \mathbb{Z}$, using periodicity (i.e. so that $\sigma_{\ell + kn} = \sigma_\ell$ for all ℓ, $k \in \mathbb{Z}$).
So we find that

$$\begin{aligned} &\int 1\{Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}\}\, h_0(\sigma_0) \dots h_{n-1}(\sigma_{n-1})\, d\big(\theta_{T_m} \circ (1\{N = n\}\,\mu_r)\big) = \\ &\int_{\{N=n\}} 1\{Z_m = x_0, Z_{m+1} = x_1, \dots Z_{m-1} = x_{n-1}\}\, h_0(\sigma_m)\, h_1(\sigma_{m+1}) \dots h_{n-1}(\sigma_{m-1})\, d\mu_r \overset{(3.12)}{=} \\ &p_{x_0,x_1} \dots p_{x_{n-1},x_0} \int_{0 < t_1 < \dots < t_n < t} h_0(t_{m+1} - t_m) \dots h_{n-m-1}(t_n - t_{n-1})\, h_{n-m}(t - t_n + t_1) \\ &\qquad h_{n-m+1}(t_2 - t_1) \dots h_{n-1}(t_m - t_{m-1})\, \frac{e^{-t}}{t}\, dt_1 \dots dt_n\, dt. \end{aligned}$$

We can perform a change of variables in the above integral, replacing $t_1, t_2, \dots t_n, t$ by $t_1, s_1, s_2, \dots s_{n-1}, s_0$, where

$$t_1 = t_1,\, t_2 = t_1 + s_1,\, t_3 = t_1 + s_1 + s_2, \dots, t_n = t_1 + s_1 + \dots + s_{n-1},$$
$$t = s_1 + \dots + s_{n-1} + s_0,$$

which bijectively maps the region $0 < t_1 < t_2 < \dots < t_n < t$ into the region $0 < t_1 < s_0$, $s_1 > 0, \dots, s_{n-1} > 0$.

So the above expression equals:

$$p_{x_0,x_1} \dots p_{x_{n-1},x_0} \int_{\substack{0<t_1<s_0 \\ s_1>0,\dots,s_{n-1}>0}} h_0(s_m) \dots h_{n-m-1}(s_{n-1}) h_{n-m}(s_0) h_{n-m+1}(s_1) \dots h_{n-1}(s_{m-1}) \frac{1}{(s_0 + \dots + s_{n-1})}\, e^{-(s_0+\dots+s_{n-1})}\, dt_1\, ds_0 \dots ds_{n-1} =$$

$$p_{x_0,x_1} \dots p_{x_{n-1},x_0} \int_{s_0>0,\dots,s_{n-1}>0} h_0(s_m) \dots h_{n-m-1}(s_{n-1}) h_{n-m}(s_0) h_{n-m+1}(s_1) \dots h_{n-1}(s_{m-1}) \frac{s_0}{(s_0 + \dots + s_{n-1})}\, e^{-(s_0+\dots+s_{n-1})}\, ds_0 \dots ds_{n-1} =$$

(by relabeling variables)

$$p_{x_0,x_1} \dots p_{x_{n-1},x_0} \int_{s_0>0,\dots,s_{n-1}>0} h_0(s_0) \dots h_{n-1}(s_{n-1}) \frac{s_{n-m}}{(s_0 + \dots + s_{n-1})}\, e^{-(s_0+\dots+s_{n-1})}\, ds_0 \dots ds_{n-1},$$

and this proves (3.34).

• (3.35):

Combining (3.34) and (3.33), we see that for $h_0, \dots, h_{n-1}$ as above and $x_0, \dots x_{n-1} \in E$, one has

$$\int 1\{Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}\}\, h_0(\sigma_0) \dots h_{n-1}(\sigma_{n-1})\, d(\theta_{T_m} \circ (1\{N = n\}\, \mu_r) =$$
$$\int 1\{Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}\}\, h_0(\sigma_0) \dots h_{n-1}(\sigma_{n-1})\, n\, \frac{\sigma_{n-m}}{\sigma_0 + \dots + \sigma_{n-1}}\, d(1\{N = n\}\, \mu_p).$$

From this we see that $\theta_{T_m} \circ (1\{N = n\}\, \mu_r)$ and $n\, \frac{\sigma_{n-m}}{\sigma_0 + \dots + \sigma_{n-1}} 1\{N = n\}\, \mu_p$ have the same (finite since $n > 1$) total mass and using Dynkin's lemma we conclude that they coincide on the σ-algebra of $L_p \cap \{N = n\}$ generated by the variables $Z_0, \dots, Z_{n-1}$, $\sigma_0, \dots, \sigma_{n-1}$. This is the full σ-algebra of measurable subsets of $L_p \cap \{N = n\}$, and (3.35) follows.

• (3.36):

On $\{N = n\}$, with $n > 1$, we set $\theta_{T_m}(\gamma) \stackrel{\text{def}}{=} \theta_{T_m(\gamma)}(\gamma)$, for $1 \le m \le n$, and note that

$F \circ \theta_{T_m} = F$ on $\{N = n\}$ (since $F \circ \theta_v = F$ for all $v \in \mathbb{R}$). As a result we find that

$$\begin{aligned}\int_{\{N=n\}} F\, d\mu_r &= \int_{\{N=n\}} \frac{1}{n} \sum_{m=1}^{n} F \circ \theta_{T_m}\, d\mu_r \\ &= \frac{1}{n} \sum_{m=1}^{n} \int F\, d\big(\theta_{T_m} \circ (1\{N=n\}\, \mu_r)\big) \\ &\overset{(3.35)}{=} \frac{1}{n} \sum_{m=1}^{n} \int_{\{N=n\}} F\, n\, \frac{\sigma_{n-m}}{\sigma_0 + \dots + \sigma_{n-1}}\, d\mu_p \\ &= \int_{\{N=n\}} F\, d\mu_p,\end{aligned}$$

and (3.36) follows. □

We now continue with the discussion of the properties of the measures on rooted and pointed loops we have introduced. Our next topic will be the restriction property.

3.3 Restriction property

This is a short section where we state and prove the restriction property. Informally said, the measures μ_r and μ_p have the property that restricting them to the set of loops that remain in U amounts to working with the modified weights and killing measure corresponding to killing when exiting U, see Remark 1.5.

When U is a subset of E, we define

(3.37) $$L_{r,U} = \{\gamma \in L_r;\ X_s(\gamma) \in U \text{ for all } s \in \mathbb{R}\} \overset{\text{notation}}{=} \{\gamma \in L_r;\ \gamma \subseteq U\},$$

Proposition 3.6. *(restriction property)*

When U is a connected (non-empty) subset of E

(3.38) $$1_{L_{r,U}} \mu_r = \mu_{r,U}, \text{ and}$$

(3.39) $$1_{L_{r,U}} \mu_p = \mu_{p,U},$$

where $\mu_{r,U}$ and $\mu_{p,U}$ respectively stand for the analogues of μ_r in (3.10) and μ_p in (3.33), when U replaces E, and U is endowed with the weights $c_{x,y}$, $x, y \in U$, and the killing measure $\widetilde{\kappa}_x = \kappa_x + \sum_{y \in E \setminus U} c_{x,y}$, $x \in U$ (cf. Remark 1.5). (When U is not connected the above identities apply to the different connected components of U, which induce a partition of $L_{r,U}$.)

Proof.

- (3.38):

When $n > 1$, $t_i > 0$, $1 \le i \le n$, $t > 0$, and $x_0, \dots, x_{n-1} \in U$, by (3.12):

(3.40) $$\begin{aligned}&\mu_r(N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}, T_1 \in t_1 + dt_1, \dots, T_n \in t_n + dt_n, \\ &\quad \zeta \in t + dt) = \\ &\frac{c_{x_0,x_1}\, c_{x_1,x_2} \dots c_{x_{n-1},x_0}}{\lambda_{x_0}\lambda_{x_1} \dots \lambda_{x_{n-1}}}\, 1\{0 < t_1 < \dots < t_n < t\}\, \frac{e^{-t}}{t}\, dt_1 \dots dt_n dt.\end{aligned}$$

Note that $\widetilde{\lambda}_x \stackrel{\text{def}}{=} \sum_{y \in U} c_{x,y} + \widetilde{\kappa}_x = \lambda_x$, for $x \in U$, and hence the expression in (3.40) equals

$$\mu_{r,U}(N = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}, T_1 \in t_1 + dt_1, \dots, T_n = t_n + dt_n, \zeta \in t + dt).$$

By (3.13), a similar equality holds when $n = 1$.

So we see that for each $n \geq 1$, $1_{L_{r,U}} \mu_r$ and $\mu_{r,U}$ coincide on the σ-algebra of $L_{r,U} \cap \{N = n\}$ generated by the variables $Z_0, \dots, Z_{n-1}$, $T_1, \dots T_{n-1}, \zeta$. By (3.8), this is the full σ-algebra of measurable subsets of $L_{r,U}$, and (3.38) follows.

- (3.39):

The argument is similar, and now uses the formula (3.33) for μ_p. □

3.4 Local times

In this section we define the local time of rooted loops and derive an identity, which will play an important role in the next chapter, when calculating the Laplace transform of the occupation field of a Poisson gas of Markovian loops.

We define the local time of the rooted loop $\gamma \in L_r$ at $x \in E$:

$$L_x(\gamma) = \int_0^{\zeta(\gamma)} 1\{X_s(\gamma) = x\}\, ds\, \frac{1}{\lambda_x}\,. \tag{3.41}$$

Note that the local time is invariant under time-shift:

$$L_x \circ \theta_v = L_x, \text{ for any } x \in E,\ v \in \mathbb{R}, \tag{3.42}$$

and also invariant under time-reversal:

$$L_x \circ \overset{\vee}{\theta} = L_x, \text{ for any } x \in E. \tag{3.43}$$

As a consequence of (3.36) and (3.42), we see that **we can indifferently use μ_r or μ_p when evaluating "expectations" of functions of $(L_x)_{x \in E}$.**

An important role is played by the next proposition, which computes the "de-singularized Laplace transform" of the field of local times of Markovian loops.

Proposition 3.7.

$$\textit{For } v \geq 0 \textit{ and } x \in E,\ \int_{\{N=1\}} (1 - e^{-vL_x})\, d\mu_r = \log\Big(1 + \frac{v}{\lambda_x}\Big). \tag{3.44}$$

For $V : E \to \mathbb{R}_+$, one has:

$$\begin{aligned} \int \big(1 - e^{-\sum_{x \in E} V(x) L_x}\big)\, d\mu_r &= \log\det(I + GV) \\ &= \log\det(I + \sqrt{V}\, G\, \sqrt{V}) \\ &= -\log\Big(\frac{\det G_V}{\det G}\Big), \end{aligned} \tag{3.45}$$

where $G_V = (V - L)^{-1}$ (the various members of (3.45) are finite, non-negative, and equal).

In particular, one has

$$\int (1 - e^{-vL_x})\, d\mu_r = \log\big(1 + v\, g(x,x)\big),\ \textit{for } v \geq 0,\ \textit{and } x \in E. \tag{3.46}$$

Remark 3.8. Note that $g_V(x,y) = (G_V 1_y)(x)$, $x, y \in E$, can be interpreted as the Green function one obtains when choosing $\kappa_x^V = \kappa_x + V(x)$, $x \in E$, as a new killing measure. □

Proof of Proposition 3.7.

• (3.44):

Since $L_x(\gamma) = 0$, when $N(\gamma) = 1$ and $Z_0(\gamma) \neq x$, and $L_x(\gamma) = \frac{\zeta(\gamma)}{\lambda_x}$, when $N(\gamma) = 1$ and $Z_0(\gamma) = x$, we find that

$$\int_{\{N=1\}} (1 - e^{-vL_x})\, d\mu_r \overset{(3.13)}{=} \int_0^\infty (1 - e^{-\frac{v}{\lambda_x} t})\, \frac{e^{-t}}{t}\, dt.$$

To compute this last integral we use the identity:

$$\text{(3.47)} \qquad \begin{array}{l} \text{for } 0 \le a < b \text{ and } 0 \le a' < b', \\ \displaystyle\int_a^b e^{-a't} - e^{-b't}\, \frac{dt}{t} = \int_a^b \int_{a'}^{b'} e^{-tt'}\, dt dt' = \int_{a'}^{b'} e^{-at'} - e^{-bt'}\, \frac{dt'}{t'}. \end{array}$$

So choosing $a = 0$ and letting $b \to \infty$, $a' = 1$, $b' = 1 + \frac{v}{\lambda_x}$, we obtain:

$$\int_{\{N=1\}} (1 - e^{-vL_x})\, d\mu_r = \int_1^{1+\frac{v}{\lambda_x}} \frac{dt'}{t'} = \log\Big(1 + \frac{v}{\lambda_x}\Big),$$

whence (3.44).

• (3.45):

We begin with a preparatory calculation. By the definition of $P_{x,x}^t$ in (2.19) and L_y in (3.41) we see that

$$E_{x,x}^t\big[1 - e^{-\sum\limits_{y\in E} V(y) L_y}\big] = P_x[X_t = x]\, \frac{1}{\lambda_x} - E_x\big[1\{X_t = x\}\, e^{-\int_0^t \frac{V}{\lambda}(X_s) ds}\big]\, \frac{1}{\lambda_x}.$$

Using Feynman-Kac's formula (1.84) (for $V = 0$, this boils down to (1.18)), we find that for $V\colon E \to \mathbb{R}_+$, $t > 0$, $x \in E$,

$$\text{(3.48)} \qquad E_{x,x}^t\big[1 - e^{-\sum\limits_{y\in E} V(y) L_y}\big] = \frac{1}{\lambda_x}\, \big(e^{t(P-I)}\, 1_x - e^{t(P-I-\frac{V}{\lambda})}\, 1_x\big)(x) \ge 0.$$

In combination with (3.10), we obtain that

$$\text{(3.49)} \qquad \int \big(1 - e^{-\sum\limits_{y\in E} V(y) L_y}\big)\, d\mu_r = \int_0^\infty Tr(e^{t(P-I)}) - Tr(e^{t(P-I-\frac{V}{\lambda})})\, \frac{dt}{t},$$

(where both sides can possibly be infinite).

We first consider the case of "small V" and the general case.

