Partial regularity for fractional harmonic maps into spheres

Joint work with Vincent Millot and Armin Schikorra

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Outline

1. “Classical” harmonic maps
2. Fractional harmonic maps
3. Energy improvement and ε-regularity
4. Minimizing $1/2$-harmonic maps

“Classical” harmonic maps

General

Assume

- $\Omega \subseteq \mathbb{R}^n$ bounded open set with smooth boundary, $n \geq 1$
- $\mathcal{N}$ smooth compact submanifold of $\mathbb{R}^d$ without boundary (for example $\mathbb{S}^{d-1}$)

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A map $u \in H^1(\Omega; \mathcal{N})$ is said to be **harmonic** in Ω if it is a **critical point of the Dirichlet energy**

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for (constrained) **outer** variations, that is to say,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{E}\left(\pi_\mathcal{N}(u + t\varphi), \Omega\right)\right]_{|t=0} = 0 \qquad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^d),$$

where $\pi_\mathcal{N}$ the nearest point projection on $\mathcal{N}$.

Harmonic map equation

- Euler-Lagrange equation: u is harmonic iff

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The application $u(x) = \frac{x}{|x|}$ from $\mathbb{R}^3$ into $\mathbb{S}^2$ is harmonic.

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Connection with minimal surfaces:

- for $n = 1$, harmonic maps are **geodesics**
- for $n = 2$, they are **branched minimal immersions**

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There exists a harmonic map $u \in H^1(B^3; \mathbb{S}^2)$ which is ***discontinuous everywhere****.*

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If u *is a* ***minimizing*** *harmonic map, i.e., satisfies*

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then $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$ *with* $\dim_{\mathscr{H}} \operatorname{sing}(u) \leq n - 3$.

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Example

Let $g \in H^1(B^3; \mathbb{S}^2) \cap C^0(\partial B^3)$ s.t. $\deg(g_{|\partial B^3}) \neq 0$. Then

$$v := \operatorname{argmin} \left\{ \mathcal{E}(u; B^3) \ : \ u \in H^1(B^3; \mathbb{S}^2) \text{ s.t. } u(x) = g(x) \text{ on } \partial B^3 \right\},$$

is a minimizing harmonic map in B^3 with a singularity.

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Theorem (Stationary harmonic maps[7])

If u is harmonic and ***stationary****, i.e. satisfies,* $\forall X \in C_c^\infty(\Omega;\mathbb{R}^n)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{E}(u \circ \Phi_t, \Omega)\Big]_{|t=0} = 0 \quad \textit{where } \Phi_t \textit{ is the integral flow of } X,$$

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Proposition (Stationarity equation)

If u is stationary harmonic in Ω then for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\mathbb{R}^n} \sum_{i=1}^n \left(|\nabla u|^2 \delta_{ij} - 2\partial_i u \cdot \partial_j u\right) \partial_i \varphi \,\mathrm{d}x = 0, \quad \forall j \in \{1, \dots, n\}$$

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Corollary (Monotonicity formula)

If u stationary harmonic in Ω, then

$$r \mapsto r^{2-n} \int_{B_r(x)} |\nabla u|^2 \,\mathrm{d}x \nearrow, \quad \forall x \in \Omega.$$

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Fractional harmonic maps

Fractional Laplacian (1)

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$$\mathcal{E}_s(u,\Omega) := \frac{\gamma_{n,s}}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

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$$\widehat{H}^s(\Omega;\mathbb{R}^d) := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^d) \ : \ \mathcal{E}_s(u,\Omega) < +\infty\right\}.$$

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$$\begin{aligned}\langle(-\Delta)^s u,\varphi\rangle_\Omega &:= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{E}_s(u+t\varphi,\Omega)\Big]_{|t=0}\\ &= \frac{\gamma_{n,s}}{2}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

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Definition (Stationary)

u is a **stationary** s-harmonic map if it is s-harmonic and satisfies, $\forall X \in C_c^\infty(\Omega; \mathbb{R}^n)$

