The singular Nirenberg problem

I will consider the problem of prescribing the Gaussian curvature (under pointwise conformal change of the metric) on surfaces with conical singularities. This question has been first raised by Troyanov in [6] and it is a generalization of the Kazdan-Warner problem for regular surfaces, known as the Nirenberg problem on the sphere.
Answer this question amounts to solve the following differential problem on a surface Σ

$$-\Delta_g u + 2K_g = 2Ke^u - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j}, \tag{1}$$

where K_g is the Gaussian curvature of the background metric g, K is the curvature we want to prescribe, p_j are the singular points and α_j the orders of the singularities of the new metric we are looking for: $\tilde{g} = e^u g$, verifying $K_{\tilde{g}} = K$.

This equation has been studied first in the case $K > 0$.I will present some new results (obtained in collaboration with R. López-Soriano) in the case K sign-changing.
When $\Sigma = S^2$, under some mild conditions on the nodal set of K we derived some sufficient conditions on K and on the singularities for the existence of solutions of (1). Even if we do not expect these conditions to be necessary, I will explain why they are somehow sharp.