• The case of a "small V":

We know by (1.29), see also above (3.22), that all eigenvalues of the self-adjoint operator P on $L^2(d\lambda)$ belong to $(-1, 1)$. The operator $P - \frac{V}{\lambda}$ is also self-adjoint on $L^2(d\lambda)$, and using the variational characterization of the largest and smallest eigenvalues of $P - \frac{V}{\lambda}$:

$$\sup\Big\{\Big(f, \Big(P - \frac{V}{\lambda}\Big) f\Big)_\lambda;\ (f,f)_\lambda = 1\Big\} \text{ and } \inf\Big\{\Big(f, \Big(P - \frac{V}{\lambda}\Big) f\Big)_\lambda;\ (f,f)_\lambda = 1\Big\},$$

respectively, we can pick $\varepsilon > 0$, so that

$$\text{(3.50)} \qquad \text{for } \|V\|_\infty < \varepsilon, \text{ all eigenvalues of } P - \frac{V}{\lambda} \text{ belong to } (-1,1).$$

We then expand e^{tP} and $e^{t(P-\frac{V}{\lambda})}$, and write:

$$\text{(3.51)} \qquad Tr(e^{t(P-I)}) - Tr(e^{t(P-I-\frac{V}{\lambda})}) = e^{-t} \sum_{k\ge 1} \frac{t^k}{k!} \Big[Tr(P^k) - Tr\Big(\Big(P - \frac{V}{\lambda}\Big)^k\Big)\Big]$$

(where we took into account the cancellation of the term $k = 0$).

Note that when $0 < a < 1$,

$$\int_0^\infty \sum_{k\ge 1} t^k \frac{a^k}{k!} e^{-t} \frac{dt}{t} = \sum_{k\ge 1} a^k \int_0^\infty \frac{t^{k-1}}{k!} e^{-t} dt = \sum_{k\ge 1} \frac{a^k}{k} = -\log(1-a) < \infty,$$

This observation combined with the fact that all eigenvalues of P and $P - \frac{V}{\lambda}$ are in absolute value smaller than 1, shows that we can insert (3.51) in the right-hand side of (3.49), and exchange summation with integrals:

$$\begin{aligned}
\int \big(1 - e^{-\sum_{y\in E} V(y) L_y}\big)\, d\mu_r &= \sum_{k\ge 1} \int_0^\infty \frac{t^k}{k!} Tr(P^k)\, e^{-t} \frac{dt}{t} \\
&\quad - \sum_{k\ge 1} \int_0^\infty \frac{t^k}{k!} Tr\Big(\Big(P - \frac{V}{\lambda}\Big)^k\Big)\, e^{-t} \frac{dt}{t} \\
\text{(3.52)} \qquad &= \sum_{k\ge 1} \frac{1}{k} Tr(P^k) - \sum_{k\ge 1} \frac{1}{k} Tr\Big(\Big(P - \frac{V}{\lambda}\Big)^k\Big) \\
\text{and as below (3.22)} \quad &= -\log\det(I-P) + \log\det\Big(I - P + \frac{V}{\lambda}\Big) \\
&= \log\det\Big(I + (I-P)^{-1} \frac{V}{\lambda}\Big).
\end{aligned}$$

By (1.37) and (1.41), we know that $(I-P)^{-1}\lambda^{-1} = G$, so

$$\det\Big(I + (I-P)^{-1} \frac{V}{\lambda}\Big) = \det(I + GV),$$

and in addition, "multiplying and dividing" by $\det(\sqrt{V})$ (i.e. by $\det(\sqrt{V+\eta})$, with $\eta > 0$, which one lets go to zero), we find that

$$\det(I + GV) = \det(I + \sqrt{V}\, G\sqrt{V}).$$

Writing further that $\det G = \det(I-P)^{-1}\det(\lambda^{-1})$ and that, since $\lambda(I-P) = -L$, cf. (1.41),

$$\det\Big(I - P + \frac{V}{\lambda}\Big) = \det(\lambda^{-1})\det(V - L) = \det(\lambda^{-1})/\det G_V,$$

we find that

$$-\log\det(I-P) + \log\det\Big(I - P + \frac{V}{\lambda}\Big) = \log\Big(\frac{\det G}{\det G_V}\Big).$$

Combining these identities, we see that we have proved (3.45) under (3.50), i.e. when V is "small".

We now treat the general case.

• The general case:

Note that the function $\beta \geq 0 \to \int_{L_r} (1 - e^{-\beta \sum_{x\in E} V(x) L_x})\, d\mu_r \in [0, \infty]$ is non-decreasing, finite for small β by the first step, and we can use the inequality $1 - e^{-(a+b)} \leq 1 - e^{-a} + 1 - e^{-b}$ for $a, b \geq 0$, to conclude that finiteness holds for all $\beta \geq 0$, that is:

$$\int_{L_r} \big(1 - e^{-\beta \sum\limits_{x\in E} V(x) L_x}\big)\, d\mu_r < \infty, \text{ for } \beta \in [0, \infty) \tag{3.53}$$

(and actually increases to $+\infty$ as $\beta \to \infty$, if V is not identically equal to 0, see also below (3.10)). In addition, as we explain below, it follows, by domination, that the function

$$\beta \in \{z \in \mathbb{C};\ Re\, z > 0\} \longrightarrow \int_{L_r} \big(1 - e^{-\beta \sum\limits_{x\in E} V(x) L_x}\big)\, d\mu_r \text{ is analytic.} \tag{3.54}$$

Indeed, we first note that the integrand in (3.54) has a modulus bounded by 2 when $\beta \in \{z \in \mathbb{C};\ Re\, z > 0\}$, and that $\mu[N > 1] < \infty$, by (3.19). There only remains to show domination on $\{N = 1\}$. To this end we observe that the integrand in (3.54) equals $\int_0^1 \beta \sum_{x\in E} V(x) L_x \exp\{-\beta \sum_{x\in E} V(x) L_x\, u\} du$, and has a modulus bounded by $|\beta| \sum_{x\in E} V(x) L_x$, for β in the same domain as above. By (3.13), $1\{N = 1\} L_z$ is μ-integrable for each $z \in E$. We have thus shown domination of the integrand in (3.54) when β remains in a compact subset of $\{z \in \mathbb{C};\ Re\, z > 0\}$. The claim (3.54) follows.

On the other hand $\sqrt{V}\, G \sqrt{V}$ is self-adjoint for the canonical scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^E$. By (1.35), all eigenvalues of $\sqrt{V}\, G \sqrt{V}$ are non-negative (because $\langle f, Gf \rangle \geq 0$, for all $f : E \to \mathbb{R}$). It follows that

$$\beta \geq 0 \longrightarrow \det(I + \beta \sqrt{V}\, G \sqrt{V}) \geq 1 \text{ is a non-decreasing function,} \tag{3.55}$$

and, in addition, that

$$\begin{aligned} &\beta > 0 \longrightarrow \log\det(I + \beta \sqrt{V}\, G \sqrt{V}) \text{ has an analytic extension} \\ &\text{to } \{z \in \mathbb{C};\ Re\, z > 0\}. \end{aligned} \tag{3.56}$$

By the first step, this function agrees with the function in (3.54) for small β in $(0, \infty)$, and hence for $\beta \in (0, \infty)$ by analyticity.

We have thus shown that

$$\int \big(1 - e^{-\beta \sum\limits_{x\in E} V(x) L_x}\big)\, d\mu_r = \log\det(I + \beta \sqrt{V}\, G \sqrt{V}), \text{ for all } \beta \geq 0, \tag{3.57}$$

and, in particular, for $\beta = 1$.

Since the equality of the three terms on the right-hand side of (3.45) has been shown below (3.52) (the proof is easily extended to the case of a general $V \geq 0$), we thus have completed the proof of (3.45).

• (3.46):

This is a special case of (3.45) when $V = v 1_{\{x\}}$ with $v \geq 0$, $x \in E$. By ordering E so that x is the first element of E, we can select the basis 1_{x_i}, $1 \leq i \leq E$, of the space of functions on E. The matrix representing $I + GV$ in this basis is triangular and has the coefficients $1 + g(x,x)\,v, 1, \dots, 1$ on the diagonal, so that $\log\det(I+GV) = \log(1+g(x,x)v)$, and (3.46) follows. □

We can now combine the restriction property, see Proposition 3.6, and the above proposition to find, as a direct application:

Proposition 3.9. *For $V : U \to \mathbb{R}_+$, one has:*

$$\begin{aligned}\int_{\{\gamma\subseteq U\}} \big(1 - e^{-\sum_{x\in E} V(x) L_x}\big)\, d\mu_r &= \log\det(I + G_U V)\\ &= \log\det(I + \sqrt{V}\, G_U \sqrt{V})\\ &= -\log\Big(\frac{\det G_{U,V}}{\det G_U}\Big),\end{aligned} \tag{3.58}$$

where we have set (see below (3.39) for notation),

$$G_{U,V} = (V - L_U)^{-1}, \text{ with } L_U f(x) = \sum_{y\in U} c_{x,y} f(y) - \lambda_x f(x), \text{ for } x \in U, \text{ and } f : U \to \mathbb{R},$$

and we have tacitly extended V by 0 outside U in the first and second line of (3.58), and the determinants in the last line of (3.58) are $U \times U$-determinants.

$$\int_{\{\gamma\subseteq U\}} (1 - e^{-vL_x})\, d\mu_r = \log\big(1 + v g_U(x,x)\big), \text{ for } v \geq 0, \text{ and } x \in U. \tag{3.59}$$

We will also record a variation on Proposition 3.7 in the case where we work with the measure $P_{x,y}$ in place of μ_r. This identity will be used in the next chapter when proving Symanzik's representation formula. Most of the work has actually already been done when proving (2.30).

Proposition 3.10. *For $V : E \to \mathbb{R}_+$ and $x, y \in E$, one has*

$$E_{x,y}\Big[\exp\Big\{-\sum_{z\in E} V(z)\, L_\infty^z\Big\}\Big] = g_V(x,y), \tag{3.60}$$

with $g_V(x,y) = (G_V 1_y)(x)$ and $G_V = (V - L)^{-1}$, as in (3.45).

Proof. By (2.30), we already know that when V is small so that $\|GV\|_\infty < 1$ (recall $V \geq 0$ here), we have

$$E_{x,y}\Big[\exp\Big\{-\sum_{z\in E} V(z)\, L_\infty^z\Big\}\Big] = \big((I+GV)^{-1} G\, 1_y\big)(x).$$

Then we observe that $(-L)^{-1} = G$, cf. (1.37), and hence $V - L = -L(I + GV)$, so that

$$G_V = (I + GV)^{-1} G. \tag{3.61}$$

We have thus proved (3.60) when the smallness assumption $\|GV\|_\infty < 1$ holds.

In the general case $V \geq 0$, $\beta V - L$ is a self-adjoint operator for the usual scalar product $\langle \cdot, \cdot \rangle$, which has all its eigenvalues > 0, when $\beta \geq 0$ (cf. (1.42), (1.39)). Using a similar identity to (3.61) in the neighborhood of $\beta_0 \geq 0$ arbitrary:

$$G_{\beta V} = (I + (\beta - \beta_0)\, G_{\beta_0 V} V)^{-1}\, G_{\beta_0 V}, \text{ for } |\beta - \beta_0| \text{ small enough}, \beta \geq 0,$$

one sees that

$$\beta > 0 \longrightarrow g_{\beta V}(x,y) \text{ has an analytic extension to a neighborhood of } (0,\infty). \tag{3.62}$$

On the other hand, by domination one sees that

$$\beta \in \{z \in \mathbb{C};\, Re\, z > 0\} \longrightarrow E_{x,y}\Big[\exp\Big\{-\beta \sum_{z\in E} V(z)\, L_\infty^z\Big\}\Big] \text{ is analytic,} \tag{3.63}$$

and coincides for small positive β with $g_{\beta V}(x,y)$.

Hence this equality also holds for all $\beta \geq 0$, and choosing $\beta = 1$, we obtain (3.60). □

3.5 Unrooted loops and the measure μ^* on unrooted loops

In the last section of this chapter we finally introduce the σ-finite measure μ_* governing loops. We also define unit weights, which are helpful with calculations involving μ_*.

We define an equivalence relation on the set L_r of rooted loops:

$$\gamma \sim \gamma' \text{ if and only if } \gamma = \theta_v(\gamma') \text{ for some } v \in \mathbb{R}, \tag{3.64}$$

and we denote by L^* the set of equivalence classes of rooted loops (also referred to as unrooted loops), and by π^* the canonical map

$$\gamma \in L_r \xrightarrow{\pi^*} \gamma^* = \pi^*(\gamma) \in L^*, \text{ the equivalence class of } \gamma \text{ for the relation } \sim \text{ in (3.64).} \tag{3.65}$$

We endow L^* with the σ-algebra (see below (3.3) for notation)

$$\mathcal{L}^* = \big\{B \subseteq L^*;\, (\pi^*)^{-1}(B) \text{ is a measurable subset of } L_r\big\}. \tag{3.66}$$

In other words, $\mathcal{L}^*$ is the largest σ-algebra on L^* such that the map $\pi^*\colon L_r \to L^*$ is measurable (we recall that L_r is endowed with the σ-algebra introduced below (3.3)).

As a consequence of (3.64), when $F\colon L^* \to \mathbb{R}$ is measurable, $F \circ \pi^*\colon L_r \to \mathbb{R}$ is invariant under all θ_v, $v \in \mathbb{R}$, and measurable. It now follows from (3.36) (and its straightforward extension to the case $n = 1$) that the measures μ_r and μ_p, see (3.10), (3.33), have the same image on L^*. We thus introduce the **loop measure** (or **unrooted loop measure**)

$$\mu^* = \pi^* \circ \mu_r = \pi^* \circ \mu_p, \tag{3.67}$$

which is straightforwardly seen to be a **σ-finite measure** on $(L^*, \mathcal{L}^*)$ (indeed, if $\zeta(\gamma^*) \stackrel{\text{def}}{=} \zeta(\gamma)$, for any $\gamma \in L_r$ with $\pi^*(\gamma) = \gamma^*$, is the duration of the unrooted loop γ^*, then for $a > 0$, $\mu^*(\zeta \geq a) = \mu_r(\zeta \geq a) < \infty$, see below (3.10)).