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Partial regularity results

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Theorem (Stationary case)

Assume $n > 2s$. *If* $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ *is* ***stationary*** s*-harmonic in* Ω*, then* $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$ *and*

- *for* $\boldsymbol{s < 1/2}$ *and* $n \geq 1$, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leq \boldsymbol{n-1}$;
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Theorem (Minimizing case)

*Assume $n > 2s$. If $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is **minimizing** s-harmonic in Ω, then $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$ and*

- *for $\boldsymbol{n \geq 3}$, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leq \boldsymbol{n-2}$;*
- *for $n = 2$, $\operatorname{sing}(u)$ is locally finite Ω;*
- *for $n = 1$, $\operatorname{sing}(u) = \emptyset$ (i.e., $u \in C^\infty(\Omega)$).*

Energy improvement and ε-regularity

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Proposition (Energy improvement)

There exists $\boldsymbol{\varepsilon}_* = \boldsymbol{\varepsilon}_*(n,s) > 0$ *and* $\boldsymbol{\tau} = \boldsymbol{\tau}(n,s) \in (0,1)$ *s.t., if* $u \in \widehat{H}^s(B_1;\mathbb{S}^{d-1})$ *is a stationary* s*-harmonic map in* B_1 *satisfying*

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Corollary (ε-regularity)

If $u \in \widehat{H}^s(B_r(x);\mathbb{S}^{d-1})$ *is a stationary* s*-harmonic map in* $B_r(x)$ *s.t.*

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then u *is Hölder-continuous in* $B_{r/2}(x)$*.*

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If $u \in H^1(\Omega; \mathbb{S}^{d-1})$ *is a stationary harmonic map in* Ω*, then* $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$*, with* $\boldsymbol{\mathscr{H}^{n-2}(\operatorname{sing}(u)) = 0}$.

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- $\forall \mathbf{x} = (x,0) \in B_R \times \{0\}$, $r \mapsto r^{2s-n}\mathbf{E}_s(u^{\mathrm{e}}, B_r^+(\mathbf{x}))$ is nondecreasing in $(0,R)$, where

$$\mathbf{E}_s(u^{\mathrm{e}}, B_r^+(\mathbf{x})) := \int_{B_r^+(\mathbf{x})} |\nabla u^{\mathrm{e}}|^2 z^{1-2s}\,\mathrm{d}\mathbf{x}$$

$$\implies u \in \mathrm{BMO}(B_R).$$

Fractional div-curl quantities

- Fractional **s-gradient:** for $u \in \widehat{H}^s(\Omega)$, we define

$$\mathrm{d_s}\, u(x,y) = \frac{\sqrt{\gamma_{n,s}}}{\sqrt{2}} \frac{u(x)-u(y)}{|x-y|^s} \in L^2_{\mathrm{od}}(\Omega)$$

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- **s-divergence:** by duality $\mathrm{div}_s : L^2_{\mathrm{od}}(\Omega) \to H^{-s}(\Omega)$ defined by

$$\langle \mathrm{div}_s F, \varphi \rangle = \int_\Omega F \odot \mathrm{d_s}\, \varphi\, \mathrm{d}x.$$

Fractional div-curl lemma

Theorem (Mazowiecka and Schikorra, 2018)

If $F \in L^2_{\mathrm{od}}(\mathbb{R}^n)$ *satisfies* $\operatorname{div}_s F = 0$ *in* $H^{-s}(\mathbb{R}^n)$ *and* $v \in H^s(\mathbb{R}^n)$*, then* $F \odot \mathrm{d_s}\, v \in \mathcal{H}^1(\mathbb{R}^n)$.

