Fine properties of monotone mappings arising in optimal transport for non quadratic costs

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This talk focuses on recent results in collaboration with Cristian E. Gutiérrez and concerning various properties of mappings arising in optimal transport problems for non quadratic costs. The cost functions considered have the form $c(x,y) = h(x-y)$, where $h \in C^2(\mathbb{R}^n)$ is convex, positively homogeneous of degree $p \geq 2$, and the Hessian $D^2h(x)$ has eigenvalues bounded away from zero and infinity for all x in the unit sphere.

A multivalued mapping $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is c-monotone if $c(\xi, x) + c(\zeta, y) \leq c(\xi, y) + c(\zeta, x)$ for all $\xi \in Tx, \zeta \in Ty$ for all $x, y \in \mathbb{R}^n$. In particular, optimal maps with respect to the cost c are c-monotone. For the quadratic cost $h(x) = |x|^2$, then c-monotonicity is the standard monotonicity $(\xi - \zeta) \cdot (x - y) \geq 0$ for all $\xi \in Tx, \zeta \in Ty$ for all $x, y \in \mathbb{R}^n$, having a large number of applications to optimization and nonlinear evolution pdes.

We prove that c-monotone mappings T are single valued a.e. and using Green's representation formulas we establish local L^∞-estimates on balls for $u(x) = Tx - Ax - b$ for each matrix A and each vector b in terms of averages of u on a slightly larger ball. As a consequence, we deduce that T is first order differentiable almost everywhere.

This research is a continuation of our very recent work in Calculus of Variations and Pdes that originated from some interesting results by M. Goldman and F. Otto concerning regularity of optimal transport maps for the quadratic cost.