The following notion introduced by Lawler-Werner [16] is convenient to handle computations with μ^*.

Definition 3.11. ***(unit weight)***

A measurable function $T\colon L_r \to [0,\infty)$ is called unit weight when

$$\int_0^{\zeta(\gamma)} T(\theta_v\gamma)\, dv = 1, \text{ for any } \gamma \in L_r. \tag{3.68}$$

Examples:

1) $T(\gamma) = \zeta(\gamma)^{-1}$, for $\gamma \in L_r$, is a trivial example of unit weight.

2) A less trivial example is the following: pick $x \in E$, and set

$$\begin{cases} T(\gamma) = \zeta(\gamma)^{-1}, \text{ if } \gamma(t) \neq x \text{ for all } t \in \mathbb{R}, \\ T(\gamma) = \dfrac{1\{\gamma(0) = x\}}{\int_0^{\zeta(\gamma)} 1\{\gamma(s) = x\}\, ds}, \text{ if } \gamma(t) = x \text{ for some } t \in \mathbb{R}. \end{cases} \tag{3.69}$$

Indeed, in the first case (notation: "$x \notin \gamma$"), one has

$$\int_0^{\zeta(\gamma)} T(\theta_v \gamma)\, dv = \zeta(\gamma)/\zeta(\gamma) = 1,$$

whereas in the second case (notation: "$x \in \gamma$"), one has

$$\int_0^{\zeta(\gamma)} T(\theta_v \gamma)\, dv = \int_0^{\zeta(\gamma)} 1\{\gamma(v) = x\}\, dv \Big/ \int_0^{\zeta(\gamma)} 1\{\gamma(s) = x\}\, ds = 1.$$

□

The interest of the above definition comes from the following

Lemma 3.12. *If T is a unit weight, then for any non-negative measurable F on L^*, one has (see (2.21) for notation):*

$$\int_{L^*} F\, d\mu^* = \sum_{x \in E} \int_{L_r} F \circ \pi^*(\gamma)\, T(\gamma)\, dP_{x,x}(\gamma)\, \lambda_x. \tag{3.70}$$

Proof. By (2.21) we see that

$$\sum_{x \in E} \int_{L_r} F \circ \pi^*(\gamma)\, T(\gamma)\, dP_{x,x}(\gamma) \overset{\text{Fubini}}{=} \int_0^\infty \sum_{x \in E} \lambda_x\, E^t_{x,x}[F \circ \pi^* T]\, dt. \tag{3.71}$$

Moreover, by (3.24), we find that for each $t > 0$,

$$\sum_{x \in E} \lambda_x\, E^t_{x,x}[F \circ \pi^*\, T] = \frac{1}{t} \int_0^t dv \Big(\sum_{x \in E} \lambda_x\, E^t_{x,x}[(F \circ \pi^* \circ \theta_v)(T \circ \theta_v)] \Big)$$

and since $F \circ \pi^* \circ \theta_v = F \circ \pi^*$, and T is a unit weight, so that $P^t_{x,x}$-a.s., $\int_0^t T \circ \theta_v\, dv = 1$, the right-hand side of (3.71) equals

$$\int_0^\infty \sum_{x \in E} \lambda_x\, E^t_{x,x}[F \circ \pi^*]\, \frac{dt}{t} \overset{(3.10)}{=} \int_{L_r} F \circ \pi^*\, d\mu_r \overset{(3.67)}{=} \int_{L^*} F\, d\mu^*,$$

and the claim (3.70) follows. □

Example:

We consider T as in (3.69) and $F \geq 0$ measurable on L^*. We write using similar notation as below (3.69):

$$\begin{aligned}
\int_{\{\gamma^* \ni x\}} F\, d\mu^* &\overset{(3.70)}{=} \sum_{y\in E} \int_{\{\gamma \ni x\}} F \circ \pi^*(\gamma)\, T(\gamma)\, dP_{y,y}(\gamma)\, \lambda_y \\
&\overset{(3.69)}{=} \sum_{y\in E} \int_{\{\gamma \ni x\}} F \circ \pi^*(\gamma)\; \frac{1\{\gamma(0)=x\}}{\int_0^{\zeta(\gamma)} 1\{\gamma(s)=x\}\, ds}\; dP_{y,y}(\gamma)\, \lambda_y \\
&= \int_{\{\gamma \ni x\}} F \circ \pi^*(\gamma) \Big(\int_0^{\zeta(\gamma)} 1\{\gamma(s)=x\}\, \frac{ds}{\lambda_x}\Big)^{-1} dP_{x,x}(\gamma) \\
&\overset{(3.41)}{=} \int_{\{\gamma \ni x\}} F \circ \pi^*(\gamma)\; L_x(\gamma)^{-1}\, dP_{x,x}(\gamma).
\end{aligned}$$

In other words, we have proved that

$$1\{\gamma^* \ni x\}\, d\mu^* = \pi^* \circ \Big(\frac{1}{L_x}\, dP_{x,x}\Big), \ \text{ for } x \in E, \tag{3.72}$$

or, alternatively, since L_x is invariant under θ_v, cf. (3.42),

$$1\{\gamma^* \ni x\}\, L_x\, \mu^* = \pi^* \circ P_{x,x}, \ \text{ for } x \in E, \tag{3.72'}$$

where we have set $L_x(\gamma^*) = L_x(\gamma)$, for any γ with $\pi^*(\gamma) = \gamma^*$.

Note that by (2.34) the total mass of $P_{x,x}$ equals $g(x,x)$, so that

$$\int_{\{\gamma^* \ni x\}} L_x\, d\mu^* = g(x,x), \ \text{ for } x \in E. \tag{3.73}$$

□

4 Poisson gas of Markovian loops

In this chapter we study the Poisson point process on the space L^* of unrooted loops with intensity measure $\alpha\mu^*$, with α a positive number. In particular, we relate the occupation field of this gas of loops to the Gaussian free field, and prove Symanzik's representation formula. At the end of the chapter we explore several precise meanings for the notion of "loops going through infinity" and relate them to random interlacements.

4.1 Poisson point measures on unrooted loops

In this section we briefly introduce the set-up for Poisson point measures on the set of loops, and recall some basic identities for the Laplace transforms of these Poisson point measures.

We consider pure point measures ω on $(L^*, \mathcal{L}^*)$, i.e. σ-finite measures of the form $\omega = \sum_{i \in I} \delta_{\gamma_i^*}$, where γ_i^*, $i \in I$, is an at most countable collection of unrooted loops, such that $\omega(A) < \infty$, for all $A = \{\gamma^* \in L^*; \, a \le \zeta(\gamma^*) \le b\}$, with $0 < a < b < \infty$, where

$$\omega(A) = \#\{i \in I; \, \gamma_i^* \in A\}, \;\; \text{for } A \in \mathcal{L}^*. \tag{4.1}$$

We introduce

$$\Omega = \text{the set of pure point measures on } (L^*, \mathcal{L}^*), \tag{4.2}$$

and endow Ω with the σ-algebra

$$\begin{array}{l} \mathcal{A} = \text{the } \sigma\text{-algebra generated by the evaluation maps} \\ \qquad \omega \in \Omega \longrightarrow \omega(A) \in \mathbb{N} \cup \{\infty\}, \;\; \text{for } A \in \mathcal{L}^*. \end{array} \tag{4.3}$$

We recall that a random variable X on some probability space is said to have Poisson distribution with parameter $\rho \in [0, \infty]$, when

$$\begin{aligned} &P[X = k] = e^{-\rho} \, \frac{\rho^k}{k!}, \;\; \text{for } k \ge 0, \text{ if } \rho < \infty, \text{ and} \\ &P[X = \infty] = 1, \;\; \text{if } \rho = \infty. \end{aligned}$$

Definition 4.1. *Given a σ-finite measure ν on $(L^*, \mathcal{L}^*)$, such that $\nu(A) < \infty$, for A as above (4.1), a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ is the* **Poisson point measure** *with intensity ν if*

(4.4) *for $A \in \mathcal{L}^*$, $\omega(A)$ is Poisson distributed with parameter $\nu(A)$,*

(4.5) *for pairwise disjoint $A_1, \dots, A_n \in \mathcal{L}^*$, $\omega(A_1), \dots, \omega(A_n)$ are independent under $\mathbb{P}$.*

We refer to [20], [22] for a more detailed discussion of Poisson point measures.

Poisson gas of Markovian loops:

Given $\alpha > 0$, we consider

$$\mathbb{P}_\alpha\text{: the Poisson point measure on } (L^*, \mathcal{L}^*) \text{ with intensity } \alpha\mu^* \tag{4.6}$$

(see (3.67) for the definition of μ^*). We will call probability measure $\mathbb{P}_\alpha$ on $(\Omega, \mathcal{A})$, the **Poisson gas of Markovian loops at level $\boldsymbol{\alpha > 0}$**. Note that under $\mathbb{P}_\alpha$ the point measure ω is a.s. infinite, but by (3.19) its restriction to $\{N > 1\} \subseteq L^*$, is a.s. a finite point measure (we use the notation $N(\gamma^*) = N(\gamma)$ for any γ with $\pi^*(\gamma) = \gamma^*$).

Lemma 4.2. *Consider a measurable function Φ: $L^* \to \mathbb{R}_+$, then*

$$\mathbb{E}_\alpha[e^{-\langle \omega, \Phi\rangle}] = \exp\Big\{ -\alpha \int_{L^*} (1-e^{-\Phi})\, d\mu^*\Big\} \tag{4.7}$$

(where $\langle \omega, \Phi\rangle = \sum_{i\in I} \phi(\gamma_i^*)$, for $\omega = \sum_{i \in I} \delta_{\gamma_i^*} \in \Omega$).

If Φ vanishes on $\{\gamma^*\colon \zeta(\gamma^*) < a\}$ for some $a > 0$, then

$$\mathbb{E}_\alpha[e^{i\langle \omega, \Phi\rangle}] = \exp\Big\{ \alpha \int_{L^*} (e^{i\Phi}-1)\, d\mu^*\Big\}. \tag{4.8}$$

Proof.

• (4.7):

When $\Phi = \sum_{i=1}^n a_i\, 1_{A_i}$, with A_i, $1 \le i \le n$, pairwise disjoint, measurable and $\mu^*(A_i) < \infty$, one has

$$\begin{aligned}
\mathbb{E}_\alpha[e^{-\langle \omega, \Phi\rangle}] &= \mathbb{E}_\alpha\Big[\prod_{i=1}^n e^{-a_i \omega(A_i)}\Big] \underset{(4.6)}{\overset{(4.4),(4.5)}{=}} \prod_{i=1}^n \Big(\sum_{k\ge 0} e^{-a_i k}\, e^{-\alpha\mu^*(A_i)}\, \frac{(\alpha\mu^*(A_i))^k}{k!}\Big) \\
&= \exp\Big\{ -\sum_{i=1}^n \alpha \mu^*(A_i)(1-e^{-a_i})\Big\} \\
&= \exp\Big\{ -\alpha \int_{L^*} (1-e^{-\Phi})\, d\mu^*\Big\},
\end{aligned} \tag{4.9}$$

so (4.7) holds.

For a general Φ: $L^* \to \mathbb{R}_+$, we can construct $\Phi_\ell \uparrow \Phi$ as $\ell \to \infty$, with each Φ_ℓ of the above type (by the usual measure-theoretic induction construction and the σ-finiteness of μ^*), and find that

$$\begin{aligned}
\mathbb{E}_\alpha[e^{-\langle \omega, \Phi\rangle}] &\underset{\text{convergence}}{\overset{\text{monotone}}{=}} \lim_{\ell\to\infty} \downarrow \mathbb{E}_\alpha[e^{-\langle \omega, \Phi_\ell\rangle}] \overset{(4.9)}{=} \lim_{\ell\to\infty} \downarrow \exp\Big\{ -\alpha \int_{L^*} (1-e^{-\Phi_\ell})\, d\mu^*\Big\} \\
&\underset{\text{convergence}}{\overset{\text{monotone}}{=}} \exp\Big\{ -\alpha \int_{L^*} (1-e^{-\Phi})\, d\mu^*\Big\}, \quad \text{whence (4.7).}
\end{aligned}$$

• (4.8):

The claim follows by similar measure theoretic induction; note that, as remarked below (3.10), $\mu^*(\zeta \ge a) < \infty$, so that $\langle \omega, \{\zeta \ge a\}\rangle < \infty$, $\mathbb{P}_\alpha$-a.s., and $e^{i\Phi} - 1$ is both bounded and μ^*-integrable. □

4.2 Occupation field

We introduce the occupation field of the Poisson gas of Markovian loops in this section, and calculate its Laplace transform. Later on we relate the law of the occupation field of the Poisson gas at level $\alpha = \frac{1}{2}$ to the free field.

For $\omega \in \Omega$, $x \in E$, we define the **occupation field** (also called **field of occupation times**) of ω at x via:

$$\begin{aligned}
\mathcal{L}_x(\omega) &= \langle \omega, L_x\rangle \in [0,\infty], \quad \text{(see (3.41), (3.42) for notation),} \\
&= \sum_{i\in I} L_x(\gamma_i^*), \ \text{if } \omega = \sum_{i\in I} \delta_{\gamma_i^*} \in \Omega\,,
\end{aligned} \tag{4.10}$$

where $L_x(\gamma^*)$ for $\gamma^* \in L^*$ is defined below (3.72').

As a consequence of Proposition 3.7 and Lemma 4.2, we obtain the following important theorem describing the Laplace transform of the field of occupation times of a Poisson gas of Markovian loops:

Theorem 4.3. $(\alpha > 0)$

For $V\colon E \to \mathbb{R}_+$, one has in the notation of (3.45):

$$\begin{aligned} \mathbb{E}_\alpha\big[e^{-\sum\limits_{x\in E} V(x)\mathcal{L}_x}\big] &= \det(I + GV)^{-\alpha} \\ &= \det(I + \sqrt{V}\, G\sqrt{V})^{-\alpha} \\ &= \Big(\frac{\det G_V}{\det G}\Big)^{\alpha}. \end{aligned} \tag{4.11}$$

For $x \in E$ and $v \geq 0$, one has

$$\mathbb{E}_\alpha[e^{-v\mathcal{L}_x}] = \big(1 + v\, g(x,x)\big)^{-\alpha}. \tag{4.12}$$

In particular, one finds that:

$$\begin{aligned} &\mathcal{L}_x \textit{ is } \Gamma\big(\alpha, g(x,x)\big)\textit{-distributed} \\ &\textit{(i.e. has density } \frac{1}{\Gamma(\alpha)}\, \frac{s^{\alpha-1}}{g(x,x)^\alpha}\; e^{-\frac{s}{g(x,x)}}\, 1\{s > 0\}\textit{)}, \end{aligned} \tag{4.13}$$

and that

$$\mathbb{P}_\alpha\textit{-a.s., } \mathcal{L}_x < \infty, \textit{ for every } x \in E. \tag{4.14}$$

Proof.