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Proposition (Local version)

There exists $C > 0$ *and* $\Lambda \in (0,1)$ *universal constant s.t., if* $F \in L^2_{\mathrm{od}}(B_r)$ *satisfies* $\operatorname{div}_s F = 0$ *in* $H^{-s}(B_r)$ *and* $v \in \widehat{H}^s(B_r)$*, then*

$$\left| \int_{\mathbb{R}^n} (F \odot \mathrm{d_s}\, v) \varphi \, \mathrm{d}x \right| \leq C \|F\|_{L^2_{\mathrm{od}}(B_{\Lambda r})} \underbrace{\|\mathrm{d_s}\, v\|_{L^2_{\mathrm{od}}(B_{\Lambda r})}}_{\simeq\ \mathcal{E}_s(v, B_{\Lambda r})} \left([\varphi]_{\mathrm{BMO}} + r^{-n} \|\varphi\|_{L^1} \right),$$

for every $\varphi \in C^\infty_c(B_{\Lambda r})$*.*

div-curl structure of the source term

- Recall
$$(-\Delta)^s u = |\mathrm{d_s}\, u|^2 u = (\mathrm{d_s}\, u \odot \mathrm{d_s}\, u)u \quad \text{in } \mathscr{D}'(\Omega)$$

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$$(-\Delta)^s u^i = \sum_{j=1}^{d} \left(\mathbf{\Omega}^{ij} \odot \mathrm{d_s}\, u^j\right) + T^i \quad \text{in } \mathscr{D}'(\Omega),$$

for all $i \in \{1, \dots, d\}$, where $u = (u^1, \dots, u^d)$, et

$$\mathbf{\Omega}^{ij}(x,y) = u^i(x)\,\mathrm{d_s}\, u^j(x,y) - u^j(x)\,\mathrm{d_s}\, u^i(x,y)$$
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Proposition (Conservation law)

$u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is s-harmonic iff $\forall i, j \in \{1, \ldots, d\}$,

$$\operatorname{div}_s \mathbf{\Omega}^{ij} = 0 \quad \text{in } H^{-s}(\Omega).$$

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for all $i \in \{1, \dots, d\}$, where $u = (u^1, \dots, u^d)$, et

$$\boldsymbol{\Omega}^{ij}(x,y) = u^i(x)\,\mathrm{d}_\mathrm{s}\, u^j(x,y) - u^j(x)\,\mathrm{d}_\mathrm{s}\, u^i(x,y)$$

$$T^i(x) = \frac{\gamma_{n,s}}{4} \int_{\mathbb{R}^n} \underbrace{|u(x)-u(y)|^2\big(u^i(x)-u^i(y)\big)}_{\leq |u(x)-u(y)|^3} \frac{\mathrm{d}y}{|x-y|^{n+2s}}$$

Proposition (Conservation law)

$u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ *is* s*-harmonic iff* $\forall i, j \in \{1, \dots, d\}$,

$$\mathrm{div}_s\, \boldsymbol{\Omega}^{ij} = 0 \quad \text{in } H^{-s}(\Omega).$$

Proof of energy improvement (1)

Proposition (Energy improvement)

There exists $\boldsymbol{\varepsilon}_* = \boldsymbol{\varepsilon}_*(n,s) > 0$ *and* $\boldsymbol{\tau} = \boldsymbol{\tau}(n,s) \in (0,1/4)$ *s.t., if* $u \in \widehat{H}^s(B_1;\mathbb{S}^{d-1})$ *is a* ***stationary*** s*-harmonic map in* B_1 *satisfying*

$$\mathcal{E}_s(u,B_1) \leq \boldsymbol{\varepsilon}_*\,,$$

then

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  - Let $w_k := \frac{u_k - [u_k]_{B_1}}{\varepsilon_k}$, then $\mathcal{E}_s(w_k, B_1) = 1$ and $\tau^{2s-n}\mathcal{E}_s(w_k, B_\tau) > \frac{1}{2}$.

Proof of energy improvement (2)

- Up to a subsequence, $w_k \xrightharpoonup{\widehat{H}^s(B_1)} w_*$, $w_k \xrightarrow{L^2(B_1)} w_*$, where $(-\Delta)^s w_* = 0$ in B_1.