• (4.11):

We first note that by (4.10), for $\omega \in \Omega$, $V\colon E \to \mathbb{R}_+$, one has

$$\sum_{x\in E} V(x)\,\mathcal{L}_x(\omega) = \langle \omega, \Phi\rangle, \text{ where } \Phi(\gamma^*) = \sum_{x\in E} V(x)\, L_x(\gamma^*), \text{ for } \gamma^* \in L^*.$$

We will use (4.7) together with (3.45). Specifically, we have

$$\mathbb{E}_\alpha\big[e^{-\sum\limits_{x\in E} V(x)\mathcal{L}_x}\big] = \mathbb{E}_\alpha\big[e^{-\langle\omega,\Phi\rangle}\big] \overset{(4.7)}{=} \exp\Big\{-\alpha \int_{L^*} (1 - e^{-\Phi})\, d\mu^*\Big\}.$$

Now, by (3.67), we obtain the identity:

$$\int_{L^*} (1 - e^{-\Phi})\, d\mu^* \overset{(3.67)}{=} \int_{L_r} \big(1 - e^{-\Phi\circ\pi^*}\big)\, d\mu_r = \int_{L_r} \big(1 - e^{-\sum\limits_{x\in E} V(x) L_x(\gamma)}\big)\, d\mu_r(\gamma).$$

As a result, we find that

$$\begin{aligned} \mathbb{E}_\alpha\big[e^{-\sum\limits_{x\in E} V(x)\mathcal{L}_x}\big] &= \exp\Big\{-\alpha \int_{L_r} \big(1 - e^{-\sum\limits_{x\in E} V(x) L_x}\big)\, d\mu_r\Big\} \\ &\overset{(3.45)}{=} \{\det(I + GV)\}^{-\alpha} \\ &= \{\det(I + \sqrt{V}\, G\sqrt{V})\}^{-\alpha} \\ &= \Big(\frac{\det G_V}{\det G}\Big)^{\alpha}, \quad \text{and (4.11) follows.} \end{aligned}$$

• (4.12):

In the special case $V = v 1_x$, with $v \geq 0$, $x \in E$, it follows by (3.46) that

$$\mathbb{E}_\alpha[e^{-v\mathcal{L}_x}] = (1 + vg(x,x))^{-\alpha},$$

and (4.12) follows.

• (4.14):

$$\mathbb{P}_\alpha[\mathcal{L}_x < \infty] = \lim_{v \to 0} \mathbb{E}_\alpha[e^{-v\mathcal{L}_x}] = \lim_{v \to 0} \big(1 + vg(x,x)\big)^{-\alpha} = 1,$$

and (4.14) follows.

• (4.13):

The Laplace transform of the $\Gamma\big(\alpha, g(x,x)\big)$-distribution is $\big(1+vg(x,x)\big)^{-\alpha}$, see [9], p. 430, and (4.13) follows. □

Remark 4.4. One can also introduce the **occupation field of non-trivial loops**:

$$\text{(4.15)} \quad \begin{aligned} \widehat{\mathcal{L}}_x(\omega) &\overset{\text{def}}{=} \langle 1\{N > 1\}\omega, L_x\rangle \\ &= \sum_{i \in I} 1\{N(\gamma_i^*) > 1\}\, L_x(\gamma_i^*), \ \ \text{if } \omega = \sum_{i \in I} \delta_{\gamma_i^*} \in \Omega, \ \ \text{and } x \in E. \end{aligned}$$

By (3.44), (3.46) we know, for instance, that for $v \geq 0$ and $x \in E$,

$$\text{(4.16)} \quad \int_{\{N>1\}} (1 - e^{-vL_x})\, d\mu_r = \log\Big(\frac{1 + vg(x,x)}{1 + \frac{v}{\lambda_x}}\Big).$$

The same proof used for (4.12) now yields that for $v \geq 0$, and $x \in E$,

$$\text{(4.17)} \quad \mathbb{E}_\alpha[e^{-v\widehat{\mathcal{L}}_x}] = \Big(\frac{1 + vg(x,x)}{1 + \frac{v}{\lambda_x}}\Big)^{-\alpha}.$$

In particular, letting $v \to \infty$ in (4.17) we find that

$$\text{(4.18)} \quad \mathbb{P}_\alpha[\widehat{\mathcal{L}}_x = 0] = \big(\lambda_x\, g(x,x)\big)^{-\alpha}, \ \text{for any } x \in E.$$

In the same vein, combining (3.44) and (3.45) yields that

$$\text{(4.19)} \quad \begin{aligned} \int_{\{N>1\}} \big(1 - e^{-\sum_{x \in E} V(x) L_x}\big)\, d\mu_r &= -\log\Big(\frac{\det G_V}{\det G}\Big) + \log\det\lambda - \log\det(\lambda + V) \\ &= -\log\frac{\det(I - P^V)^{-1}}{\det(I-P)^{-1}} = \log\frac{\det(I - P^V)}{\det(I - P)}, \end{aligned}$$

where $P^V f(x) \overset{\text{def}}{=} \sum_{y \in E} \frac{c_{x,y}}{\lambda_x + V(x)}\ f(y)$, for $x \in E$, and $f\colon\ E \to \mathbb{R}$, and we used the equalities

$$\det G = \det(I - P)^{-1} \det \lambda^{-1} \ \text{ and } \ \det G_V = \det(I - P^V)^{-1} \det(\lambda + V)^{-1}$$

(recall $G_V = (V-L)^{-1}$ and note that $V - L = (\lambda + V)(I - P^V)$), as well as the identity

$$\int_{\{N=1\}} \big(1 - e^{-\sum_{x\in E} V(x)L_x}\big)\, d\mu_r = \sum_{x\in E} \int_{\{N=1\}} \big(1 - e^{-V(x)L_x}\big)\, d\mu_r,$$

since $L_x(\gamma) = 0$, for all $x \neq \gamma(0)$, when $N(\gamma) = 1$. We refer to Remark 3.8 for the probabilistic interpretation of G_V, a similar interpretation holds for P^V as well. By the same proof as in (4.11), we thus see that

$$\mathbb{E}_\alpha\big[e^{-\sum_{x\in E} V(x)\widehat{\mathcal{L}}_x}\big] = \Big(\frac{\det(I-P^V)}{\det(I-P)}\Big)^{-\alpha}, \quad \text{for } V\colon\, E \to \mathbb{R}. \tag{4.20}$$

Letting $V(x) \uparrow \infty$ for every $x \in E$, we find that

$$\mathbb{P}_\alpha[\widehat{\mathcal{L}}_x = 0, \ \text{for all } x] = \big(\det(I-P)\big)^\alpha, \tag{4.21}$$

a formula which also follows from the identity

$$\mathbb{P}_\alpha[\omega(N>1) = 0] \underset{(4.4)}{\overset{(3.67)}{=}} \exp\{-\alpha\, \mu_r(N>1)\} \overset{(3.19)}{=} \big(\det(I-P)\big)^\alpha. \tag{4.22}$$

□

Occupation time of the Poisson gas of Markovian loops and the free field:

We now want to relate the field of occupation times $\mathcal{L}_x$, $x \in E$, to the Gaussian free field, i.e. cf. (2.2), the unique probability P^G on $\mathbb{R}^E$, under which

(4.23) the canonical coordinates φ_x, $x \in E$, are a centered Gaussian field with covariance $E^G[\varphi_x\, \varphi_y] = g(x,y)$, for $x, y \in E$.

Here is the crucial link, due to Le Jan [17], between the occupation times of the Poisson gas of Markovian loops with the choice $\alpha = \frac{1}{2}$ and the free field (see also (0.11)):

Theorem 4.5.

$$(\mathcal{L}_x)_{x\in E} \ \textit{under } \mathbb{P}_{\alpha=\frac{1}{2}}, \textit{ has same law as } \Big(\frac{1}{2}\,\varphi_x^2\Big)_{x\in E} \ \textit{under } P^G. \tag{4.24}$$

Proof. We consider $V\colon\, E \to \mathbb{R}_+$. On the one hand, we know by (4.11) that

$$\mathbb{E}_{\frac{1}{2}}\Big[\exp\Big\{-\sum_{x\in E} V(x)\mathcal{L}_x\Big\}\Big] = \det(I+GV)^{-\frac{1}{2}}. \tag{4.25}$$

On the other hand, by (2.59) and the fact that $V-L$ is positive definite, see (1.39), (1.44), we find that

$$E^G\Big[\exp\Big\{-\sum_{x\in E} \frac{V(x)}{2}\,\varphi_x^2\Big\}\Big] = \det\,(I+GV)^{-\frac{1}{2}}. \tag{4.26}$$

As a result of (4.25) and (4.26), the Laplace transforms of $\mathcal{L}_x$, $x \in E$, under $\mathbb{P}_{\frac{1}{2}}$, and of $\frac{1}{2}\,\varphi_x^2$, $x \in E$, under P^G coincide and this yields (4.24). □

The above result somehow complements the picture stemming from the isomorphism theorems of Dynkin and Eisenbaum, see (2.33) and (2.43). In particular, for any $x, y \in E$,

$$(4.27)\qquad \begin{array}{l} (L^z_\infty + \mathcal{L}_z)_{z\in E} \ \text{ under } P_{x,y} \otimes \mathbb{P}_{\frac{1}{2}}, \text{ has the same "law" as} \\ \Big(\frac{1}{2}\,\varphi_z^2\Big)_{z\in E} \ \text{ under } \varphi_x\,\varphi_y\,P^G \ (\leftarrow \text{signed measure when } x \neq y!). \end{array}$$

As we will now see (4.27) and (4.24) are very close to (0.10), i.e. the representation formula of Symanzik for the moments of random fields of the type (0.9), which we had mentioned in the Introduction.

4.3 Symanzik's representation formula

In this section we state and prove Symanzik's representation formula. It expresses the moments of certain interacting fields in terms of a Poisson gas of loops and a collection of paths interacting with a random potential. In a way this formula embodies one of the starting points for the various developments covered in these notes.

We begin with a lemma concerning even moments of centered Gaussian vectors, which will be used in the derivation of Symanzik's formula.

Lemma 4.6. *Let $k \geq 1$, and $(\psi_1, \dots, \psi_{2k})$ be a centered Gaussian vector with covariance matrix $A_{i,j}$, $1 \leq i, j \leq 2k$, then*

$$(4.28)\qquad E\Big[\prod_{i=1}^{2k} \psi_i\Big] = \sum_{D_1\cup\dots\cup D_k = \{1,\dots,2k\}} \ \prod_{i=1}^{k} \operatorname{cov}(D_i),$$

where the sum is over all pairings $D_1, \dots, D_k$ of $\{1, \dots, 2k\}$, i.e. over all unordered partitions of $\{1, \dots, 2k\}$ into k disjoint sets each containing two elements, and where $\operatorname{cov}(D) \stackrel{\text{def}}{=} A_{s,t}$, *for $D = \{s,t\}$. (Note that some ψ_i, $1 \leq i \leq 2k$, may coincide.)*

Proof. For $\lambda_1, \dots, \lambda_{2k} \in \mathbb{R}$,

$$E\Big[\exp\Big\{\sum_{i=1}^{2k} \lambda_i\,\psi_i\Big\}\Big] = \exp\Big\{\frac{1}{2}\,E\Big[\Big(\sum_{i=1}^{2k} \lambda_i\,\psi_i\Big)^2\Big]\Big\},$$

so that by taking $2k$ partial derivatives,

$$\begin{aligned} E\Big[\prod_{i=1}^{2k} \psi_i\Big] &= \frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_{2k}} \ \exp\Big\{\frac{1}{2}\,E\Big[\Big(\sum_{i=1}^{2k} \lambda_i\,\psi_i\Big)^2\Big]\Big\}\Big|_{\lambda_1,\dots,\lambda_{2k}=0} \\ &= \sum_{n\geq 0} \frac{1}{2^n\,n!}\ \frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_{2k}} \Big(\sum_{1\leq i,j\leq 2k} \lambda_i\,\lambda_j\,A_{i,j}\Big)^n\Big|_{\lambda_1,\dots,\lambda_{2k}=0}. \end{aligned}$$

In the above series, only the term $n = k$ survives and when looking at

$$\frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_{2k}} \Big(\sum_{1\leq i,j\leq 2k} \lambda_i\,\lambda_j\,A_{i,j}\Big)^k\Big|_{\lambda_1,\dots,\lambda_{2k}=0},$$

the only terms to survive are such that each of the k factors gets hit by two (distinct) derivatives, which keep alive exactly two terms (corresponding to the choice of order between i and j) in each factor. Keeping in mind that pairing corresponds to unordered partitions we obtain

$$\frac{1}{2^k}\ \frac{1}{k!}\ \frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_{2k}}\Big(\sum_{1\le i,j\le 2k}\lambda_i\,\lambda_j\,A_{i,j}\Big)^k\Big|_{\lambda_1=\dots=\lambda_{2k}=0} = \text{ the right-hand side of (4.28).}$$

Our claim (4.28) follows. □

Remark 4.7. The Feynman graphs (or diagrams), see [11], p. 146, provide a graphical representation of the "pairings" in the above formula (4.28). One attaches a half-edge to each of $2k$ distinct vertices and one chooses a match for each vertex so as to obtain one such pairing corresponding to a graph on the $2k$ vertices, where each vertex belongs to exactly one edge.

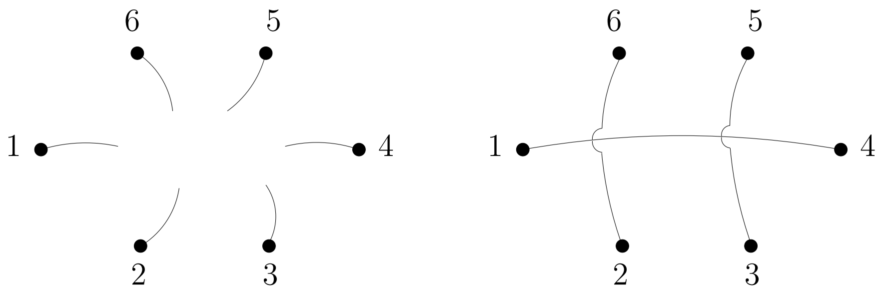


Fig. 4.1 When $2k = 6$, an example of a Feynman diagram on the right-hand side where the half-edges are paired together. It illustrates an unordered pairing corresponding to $D_1 = \{2,6\}$, $D_2 = \{1,4\}$, $D_3 = \{3,5\}$, see also Fig. 4.2.

□

We now consider a probability ν on $\mathbb{R}_+$, and define as below (0.9)

(4.29) $$h(u) = \int_0^\infty e^{-vu}\,d\nu(v), \text{ for } u \ge 0,$$

the Laplace transform of ν.

We are interested in the probability measure on $\mathbb{R}^E$

(4.30) $$\overline{P}^{G,h} = \frac{1}{Z_h}\ \exp\Big\{-\frac{1}{2}\,\mathcal{E}(\varphi,\varphi)\Big\}\prod_{x\in E} h\Big(\frac{\varphi_x^2}{2}\Big)\,d\varphi,$$

where the normalizing constant Z_h is given by the formula:

(4.31) $$Z_h = \int_{\mathbb{R}^E}\exp\Big\{-\frac{1}{2}\,\mathcal{E}(\varphi,\varphi)\Big\}\prod_{x\in E} h\Big(\frac{\varphi_x^2}{2}\Big)\,d\varphi\ \ (\text{with } d\varphi = \prod_{x\in E} d\varphi_x).$$

Similar measures arise in mathematical physics, in the context of Euclidean quantum field theory, see [11]. In this context a natural choice for h would be $h(u) = e^{-\lambda u^2+\sigma u}$, with $\lambda > 0$ and $\sigma \in \mathbb{R}$, see Chapter 17 of [11]. The assumption (4.29) however rules out such a choice since it implies that $\log h$ is convex (by Hölder's inequality). Nevertheless, it simplifies the presentation made below.

We write $\langle \text{—} \rangle_h$ for the expectation relative to $\overline{P}^{G,h}$, i.e.

$$\langle F \rangle_h = \frac{\int_{\mathbb{R}^E} F(\varphi)\, e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in E} h(\frac{\varphi_x^2}{2})\, d\varphi}{\int_{\mathbb{R}^E} e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in E} h(\frac{\varphi_x^2}{2})\, d\varphi} \tag{4.32}$$

when, for instance, F: $\mathbb{R}^E \to \mathbb{R}$ is a bounded measurable function. Note that when $\nu = \delta_0$, then $h = 1_E$, and one recovers the free field:

$$\overline{P}^{G,h=1_E} = P^G, \text{ in the notation of (2.2).} \tag{4.33}$$

Symanzik's formula will provide a representation of the moments of the random field governed by $\overline{P}^{G,h}$ in terms of the occupation field of a Poisson gas of loops and of the local times of walks interacting via random potentials. A variant of the formula is, for instance, used in Section 3 of [2], to obtain bounds on critical temperatures.

As a last ingredient, we introduce an auxiliary probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$ endowed with a collection $V(x, \widetilde{\omega})$, $x \in E$, of non-negative random variables (the random potentials) such that

(4.34) under $\widetilde{\mathbb{P}}$, the variables $V(x,\widetilde{\omega})$, $x \in E$, are (non-negative) i.i.d. ν-distributed.

Let us point out that the probability $\mathbb{Q}$ in (0.10) coincides with $\widetilde{\mathbb{P}} \otimes \mathbb{P}_{\alpha=\frac{1}{2}}$. We are now ready to state and prove Symanzik's formula.

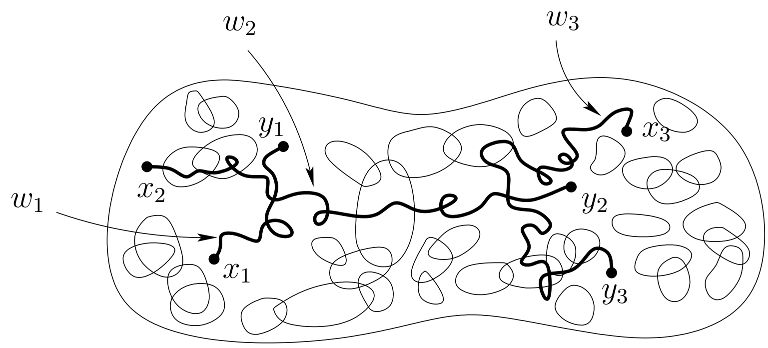


Fig. 4.2: The paths w_1, w_2, w_3 in E, and the gas of loops interact through the random potentials ($k = 3$, and the $z_1, \dots, z_6$ are distinct).

Theorem 4.8. *(Symanzik's representation formula)*

For any $k \ge 1$, $z_1, \dots, z_{2k} \in E$, one has

$$\langle \varphi_{z_1} \dots \varphi_{z_{2k}} \rangle_h = \sum_{\substack{\text{pairings of} \\ \{1,\dots,2k\}}} \frac{E_{x_1,y_1} \otimes \dots \otimes E_{x_k,y_k} \otimes \widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}} \big[e^{-\sum_{x\in E} V(x,\widetilde{\omega})(\mathcal{L}_x(\omega) + L^x_\infty(w_1) + \dots + L^x_\infty(w_k))} \big]}{\widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}} \big[e^{-\sum_{x\in E} V(x,\widetilde{\omega})\mathcal{L}_x(\omega)} \big]}, \tag{4.35}$$

where $\{x_i, y_i\} = \{z_\ell;\, \ell \in D_i\}$, $1 \le i \le k$, and $D_1, \dots, D_k$ stands for the (unordered) pairing of $\{1, \dots, 2k\}$, and w_i denotes the integration variable under P_{x_i,y_i}.

Proof. By (3.60), we know that for $\widetilde{\omega} \in \widetilde{\Omega}$, $x, y \in E$,

$$E_{x,y}\big[e^{-\sum_{z\in E} V(z,\widetilde{\omega}) L^z_\infty}\big] = g_{V(\cdot,\widetilde{\omega})}(x,y), \tag{4.36}$$

which is a symmetric function of x, y, so that the expression under the sum in the right-hand side of (4.35) only depends on the pairing and is therefore well-defined.

Let F be a bounded measurable function on $\mathbb{R}^E$. Denote by $(F)_h$ the numerator of (4.32) so that $\langle F\rangle_h = \frac{(F)_h}{(1)_h}$. We have the identity:

$$\begin{aligned}
(F)_h &\overset{(4.34)}{=} \widetilde{\mathbb{E}}\Big[\int_{\mathbb{R}^E} F(\varphi)\, e^{-\frac{1}{2}[\mathcal{E}(\varphi,\varphi)+\sum_{x\in E} V(x,\widetilde{\omega})\varphi_x^2]}\, d\varphi\Big] \\
&= \widetilde{\mathbb{E}} \otimes E^{G,V(\cdot,\widetilde{\omega})}\big[F(\varphi)(2\pi)^{\frac{|E|}{2}}(\det G_{V(\cdot,\widetilde{\omega})})^{\frac{1}{2}}\big] \\
&\overset{(4.11)}{=} \widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}} \otimes E^{G,V(\cdot,\widetilde{\omega})}\big[F(\varphi)\, e^{-\sum_{x\in E} V(x,\widetilde{\omega})\mathcal{L}_x(\omega)}\big](2\pi)^{\frac{|E|}{2}}\,(\det G)^{\frac{1}{2}},
\end{aligned} \tag{4.37}$$

where the second line is a consequence of (2.5), (2.36), and $(V(\cdot,\widetilde{\omega})-L)^{-1} \overset{\text{below (3.45)}}{=} G_{V(\cdot,\widetilde{\omega})}$. As a result of (4.28), we thus find that

$$\begin{aligned}
&(\varphi_{z_1}\dots\varphi_{z_{2k}})_h = \\
&\sum_{\substack{\text{pairings of}\\ \{1,\dots,2k\}}} \widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}}\big[G_{V(\cdot,\widetilde{\omega})}(x_1,y_1)\dots G_{V(\cdot,\widetilde{\omega})}(x_k,y_k)\, e^{-\sum_{x\in E} V(x,\widetilde{\omega})\mathcal{L}_x(\omega)}\big]\times \\
&(2\pi)^{\frac{|E|}{2}}(\det G)^{\frac{1}{2}} \overset{(4.36)}{=} \\
&\sum_{\substack{\text{pairings of}\\ \{1,\dots,2k\}}} E_{x_1,y_1} \otimes \dots \otimes E_{x_k,y_k} \otimes \widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}}\big[e^{-\sum_{x\in E} V(x,\widetilde{\omega})(\mathcal{L}_x(\omega)+L^x_\infty(w_1)+\dots+L^x_\infty(w_k))}\big]\times \\
&(2\pi)^{\frac{|E|}{2}}(\det G)^{\frac{1}{2}}.
\end{aligned} \tag{4.38}$$

In the same way, we find by (4.37) that

$$(1)_h = \widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}}\big[e^{-\sum_{x\in E} V(x,\widetilde{\omega})\mathcal{L}_x(\omega)}\big](2\pi)^{\frac{|E|}{2}}(\det \mathrm{G})^{\frac{1}{2}}. \tag{4.39}$$

Taking the ratio of (4.38) and (4.39) precisely yields (4.35). □

Remark 4.9. When $k = 1$, Symanzik's representation formula (4.35) becomes

$$\langle\varphi_x\,\varphi_y\rangle_h = \frac{E_{x,y} \otimes \widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}}\big[e^{-\sum_{z\in E} V(z,\widetilde{\omega})(\mathcal{L}_z(\omega)+L^z_\infty(w))}\big]}{\widetilde{\mathbb{E}} \otimes \mathbb{E}_{\frac{1}{2}}\big[e^{-\sum_{z\in E} V(z,\widetilde{\omega})\mathcal{L}_z(\omega)}\big]}.$$

We explain below another way to obtain this identity. Recall that (4.27) combines Dynkin's isomorhism theorem and the identity in law of $(\mathcal{L}_x)_{x\in E}$ under $\mathbb{P}_{\frac{1}{2}}$, with $(\frac{1}{2}\,\varphi_x^2)_{x\in E}$ under P^G, stated in Theorem 4.5.

By (4.27) the numerator equals $\widetilde{\mathbb{E}} \otimes E^G[\varphi_x\,\varphi_y\, e^{-\frac{1}{2}\sum_{z\in E} V(z,\widetilde{\omega})\varphi_z^2}]$, whereas by (4.24) the denominator equals $\widetilde{\mathbb{E}} \otimes E^G[e^{-\frac{1}{2}\sum_{z\in E} V(z,\widetilde{\omega})\varphi_z^2}]$. Keeping in mind (4.29) and (4.34), one easily recovers the left-hand side of the above quality. □

4.4 Some identities

In this section we discuss some further formulas concerning the Poisson gas of Markovian loops. In particular given two disjoint subsets of E, we derive a formula for the probability that no loop of the Poisson gas visits both subsets. In the next section, as an application, we will link the so-called random interlacements with various notions of "loops going through infinity" for the Poisson cloud of Markovian loops.

Given $U \subseteq E$, we can consider the field of occupation times of loops contained in U:

$$\begin{aligned}\mathcal{L}_x^U(\omega) &= \langle 1\{\gamma^* \subseteq U\}\,\omega,\, L_x\rangle \\ &= \sum_{i\in I} 1\{\gamma_i^* \subseteq U\}\, L_x(\gamma_i^*),\ \text{if}\ \omega = \sum_{i\in I}\delta_{\gamma_i^*} \in \Omega,\ x\in E,\end{aligned} \tag{4.40}$$

and we used the slightly informal notation $1\{\gamma_i^* \subseteq U\}$ in place of $1\{\gamma_i \in L_{r,U}\}$, where $\gamma_i \in L_r$ is any rooted loop such that $\pi^*(\gamma_i) = \gamma_i^*$, and $L_{r,U}$ has been defined in (3.37).

Proposition 4.10. ($\alpha > 0$)

Given $K \subseteq E$, $U = E\backslash K$, and $V\colon E \to \mathbb{R}_+$,

$$\begin{aligned}\mathbb{E}_\alpha\big[e^{-\sum_{x\in E} V(x)(\mathcal{L}_x - \mathcal{L}_x^U)}\big] &= \Big(\frac{\det G_V}{\det G}\ \frac{\det G_U}{\det G_{U,V}}\Big)^\alpha \\ &= \Big(\frac{\det_{K\times K} G_V}{\det_{K\times K} G}\Big)^\alpha\end{aligned} \tag{4.41}$$

(where $\det_{K\times K} A \overset{\text{def}}{=} \det(A_{|K\times K})$, for A an $E\times E$-matrix, and we write $\det G_U$, resp. $\det G_{U,V}$, in place of $\det_{U\times U} G_U$, resp. $\det_{U\times U} G_{U,V}$, with the notation from below (3.58)).

If $K_1 \cap K_2 = \phi$, and $U_i = E\backslash K_i$, for $i = 1, 2$, then

$$\begin{aligned}\mathbb{P}_\alpha[\textit{no loop intersects both } K_1 \textit{ and } K_2] &= \Big(\frac{\det G}{\det G_{U_1}}\ \frac{\det G_{U_1\cap U_2}}{\det G_{U_2}}\Big)^{-\alpha}, \\ &= \Big(\frac{\det_{K_1\times K_1} G_{U_2}}{\det_{K_1\times K_1} G}\Big)^\alpha,\end{aligned} \tag{4.42}$$

(and we used a similar convention as above for $\det G_U$).

Proof.