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Minimizing $1/2$-harmonic maps

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Theorem ($d \geq 3$)

If $u \in \widehat{H}^{1/2}(\Omega; \mathbb{S}^{d-1})$ is a minimizing $1/2$-harmonic map, where $d \geq 3$, then $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$ with $\dim_{\mathscr{H}} \operatorname{sing}(u) \leq \boldsymbol{n-3}$.

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Theorem ($n = d = 2$)

- *$u_*(x) = x/|x|$ from $\mathbb{D}$ into $\mathbb{S}^1$ is a **minimizing** $1/2$-harmonic map.*
- *u_* is the **only 0-homogeneous minimizing** $1/2$-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^1$, up to an orthogonal transformation.*
- *If $u \in \widehat{H}^{1/2}(\Omega \subseteq \mathbb{R}^2; \mathbb{S}^1)$ is a minimizing $1/2$-harmonic map, then the **topological degree** of u around its singularities is ± 1.*

Proof $d \geq 3$ (1)

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  - ↳ $w_g(\partial\mathbb{D}) \subseteq \mathbb{S}^{d-1}$ and $\partial_\nu w_g = (-\Delta)^{1/2} g \perp \mathrm{Tan}(g = w_g, \mathbb{S}^{d-1})$ on $\partial\mathbb{D}$
    $\implies w_g(\overline{\mathbb{D}})$ is a flat equatorial disk (Fraser and R. Schoen; Nitsche)
    for example $w_g(\overline{\mathbb{D}}) = \overline{\mathbb{D}} \times \{0\}^{d-2}$, and $g(\mathbb{S}^1) = \mathbb{S}^1 \times \{0\}^{d-2}$.

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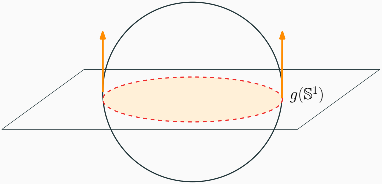

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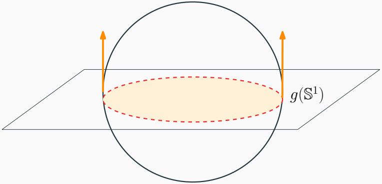

- Positive second variation (by minimality) gives

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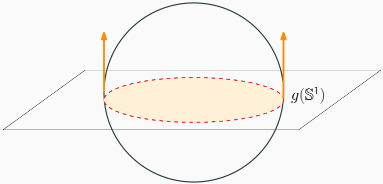

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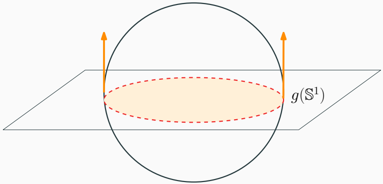

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- $\mathcal{E}_{1/2}(g, \mathbb{S}^1) = \pi \left|\deg(g, \mathbb{S}^1)\right|$
  $\implies \mathcal{E}_{1/2}(g, \mathbb{S}^1) = 0 \implies g$ constant.

Perspectives

- Improvement of the upper bound on $\dim_{\mathscr{H}} \mathrm{sing}(u)$ for minimizing s-harmonic maps into $\mathbb{S}^{d-1}$ for $s > 1/2$?
- Minimality of $x/|x|$ when $s \neq 1/2$
- Construction of a $1/2$-harmonic map from $\mathbb{D}$ into $\mathbb{S}^1$ discontinuous everywhere
- Study of nonlocal PDE (fractional versions of Ginzburg-Landau, Cahn-Hilliard, ...)