- (4.41):

Observe that $\mathcal{L}_x - \mathcal{L}_x^U$, $x \in E$, is the field of occupation times of loops which are not contained in U, whereas $\mathcal{L}_x^U$, $x \in U$, is the field of occupation times of loops which are contained in U:

$$\mathcal{L}_x - \mathcal{L}_x^U = \langle 1_{\{\gamma^*\subseteq U\}^c}\,\omega, L_x\rangle,\ x\in E,$$

$$\mathcal{L}_x^U = \langle 1_{\{\gamma^*\subseteq U\}}\,\omega, L_x\rangle,\ \ x\in E,$$

and by (4.5) we see that they constitute independent collections of random variables. So we find that for $V\colon E \to \mathbb{R}_+$,

$$\Big(\frac{\det G_V}{\det G}\Big)^\alpha \overset{(4.11)}{=} \mathbb{E}_\alpha\Big[e^{-\sum\limits_{x\in E} V(x)\mathcal{L}_x}\Big] \overset{\text{independence}}{=} \mathbb{E}_\alpha\Big[e^{-\sum\limits_{x\in E} V(x)(\mathcal{L}_x - \mathcal{L}_x^U)}\Big] \times \mathbb{E}_\alpha\Big[e^{-\sum\limits_{x\in E} V(x)\mathcal{L}_x^U}\Big]$$

$$\underset{(4.7)}{\overset{(3.58)}{=}} \mathbb{E}_\alpha\Big[e^{-\sum\limits_{x\in E} V(x)(\mathcal{L}_x - \mathcal{L}_x^U)}\Big] \Big(\frac{\det G_{U,V}}{\det G_U}\Big)^\alpha.$$

The equality in the first line of (4.41) follows.

To prove the second equality in (4.41) we will use the next lemma:

Lemma 4.11. *(Jacobi's determinant identity)*

Assume that A is a $(k+\ell)\times(k+\ell)$-matrix which is invertible and that

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} B & C \\ D & E \end{bmatrix},$$

where B and W are $k \times k$ matrices, then

$$\det Z = \det A \det B. \tag{4.43}$$

Proof. We know that

$$I = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} BW + CY & BX + CZ \\ DW + EY & DX + EZ \end{bmatrix},$$

and hence

$$BX + CZ = 0 \ (k \times \ell\text{-matrix}) \text{ and } DX + EZ = I \ (\ell \times \ell\text{-matrix}).$$

As a result, we find that

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix} \begin{bmatrix} I & X \\ 0 & Z \end{bmatrix} = \begin{bmatrix} B & BX + CZ \\ D & DX + EZ \end{bmatrix} = \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}.$$

Taking determinants, we find that

$$\det A^{-1} \det Z = \det B,$$

and the claim (4.43) follows. □

We choose $A = -L$, $A^{-1} = G$, $B = G_{|K\times K}$, $Z = -L_{|U\times U}$, so that

$$\det Z = \det(-L_{|U\times U}) = (\det G_U)^{-1}, \ \det A = (\det G)^{-1},$$

and (4.43) yields

$$\frac{\det G}{\det G_U} = \det G_{|K\times K} \ \big(\overset{\text{notation}}{=} \det\nolimits_{K\times K} G\big). \tag{4.44}$$

In the same way, with $A = V - L$, $A^{-1} = G_V$, $B = G_{V|K\times K}$, $Z = (V-L)_{U\times U}$, we find that

$$\frac{\det G_V}{\det G_{U,V}} = \det G_{V|K\times K}. \tag{4.45}$$

Coming back to the expression after the first equality in (4.41), we see that this expression equals $(\frac{\det G_{V|K\times K}}{\det G_{|K\times K}})^\alpha$, and this completes the proof of (4.41).

• (4.42):

We first note that the left-hand side of (4.42) equals

$$\begin{aligned}&\mathbb{P}_\alpha\Big[\Big\langle \omega, \Big(\sum_{x\in K_1} L_x\Big)\Big(\sum_{x\in K_2} L_x\Big)\Big\rangle = 0\Big] = \\ &\mathbb{P}_\alpha\big[\Big\langle 1\{N>1\}\,\omega, \Big(\sum_{x\in K_1} L_x\Big)\Big(\sum_{x\in K_2} L_x\Big)\Big\rangle = 0\Big] \overset{(4.4)}{=} \\ &\exp\Big\{-\alpha\mu^*\big[\gamma^* : \sum_{x\in K_1} L_x(\gamma^*) > 0 \text{ and } \sum_{x\in K_2} L_x(\gamma^*) > 0, \text{ and } N(\gamma^*) > 1\big]\Big\}.\end{aligned} \tag{4.46}$$

Now we have $\mu^*[N(\gamma^*) > 1] < \infty$ (it equals $-\log\det(I-P)$ by (3.19) and (3.67)), so we can write:

$$\begin{aligned}&\mu^*\big[N>1, \sum_{x\in K_1} L_x > 0 \text{ and } \sum_{x\in K_2} L_x > 0\big] = \\ &\mu^*\big[N>1, \sum_{x\in K_1} L_x > 0\big] + \mu^*\big[N>1, \sum_{x\in K_2} L_x > 0\big] \\ &-\mu^*\big[N>1, \sum_{x\in K_1\cup K_2} L_x > 0\big].\end{aligned} \tag{4.47}$$

The next lemma will be useful to evaluate the above terms.

Lemma 4.12. $(\alpha > 0, K \subseteq E, U = E\backslash K)$

With the notation of (4.15), one has

$$\mathbb{P}_\alpha\big[\sum_{x\in K}\widehat{\mathcal{L}}_x = 0\big] = \Big(\det G_{|K\times K}\prod_{x\in K}\lambda_x\Big)^{-\alpha} = \Big(\frac{\det G}{\det G_U}\prod_{x\in K}\lambda_x\Big)^{-\alpha}, \tag{4.48}$$

$$\mu^*\big[N>1, \sum_{x\in K} L_x > 0\big] = \log\Big(\det G_{|K\times K}\prod_{x\in K}\lambda_x\Big). \tag{4.49}$$

Proof. Note that (4.49) is a direct consequence of (4.48) and the identity

$$\mathbb{P}_\alpha\big[\sum_{x\in K}\widehat{\mathcal{L}}_x = 0\big] \overset{(4.4)}{=} \exp\Big\{-\alpha\mu^*\big[N>1, \sum_{x\in K} L_x > 0\big]\Big\}.$$

We hence only need to prove (4.48). We use (4.20) with the choice $V = \rho 1_K$ where $\rho \uparrow \infty$. We then find that

$$\mathbb{P}_\alpha\big[\sum_{x\in K}\widehat{\mathcal{L}}_x = 0\big] = \lim_{\lambda\to\infty} \mathbb{E}_\alpha\big[e^{-\rho\sum_{x\in K}\widehat{\mathcal{L}}_x}\big] \overset{(4.20)}{=} \lim_{\lambda\to\infty}\Big(\frac{\det I - P^{V=\rho 1_K}}{\det I - P}\Big)^{-\alpha}. \tag{4.50}$$

Observe that $P^{V=\rho 1_K} \underset{\lambda\to\infty}{\longrightarrow} 1_U\,P$ (i.e. $\lim_{\rho\to\infty} P^{V=\rho 1_K} f(x) = 1_U(x)(Pf)(x)$, for all $x\in E$), by the definition of P^V below (4.19). The matrix for $I - 1_U\,P$ is block diagonal:

$$\left[\begin{array}{c|c} I & 0 \\ \hline & \end{array}\right] \leftarrow (I-P)_{|U\times U},$$

and, therefore,

$$\lim_{\lambda\to\infty}\Big(\frac{\det(I-P^{V=\lambda 1_K})}{\det(I-P)}\Big)^{-\alpha}=\Big(\frac{\det(I-P)_{|U\times U})}{\det(I-P)}\Big)^{-\alpha}.$$

Now, $\det G_U=(\det(-L)_{|U\times U})^{-1}=(\prod_{x\in U}\lambda_x)^{-1}(\det(I-P)_{|U\times U})^{-1}$, and similarly,

$$\det G=\Big(\prod_{x\in E}\lambda_x\Big)^{-1}\big(\det(I-P)\big)^{-1}.$$

Coming back to (4.50), we have shown that

$$\mathbb{P}_\alpha\Big[\sum_{x\in K}\widehat{\mathcal{L}}_x=0\Big]=\left(\frac{\det G\prod\limits_{x\in E}\lambda_x}{\det G_U\prod\limits_{x\in U}\lambda_x}\right)^{-\alpha}=\Big(\frac{\det G}{\det G_U}\prod_{x\in K}\lambda_x\Big)^{-\alpha},$$

and the proof of (4.48) is completed with (4.44). □

We now return to (4.46), (4.47), and find that (recall $K_1\cap K_2=\phi$)

$$\begin{aligned}&\mathbb{P}_\alpha[\text{no loop intersects both } K_1 \text{ and } K_2]=\\&\Big(\frac{\det G}{\det G_{U_1}}\;\frac{\det G}{\det G_{U_2}}\;\frac{\det G_{U_1\cap U_2}}{\det G}\Big)^{-\alpha}=\Big(\frac{\det G}{\det G_{U_1}}\;\frac{\det G_{U_1\cap U_2}}{\det G_{U_2}}\Big)^{-\alpha}\\&\overset{(4.44)}{=}\Big(\frac{\det_{K_1\times K_1}G}{\det_{K_1\times K_1}G_{U_2}}\Big)^{-\alpha},\end{aligned}$$

since $K_1=U_2\backslash(U_1\cap U_2)$. This concludes the proof of (4.42) and of Proposition 4.10. □

Special case: loops going through a point

We specialize the above formula (4.42) to find the probability that loops in the Poisson cloud going through a base point x all avoid some K not containing x:

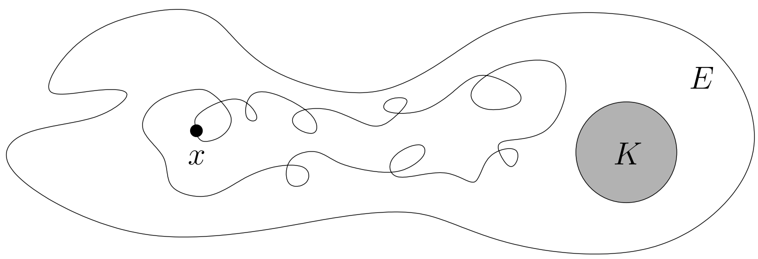


Fig. 4.3

Corollary 4.13. ($\alpha>0$)

Consider $x\in E$, and $K\subseteq E$ not containing x, then

$$\begin{aligned}&\mathbb{P}_\alpha[\textit{all loops going through } x \textit{ do not intersect } K]=\\&\Big(1-\frac{E_x[H_K<\infty,g(X_{H_K},x)]}{g(x,x)}\Big)^{\alpha}.\end{aligned}\tag{4.51}$$

Proof. In the notation of (4.42) we pick $K_1 = \{x\}$, $K_2 = K$. Setting $U = E\backslash K$, (4.42) yields that the left-hand side of (4.51) equals

$$\Big(\frac{g_U(x,x)}{g(x,x)}\Big)^\alpha \overset{(1.49)}{=} \Big(\frac{g(x,x) - E_x[H_K < \infty, g(X_{H_K}, x)]}{g(x,x)}\Big)^\alpha,$$

and (4.51) follows. □

4.5 Some links between Markovian loops and random interlacements

In this section we discuss various limiting procedures making sense of the notion of "loops going through infinity", and see random interlacements appear as a limit object.

We begin with the case of $\mathbb{Z}^d$, $d \ge 3$. Random interlacements have been introduced in [27], and we refer to [27] for a more detailed discussion of the Poisson point process of random interlacements. We will recover random interlacements on $\mathbb{Z}^d$, $d \ge 3$, by the consideration of "loops going through infinity". More precisely, we consider $d \ge 3$, and

(4.52) U_n, $n \ge 1$, a non-decreasing sequence of finite connected subsets of $\mathbb{Z}^d$, with $\bigcup_n U_n = \mathbb{Z}^d$,

as well as

(4.53) $$x_* \in \mathbb{Z}^d, \text{ a "base point"}.$$

For fixed $n \ge 1$, we endow the connected subset U_n, playing the role of E in (1.1), with the weights:

$$c^n_{x,y} = \frac{1}{2d}\, 1\{|x-y| = 1\}, \text{ for } x, y \in U_n,$$

and with the killing measure:

$$\kappa^n_x = \sum_{y \in \mathbb{Z}^d \backslash U_n} \frac{1}{2d}\, 1\{|x-y| = 1\}, \text{ for } x \in U_n,$$

very much in the spirit of what is done in Example 2) above (1.18) (except for the fact we now replace 1 by $\frac{1}{2d}$). Note that $\lambda^n_x = \sum_{y \in U_n} c^n_{x,y} + \kappa^n_x = 1$, for all $x \in U_n$.

We write Ω_n for the space corresponding to (4.2), of pure point measures on the set of unrooted loops contained in U_n, and $\mathbb{P}^n_\alpha$ for the corresponding Poisson gas of Markovian loops at level α, see (4.6).

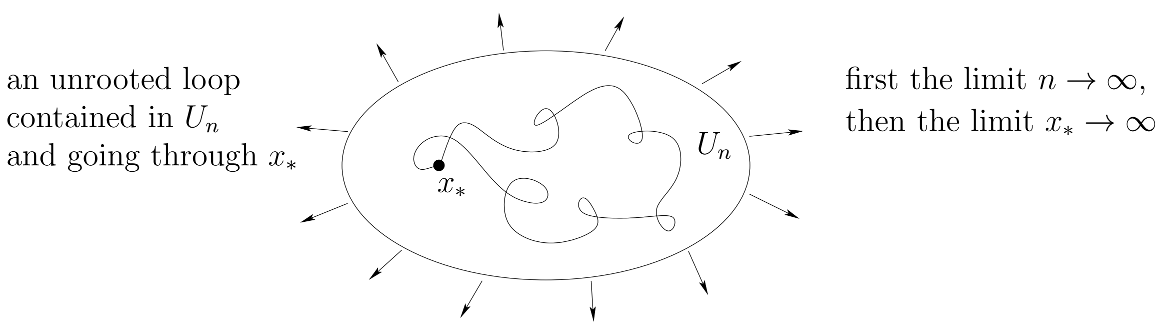


Fig. 4.4

We want to successively take the limit $n \to \infty$, and then $x_* \to \infty$. The first limit corresponds to the construction of a Poisson gas of unrooted loops on $\mathbb{Z}^d$. We will not really discuss this Poisson measure, which can be defined in a rather similar fashion to what we have done at the beginning of this chapter, but of course escapes the set-up of a finite state space E with weights and killing measure satisfying (1.1) - (1.5).

For the second limit (i.e. $x_* \to \infty$), we will also adjust the level α as a function of x_*. The fashion in which we tune α to x_* is dictated by the Green function of simple random walk on $\mathbb{Z}^d$:

$$g_{\mathbb{Z}^d}(x,y) = E_x^{\mathbb{Z}^d}\Big[\int_0^\infty 1\{X_t = y\}dt\Big], \text{ for } x,y \in \mathbb{Z}^d, \tag{4.54}$$

where $P_x^{\mathbb{Z}^d}$ denotes the canonical law of continuous-time simple random walk with jump rate 1 on $\mathbb{Z}^d$ starting at x, and X_t, $t \geq 0$, the canonical process. Taking advantage of translation invariance we introduce the function

$$g(x) \overset{\text{def}}{=} g_{\mathbb{Z}^d}(0,x), \text{ for } x \in \mathbb{Z}^d \text{ (so } g_{\mathbb{Z}^d}(x,y) = g(y-x)). \tag{4.54'}$$

The function $g(\cdot)$ is known to be positive, finite (recall $d \geq 3$), symmetric, i.e. $g(-x) = g(x)$, and has the asymptotic behavior

$$\begin{aligned} & g(x) \sim c_d\,|x|^{-(d-2)}, \text{ as } x \to \infty, \\ & \text{where } c_d = \frac{2}{(d-2)\,|B(0,1)|} = \frac{d}{2}\,\Gamma\Big(\frac{d}{2}-1\Big)\,\frac{1}{\pi^{\frac{d}{2}}}\,, \end{aligned} \tag{4.55}$$

where $|x|$ stands for the Euclidean norm of x, and $|B(0,1)|$ for the volume of the unit ball of $\mathbb{R}^d$ (see for instance [15], p. 31).

We will choose α according to the formula:

$$\alpha = u\,\frac{g(0)}{c_d^2}\,|x_*|^{2(d-2)}, \text{ with } u \geq 0. \tag{4.56}$$

We introduce for $\omega \in \Omega_n$, the subset of U_n of points visited by the unrooted loops in the support of the pure point measure ω, which pass through the base point x_*:

$$\begin{aligned} \mathcal{J}_{n,x_*}(\omega) = \big\{z \in U_n;\ & \text{there is a } \gamma^* \text{ in the support of } \omega, \\ & \text{which goes through } x_* \text{ and } z\big\}, \quad \text{for } \omega \in \Omega_n. \end{aligned} \tag{4.57}$$

Note that $\mathcal{J}_{n,x_*}(\omega) = \phi$, when $x_* \notin U_n$, and $\mathcal{J}_{n,x_*}(\omega) \ni x_*$, when at least one γ^* in the support of ω goes through x_*.

For the next result we will use the fact that (1.57) and (1.58) in the case of continuous-time simple random walk with jump rate 1 on $\mathbb{Z}^d$ take the following form:

$$\begin{aligned} & \text{when } K \text{ is a finite subset of } \mathbb{Z}^d, \\ & P_x^{\mathbb{Z}^d}[H_K < \infty] = \sum_{y \in K} g_{\mathbb{Z}^d}(x,y)\,e_K(y), \text{ for } x \in \mathbb{Z}^d \\ & (\text{with } \widetilde{H}_K \text{ as in (1.45)}), \end{aligned} \tag{4.58}$$

where the equilibrium measure

$$\begin{aligned} & e_K(y) = P_y^{\mathbb{Z}^d}[\widetilde{H}_K = \infty]\,1_K(y),\ y \in \mathbb{Z}^d \\ & (\text{with } \widetilde{H}_K \text{ as in (1.45)}), \end{aligned}$$

is the unique measure supported on K such that the equality in (4.58) holds for all $x \in K$. Its mass $\mathrm{cap}_{\mathbb{Z}^d}(K)$ is the capacity of K.

The next theorem relates the so-called "**random interlacement at level $\boldsymbol{u}$**" to the the set $\mathcal{J}_{n,x_*}$ when $n \to \infty$, and then $x_* \to \infty$, under the measure $\mathbb{P}^n_\alpha$, with α as in (4.56).

In this set of notes we will not introduce the full formalism of the Poisson point process of random interlacements but only content ourselves with the description of the random interlacement at level u, see Remark 4.15 below.

Theorem 4.14. ($d \geq 3$)

For $u \geq 0$ and $K \subseteq \mathbb{Z}^d$ finite, one has

$$\lim_{x_* \to \infty} \lim_{n \to \infty} \mathbb{P}^n_{\alpha = u \frac{g(0)}{c_d^2} |x_*|^{2(d-2)}} [\mathcal{J}_{n,x_*} \cap K = \phi] = e^{-u \, \mathrm{cap}_{\mathbb{Z}^d}(K)}. \tag{4.59}$$

Proof. By (4.51) we have, as soon as $x_* \in U_n$ and $x_* \notin K$,

$$\begin{aligned} \mathbb{P}^n_\alpha[\mathcal{J}_{n,x_*} \cap K = \phi] &= \mathbb{P}^n_\alpha[\text{all loops going through } x_* \text{ do not meet } K] \\ &= \Big(1 - \frac{E^{\mathbb{Z}^d}_{x_*}[H_K < T_{U_n}, g_{U_n}(X_{H_K}, x_*)]}{g_{U_n}(x_*, x_*)}\Big)^\alpha, \end{aligned}$$

with $g_{U_n}(\cdot,\cdot)$ the Green function of simple random walk on $\mathbb{Z}^d$ killed when exiting U_n. Clearly, by monotone convergence,

$$g_{U_n}(x,y) \uparrow g_{\mathbb{Z}^d}(x,y), \text{ for } x,y \in \mathbb{Z}^d, \text{ when } n \to \infty.$$

So we see that when $x_* \notin K$:

$$\lim_n \mathbb{P}^n_\alpha[\mathcal{J}_{n,x_*} \cap K = \phi] = \Big(1 - \frac{E^{\mathbb{Z}^d}_{x_*}[H_K < \infty, g_{\mathbb{Z}^d}(X_{H_K}, x_*)]}{g_{\mathbb{Z}^d}(x_*, x_*)}\Big)^\alpha \tag{4.60}$$

(the formula holds also when $x_* \in K$).

Now $g_{\mathbb{Z}^d}(x_*, x_*) = g(0)$, and, as $x_* \to \infty$, we have by (4.55)

$$\begin{aligned} E^{\mathbb{Z}^d}_{x_*}[H_K < \infty, g_{\mathbb{Z}^d}(X_{H_K}, x_*)] &\overset{(4.55)}{\sim} P^{\mathbb{Z}^d}_{x_*}[H_K < \infty] \, c_d \, |x_*|^{-(d-2)} \\ &\underset{(4.55)}{\overset{(4.58)}{\sim}} (c_d \, |x_*|^{-(d-2)})^2 \, \mathrm{cap}_{\mathbb{Z}^d}(K), \end{aligned}$$

and, in particular, with α as in (4.56),

$$\lim_{x_* \to \infty} \frac{\alpha}{g(0)} \, E^{\mathbb{Z}^d}_{x_*}[H_K < \infty, \, g_{\mathbb{Z}^d}(X_{H_K}, x_*)] = u \, \mathrm{cap}_{\mathbb{Z}^d}(K).$$

Coming back to (4.60) we readily obtain (4.59). $\square$

Remark 4.15. One can define a translation invariant random subset of $\mathbb{Z}^d$ denoted by $\mathcal{I}^u$, the so-called *random interlacement at level* u, see [27], with distribution characterized by the identity:

$$\mathbb{P}[\mathcal{I}^u \cap K = \phi] = e^{-u \, \mathrm{cap}_{\mathbb{Z}^d}(K)}, \text{ for all } K \subseteq \mathbb{Z}^d \text{ finite.} \tag{4.61}$$

Coming back to (4.59), note that for any disjoint finite subsets K, K' of $\mathbb{Z}^d$ one has by an inclusion-exclusion argument:

$$\begin{array}{l}\mathbb{P}_\alpha^n[\mathcal{J}_{n,x_*} \cap K = \phi \ \text{ and } \ \mathcal{J}_{n,x_*} \supseteq K'] = \\ \mathbb{E}_\alpha^n\Big[\prod_{x \in K} 1_{\mathcal{J}^c_{n,x_*}}(x) \prod_{x \in K'} \big(1 - 1_{\mathcal{J}^c_{n,x_*}}(x)\big)\Big] = \\ \sum_{A \subseteq K'} (-1)^{|A|}\, \mathbb{P}_\alpha^n[\mathcal{J}_{n,x_*} \cap (K \cup A) = \phi],\end{array}$$

where we expanded the last product in the second line to find the last line.

In the same fashion, we see that for disjoint finite subsets K, K' of $\mathbb{Z}^d$ we have

$$\begin{array}{l}\mathbb{P}[\mathcal{I}^u \cap K = \phi \ \text{ and } \ \mathcal{I}^u \supseteq K'] = \\ \sum_{A \subseteq K'} (-1)^{|A|}\, \mathbb{P}[\mathcal{I}^u \cap (K \cup A) = \phi] = \sum_{A \subseteq K'} (-1)^{|A|}\, e^{-u\, \mathrm{cap}_{\mathbb{Z}^d}(K \cup A)}.\end{array}$$

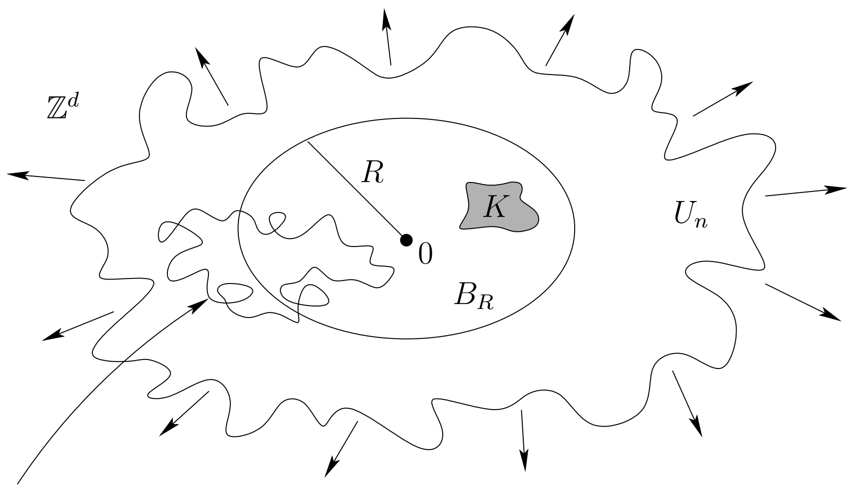
As a result, Theorem 4.14 can be seen to imply that under the measure $\mathbb{P}^n_{\alpha = u \frac{g(0)}{c_d^2} |x_*|^{2(d-2)}}$, the law of $\mathcal{J}_{n,x_*}$ converges in an appropriate sense (i.e. convergence of all finite dimensional marginal distributions) to the law of $\mathcal{I}^u$, as $n \to \infty$, and then $x_* \to \infty$. □

We continue with the discussion of **links** between **random interlacements** and "**loops going through infinity**" in the Poisson cloud of Markovian loops.

We begin with a variation on (4.59) in the context of $\mathbb{Z}^d$, $d \geq 3$, where we will give a different meaning to the informal notion of "loops going through infinity".

We consider a sequence U_n, $n \geq 1$, as in (4.52) of finite connected subsets of $\mathbb{Z}^d$, $d \geq 3$, which increases (in the wide sense) to $\mathbb{Z}^d$. The role of the base point x_*, cf. (4.53), is now replaced by the complement of the Euclidean ball:

(4.62) $$B_R \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^d;\ |x| \leq R\}, \quad \text{with } R > 0.$$



an unrooted loop contained in U_n and touching B_R^c

first the limit $n \to \infty$, then the limit $R \to \infty$

Fig. 4.5

By analogy with (4.57), we introduce for $\omega \in \Omega_n$,

$$\mathcal{K}_{n,R}(\omega) = \{z \in U_n;\ \text{there is a } \gamma^* \text{ in the support of } \omega, \text{ which goes through } B_R^c \text{ and } z\}. \tag{4.63}$$

We now choose α according to

$$\alpha = u\, \frac{R^{d-2}}{c_d}, \text{ with } u \ge 0, \text{ and } c_d \text{ as in (4.55).} \tag{4.64}$$

The corresponding statement to (4.59) is now the following. By the argument of Remark 4.15, it can be interpreted as a convergence of the law of $\mathcal{K}_{n,R}$ to the law of $\mathcal{I}^u$, as $n \to \infty$ and then $R \to \infty$.