Thank you.

div-curl structure of the RHS

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- Recall
$$(-\Delta)^s u = |\mathrm{d}_\mathrm{s}\, u|^2 u = (\mathrm{d}_\mathrm{s}\, u \odot \mathrm{d}_\mathrm{s}\, u) u$$
- Rewriting of the source term. For $i, j \in \{1, \dots, d\}$
$$\begin{aligned}
&|\mathrm{d}_\mathrm{s}\, u^j|^2(x) u^i(x) \\
&\quad= \int_{\mathbb{R}^n} \Big(u^i(x)\, \mathrm{d}_\mathrm{s}\, u^j(x,y) \Big)\, \mathrm{d}_\mathrm{s}\, u^j(x,y) \frac{\mathrm{d}y}{|x-y|^n} \\
&\quad= \int_{\mathbb{R}^n} \underbrace{\Big(u^i(x)\, \mathrm{d}_\mathrm{s}\, u^j(x,y) - u^j(x)\, \mathrm{d}_\mathrm{s}\, u^i(x,y) \Big)}_{\boldsymbol{\Omega}^{ij}} \mathrm{d}_\mathrm{s}\, u^j(x,y) \frac{\mathrm{d}y}{|x-y|^n} \\
&\qquad\quad + \int_{\mathbb{R}^n} \Big(\mathrm{d}_\mathrm{s}\, u^j \odot \mathrm{d}_\mathrm{s}\, u^i \Big) u^j \,\mathrm{d}x \frac{\mathrm{d}y}{|x-y|^n}
\end{aligned}$$
- Since $|u| = 1$, we have
$$\sum_{j=1}^{d} \Big(\mathrm{d}_s u^j \odot \mathrm{d}_s u^i \Big) u^j = \underbrace{\frac{\gamma_{n,s}}{4} \int_{\mathbb{R}^n} \overbrace{|u(x)-u(y)|^2 (u^i(x) - u^i(y))}^{\le |u(x)-u(y)|^3} \frac{\mathrm{d}y}{|x-y|^n}}_{T^i}$$

Embeddings between $\mathcal{Q}_{\mathbf{p}}^{\boldsymbol{\alpha},\mathbf{q}}$-spaces

Embeddings between $\mathcal{Q}_{\mathbf{p}}^{\boldsymbol{\alpha},\mathbf{q}}$-spaces

- We have
$$|\mathrm{RHS}| \leq o(1) + C\varepsilon_k^2 [w_k]^3_{W^{s/3,6}(B_{r_0})} \|w_k\|_{H^s(B_1)}$$
- $\mathcal{Q}_p^{\alpha,q}(\mathbb{R}^n)$ seminormed space made of measurable functions f s.t.
$$[f]_{\mathcal{Q}_p^{\alpha,q}(\mathbb{R}^n)} := \sup_Q |Q|^{1/p-1/q} \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/q} < \infty \, .$$
- $\dot{F}^{s,u}_{M_{p,q,\lambda}}(\mathbb{R}^n)$ Triebel-Lizorkin-Morrey-Lorentz spaces. We have
$$\mathcal{Q}^{\alpha_1,q_1}_{\frac{nq_1}{\lambda}}(\mathbb{R}^n) = \dot{F}^{\alpha_1,\frac{n-\lambda}{nq_1}}_{q_1,q_1}(\mathbb{R}^n) \hookrightarrow \dot{F}^{\alpha_2,\frac{n-\lambda}{nq_2}}_{q_2,q_2}(\mathbb{R}^n) = \mathcal{Q}^{\alpha_2,\frac{n-\lambda}{nq_2}}_{q_2,q_2}(\mathbb{R}^n),$$
for all $0 < \alpha_1 < \alpha_2 < 1$, $1 \leq q_2 < q_1 < \infty$ and $0 < \lambda \leq n$.
- In particular
$$\mathcal{Q}^{s,2}_{n/s}(\mathbb{R}^n) \hookrightarrow \mathcal{Q}^{s/3,6}_{3n/s}(\mathbb{R}^n),$$
i.e.,
$$\sup_{B_r(x)\subseteq\mathbb{R}^n} r^{\frac{2s-n}{3}} [f]^2_{W^{s/3,6}(B_r(x))} \leq C \sup_{B_r(x)\subseteq\mathbb{R}^n} r^{2s-n} [f]^2_{H^s(B_r(x))}.$$