Theorem 4.16. $(d \ge 3)$

For $u \ge 0$ and $K \subseteq \mathbb{Z}^d$ finite, one has

$$\lim_{R\to\infty}\ \lim_{n\to\infty}\ \mathbb{P}^n_{\alpha = u\,\frac{R^{d-2}}{c_d}}\,[\mathcal{K}_{n,R} \cap K = \phi] = e^{-u\,\mathrm{cap}_{\mathbb{Z}^d}(K)}. \tag{4.65}$$

Proof. We assume that R is large enough so that $K \subseteq B_R$ and n sufficiently large so that $B_R \subsetneq U_n$. In the notation of (4.42), we chose $K_1 = K$ and $K_2 = U_n \backslash B_R$, so that $K_1 \cap K_2 = \phi$. Then (4.42) yields that

$$\mathbb{P}^n_\alpha[\mathcal{K}_{n,R} \cap K = \phi] = \Big(\frac{\det_{K\times K} G_{B_R}}{\det_{K\times K} G_{U_n}}\Big)^\alpha.$$

By (1.49), we write

$$\det_{K\times K} G_{B_R} = \det(A_n - B_n^{(R)}),$$

where A_n is the $K \times K$-matrix:

$$A_n(x,y) = g_{U_n}(x,y), \text{ for } x,y \in K,$$

and $B_n^{(R)}$, the $K \times K$-matrix

$$B_n^{(R)}(x,y) = E_x^{\mathbb{Z}^d}[H_{B_R^c} < T_{U_n},\ g_{U_n}(X_{H_{B_R^c}}, y)], \text{ for } x,y \in K.$$

Likewise, by the above definitions, we find that

$$\det_{K\times K} G_{U_n} = \det(A_n)\,.$$

When $n \to \infty$,

$$\lim_n\ A_n = A, \qquad \text{where } A(x,y) = g_{\mathbb{Z}^d}(x,y), \text{ for } x,y \in K \tag{4.66}$$

$$\lim_n\ B_n^{(R)} = B^{(R)}, \text{ where } B^{(R)}(x,y) = E_x^{\mathbb{Z}^d}[H_{B_R^c} < \infty,\ g_{\mathbb{Z}^d}(X_{H_{B_R^c}}, y)], \text{ for } x,y \in K. \tag{4.67}$$

The matrix A is known to be invertible (one can base this on a similar calculation as in the proof of (1.35), see also [25], P2, p. 292). So we find that:

$$\lim_{n\to\infty}\ \mathbb{P}^n_\alpha[\mathcal{K}_{n,R} \cap K = \phi] = \Big(\frac{\det(A - B^{(R)})}{\det A}\Big)^\alpha = \big(\det(I - A^{-1} B^{(R)})\big)^\alpha. \tag{4.68}$$

For $x \in K$, $P_x^{\mathbb{Z}^d}$-a.s., $H_{B_R^c} < \infty$, and $X_{H_{B_R^c}} \in \partial B_R$, so that

$$B^{(R)}(x,y) = E_x^{\mathbb{Z}^d}[g_{\mathbb{Z}^d}(X_{H_{B_R^c}}, y)] \stackrel{(4.55)}{\sim} \frac{c_d}{R^{d-2}}, \ \text{ for } x, y \in K, \text{ as } R \to \infty.$$

It follows that

$$\det(I - A^{-1}\, B^{(R)}) = 1 - \frac{c_d}{R^{d-2}}\, Tr(A^{-1}\, \mathbf{1}_{K\times K}) + o\Big(\frac{1}{R^{d-2}}\Big), \text{ as } R \to \infty, \tag{4.69}$$

where $\mathbf{1}_{K\times K}$ denotes the $K \times K$ matrix with all coefficients equal to 1.

Coming back to (4.58), we see that $A^{-1}\mathbf{1}_{K\times K} = C$, where C is the $K \times K$-matrix with coefficients $C(x,y) = e_K(x)$, for $x, y \in K$. Since $\sum_{x\in K} e_K(x) = \mathrm{cap}_{\mathbb{Z}^d}(K)$, we have found that

$$\det(I - A^{-1}\, B^{(R)}) = 1 - \frac{c_d}{R^{d-2}}\, \mathrm{cap}_{\mathbb{Z}^d}(K) + o\Big(\frac{1}{R^{d-2}}\Big), \text{ as } R \to \infty. \tag{4.70}$$

Inserting this formula into (4.68), with α as in (4.64), immediately yields (4.65). □

Complement: random interlacements and Poisson gas of loops coming from infinity on a transient weighted graph

So far we only discussed links between random interlacements and a Poisson cloud of "loops going though infinity", in the case of $\mathbb{Z}^d$, $d \geq 3$.

We now discuss another construction, which applies to the general set-up of an (infinite) transient weighted graph with no killing.

We consider a countable (in particular infinite) set Γ endowed with non-negative weights $c_{x,y}$, $x, y \in \Gamma$ (i.e. satisfying (1.2) with Γ in place of E), so that

(4.71) Γ endowed with the set of edges consisting of $\{x, y\}$ such that $c_{x,y} > 0$, is connected, locally finite, (i.e. each $x \in \Gamma$ has a finite number of neighbors),

and

(4.72) the simple random walk with jump rate 1 on Γ induced by these weights $c_{x,y}$, $x, y \in \Gamma$, is transient.

This is what we mean by a **transient weighted graph** (i.e. with no killing).

We consider, as in (4.52),

(4.73) U_n, $n \geq 1$, a non-decreasing sequence of finite connected subsets of Γ increasing to Γ (i.e. $\bigcup_{n\geq 1} U_n = \Gamma$ and $U_n \subseteq U_{n+1}$),

as well as a point

(4.74) x_* not in Γ (which will play the role of the point "at infinity for each U_n").

We consider the finite graph with vertex set $E_n = U_n \cup \{x_*\}$, endowed with the **weights obtained by collapsing U_N^c on x_***:

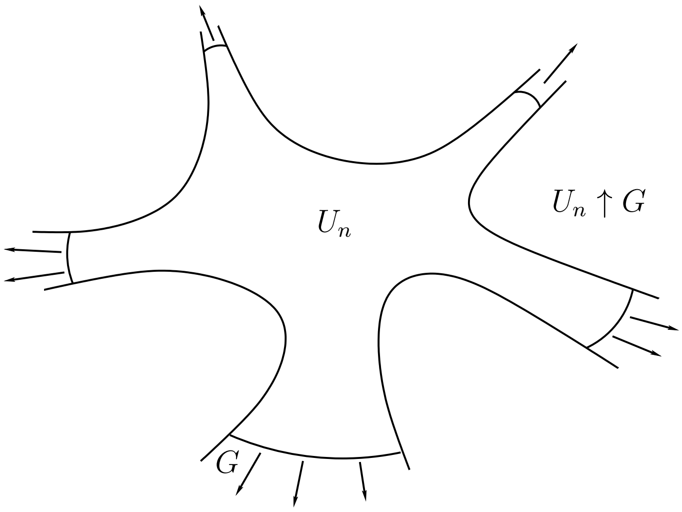


Fig. 4.6

$$(4.75)\qquad \begin{aligned} c^n_{x,y} &= c_{x,y}, \text{ when } x, y \in U_n \\ c^n_{x_*,y} &= c^n_{y,x_*} = \sum_{x \in G \setminus U_n} c_{x,y}, \text{ when } y \in U_n. \end{aligned}$$

In addition we choose on E_n the killing measure κ^n_x, $x \in E_n$, concentrated on x_*, so that

$$(4.76)\qquad \begin{aligned} \kappa^n_{x_*} &= \lambda_n > 0, \text{ with } \lim \lambda_n = \infty, \\ \kappa^n_x &= 0, \text{ for } x \in U_n = E_n \setminus \{x_*\}. \end{aligned}$$

For the continuous-time walk on Γ with jump rate 1, one can show that when K is a finite subset of G, setting $\lambda^0_x = \sum_{y \in \Gamma} c_{x,y}$, for $x \in \Gamma$, and $g_\Gamma(\cdot, \cdot)$ for the Green function (i.e. $g_\Gamma(x, y) = \frac{1}{\lambda^0_y} E^\Gamma_x[\int_0^\infty 1\{X_t = y\} dt]$, for $x, y \in \Gamma$),

$$(4.77)\qquad P^\Gamma_x[H_K < \infty] = \sum_{y \in K} g_\Gamma(x, y)\, e_K(y), \text{ for } x \in \Gamma,$$

where e_K is the equilibrium measure of K:

$$(4.78)\qquad e_K(y) = P^\Gamma_y[\widetilde{H}_K = \infty]\, 1_K(y)\, \lambda^0_y, \text{ for } y \in \Gamma$$

(for instance one approximates the left-hand side of (4.77) by $P^\Gamma_x[H_K < T_{U_n}]$, with $n \to \infty$, and applies (1.57), (1.53) to the walk killed when exiting U_n).

The total mass of e_K is the capacity of K:

$$(4.79)\qquad \text{cap}_\Gamma(K) = \sum_{y \in K} e_K(y).$$

We write Ω_n for the space of unrooted loops on E_n and $\mathbb{P}^n_\alpha$ for the Poisson gas of Markovian loops at level $\alpha > 0$, on the above finite set E_n endowed with the weights c^n in (4.75) and the killing measure κ^n in (4.76).

We also introduce the random subset of U_n:

$$\mathcal{J}_n(\omega) = \{z \in U_n; \text{ there is a } \gamma^* \text{ in the support of } \omega \text{ which goes through } x_* \text{ and } z\}. \tag{4.80}$$

We now specify α via the formula, see (4.76),

$$\alpha = u\,\lambda_n, \text{ with } u \geq 0. \tag{4.81}$$

The statement corresponding to (4.59) and (4.65), which in the present context links the Poisson gas of loops on E_n going through "the point x_* at infinity", with the interlacement at level u on G is coming next. We refer to Remark 1.4 of [27] and [29] for a more detailed description of the Poisson point process of random interlacements in this context.

Theorem 4.17. *For $u \geq 0$ and $K \subseteq G$ finite, one has*

$$\lim_{n\to\infty} \mathbb{P}^n_{\alpha=u\lambda_n}[\mathcal{J}_n \cap K = \phi] = e^{-u\,\mathrm{cap}_G(K)}. \tag{4.82}$$

Proof. For large n, $K \subseteq U_n$, and we can write by (4.51):

$$\mathbb{P}^n_\alpha[\mathcal{J}_n \cap K = \phi] = \Big(1 - \frac{E^n_{x_*}[H_K < \infty,\, g_n(X_{H_K}, x_*)]}{g_n(x_*, x_*)}\Big)^\alpha,$$

where P^n_x stands for the law of the walk on E_n with unit jump rate, starting at $x \in E_n$, attached to the weights and killing measure in (4.75), (4.76) and $g_n(\cdot,\cdot)$ for the corresponding Green function.

By (2.71), we know that

$$g_n(x_*, z) = \lambda_n^{-1}, \text{ for all } z \in E_n, \tag{4.83}$$

and, as a result,

$$\begin{aligned} \mathbb{P}^n_\alpha[\mathcal{J}_n \cap K = \phi] &= \big(1 - P^n_{x_*}[H_K < \infty]\big)^\alpha \\ &\underset{(4.83)}{\overset{(1.57)}{=}} \Big(1 - \frac{1}{\lambda_n}\,\mathrm{cap}_n(K)\Big)^\alpha, \end{aligned} \tag{4.84}$$

where $\mathrm{cap}_n(K)$ stands for the capacity of K in E_n.

By (1.53) and the fact that κ^n vanishes on U_n, we know that

$$\mathrm{cap}_n(K) = \sum_{x\in K} P^n_x[\widetilde{H}_K = \infty]\,\lambda^0_x. \tag{4.85}$$

In addition, we know that P^n_x-a.s.,

$$\{\widetilde{H}_K = \infty\} = \{H_{x_*} < \widetilde{H}_K\} \cap \theta^{-1}_{H_{x_*}}(\{H_K = \infty\}),$$

because U_n is finite and the walk is only killed at x_*. So applying the strong Markov property at time H_{x_*} we find that:

$$\begin{aligned} P^n_x[\widetilde{H}_K = \infty] &= P^n_x[H_{x_*} < \widetilde{H}_K] \times P^n_{x_*}[H_K = \infty] \\ &= P^\Gamma_x[T_{U_n} < \widetilde{H}_K] \times (1 - P^n_{x_*}[H_K < \infty]), \end{aligned}$$

using the fact that the walk on G and on E_n "agree up to time T_{U_n}". Note, in addition, that

$$P^n_{x_*}[H_K < \infty] \underset{(1.53)}{\overset{(1.57)}{\leq}} \sum_{y \in K} g_n(x_*, y)\, \lambda^0_y \overset{(4.83)}{=} \frac{1}{\lambda_n} \sum_{y \in K} \lambda^0_y \underset{n \to \infty}{\overset{(4.76)}{\longrightarrow}} 0,$$

and that

$$P^\Gamma_x[T_{U_n} < \widetilde{H}_K] \downarrow P^\Gamma_x[\widetilde{H}_K = \infty], \text{ as } n \to \infty.$$

Coming back to (4.85), we have shown that

$$\lim_n \, \mathrm{cap}_n(K) = \sum_{x \in K} P^\Gamma_x[\widetilde{H}_K = \infty]\, \lambda^0_x \underset{(4.79)}{\overset{(4.78)}{=}} \mathrm{cap}_\Gamma(K). \tag{4.86}$$

If we now insert this identity in (4.84) and keep in mind that $\alpha = u\lambda_n$, we readily find (4.82). □

Remark 4.18.

1) By a similar argument as described in Remark 4.15, the above theorem can be seen to imply that under $\mathbb{P}^n_{\alpha = u\lambda_n}$, the law of $\mathcal{J}_n$ converges to the law of $\mathcal{I}^u$ in the sense of finite dimensional marginal distributions, as n goes to infinity.

2) A variation on the approximation scheme, which we employed to approximate random interlacements on a transient weighted graph, can be used to prove an isomorphism theorem for random interlacements, see [28]. One can define the random field $(L_{x,u})_{x \in \Gamma}$ of occupation times of continuous-time random interlacements at level u (this random field is governed by a probability denoted by $\mathbb{P}$). One can also define the canonical law P^G on $\mathbb{R}^\Gamma$ of the Gaussian free field attached to the transient weighted graph under consideration: under P^G the canonical field $(\varphi_x)_{x \in \Gamma}$ is a centered Gaussian field with covariance $E^G[\varphi_x \, \varphi_y] = g_\Gamma(x, y)$, for $x, y \in \Gamma$, with $g_\Gamma(\cdot, \cdot)$ the Green function. The isomorphism theorem from [28] states that

$$\begin{aligned} &\Big(L_{x,u} + \frac{1}{2}\, \varphi_x^2\Big)_{x \in \Gamma} \text{ under } \mathbb{P} \otimes P^G, \text{ has the same law as} \\ &\Big(\frac{1}{2}\, (\varphi_x + \sqrt{2u})^2\Big)_{x \in \Gamma} \text{ under } P^G. \end{aligned} \tag{4.87}$$

The above identity in law is intimately related to the generalized second Ray-Knight theorem, see Theorem 2.17, and characterizes the law of $(L_{x,u})_{x \in \Gamma}$. □

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