NOTES ON CONTACT CONVEXITY

1. Characteristic foliation

Let M be a contact manifold of dimension $2n+1$ with a co-oriented contact structure $\xi = \{\alpha = 0\}$. Note that if $n = 2k+1$ then any contact structure defines an orientation of the manifold, and if $n = 2k+1$ then a choice of a co-orientation defines an orientation of the manifold because for odd values of n a contact structure defines an orientation of contact planes.

Let $\Sigma \subset M$ be a co-oriented hypersurface. As M is oriented by the volume form $\alpha \wedge d\alpha^n$, the co-orientation of Σ defines its orientation. At any point $p \in \Sigma$, where $\xi(p) \pitchfork T_p\Sigma$, there is defined a *characteristic line* $\lambda_p \subset \xi(p) \cap T_p\Sigma = \operatorname{Ker}\left(d\alpha|_{\xi(p)\cap T_p\Sigma}\right)$. Note that the co-orientation of Σ defines a co-orientation of $\xi_p \cap T_p\Sigma$ in $\xi(p)$. We orient λ_p by the vector which is $d\alpha_p|_{\xi(p)}$-dual to a form on $\xi(p)$ annihilating $\xi_p \cap T_p\Sigma$ and defining its co-orientation. Thus, the line field λ, which is defined in the complement of the tangency locus T of ξ and Σ, integrates to a singular foliation on Σ with singularities at the points of T. We will keep the notation ℓ for this foliation, and write ℓ_Σ, ℓ_ξ or $ell_{\xi,\Sigma}$ when it will be important to stress the dependence of ℓ on Σ, ξ, or both.

The singular locus T naturally can be presented as a union of disjoint closed subsets, $T = T_+ \cup T_-$, where T_+ (resp. T_-) consists of *positive*, where the orientations of ξ_p and $T_p(\Sigma)$ coiuncide (resp. opposite). On a neighborhood $U_\pm \supset T_\pm$ the form $d\alpha|_{U_\pm}$ is symplectic. We define a vector field X on Σ directing ℓ as equal to the Liouville field, $d\alpha|_{U_+}$-dual to $\alpha|_{U_+}$ on U_+,negative of the Liouville field, $d\alpha|_{U_-}$-dual to $\alpha|_{U_-}$ on U_-, and extend it to the rest of Σ as any non-vanishing vector field defining the given orientation of ℓ.

All singularities of X can be made non-degenerate by a C^∞-small perturbation of Σ, see e.g [1]. It will then automatically follow that they are hyperbolic, i.e. for each zero p the linearization d_pX is non-degenerate and has no pure imaginary eigenvalues.

2. Lyapunov functions

Let X be a vector field on a compact manifold Σ. A function $\phi : \Sigma \to \mathbb{R}$ is called *Lyapunov* for X if

$$d\phi(X) \geq C(||X||^2 + ||d\phi||^2)$$

for a positive constant C. We assume here that Σ is endowed with a background Riemannian metric.

Suppose that X has isolated non-degenerate hyperbolic zeroes (i.e the linearizations of X at zeroes have no pure imaginary eigenvalues. A neighborhood of a hyperbolic zero always admits a Lyapunov function.

Date: May 29, 2022.

1991 *Mathematics Subject Classification.* Primary ; Secondary .

In the case of a vector field directing a characteristic foliation stable manifolds of singular points of X are isotropic (with respect to the form $d\alpha|_\Sigma$) for positive zeroes and co-isotropic for the negative ones, see [1], and hence the local Lyapunov function on $\mathcal{O}p\,(T_+ \cup T_-)$, have critical points of index $\leq n$ at positive points, and index $\geq n$ in the negative ones. Let us stress the point, that index n critical points can be either negative or positive.

We call a Lyapunov function $f : \Sigma \to \mathbb{R}$ for X *good* if there exists a regular value c such that all positive zeroes of X are in $f > c$ and all negative ones are in $f < c$.

We say that the characteristic foliation on a hypersurface Σ in a contact manifold (M, ξ) is of *Morse-Smale type* if the vector field X directing characteristic foliation λ on Σ has isolated non-degenerate zeroes, all trajectories of X originate and terminate at zeroes of X, and the flow of X satisfies the Morse-Smale condition.

Lemma 2.1. *Suppose that the characteristic foliation λ on Σ is of Morse-Smale type. Then the vector field X directing λ admits a good global Lyapunov function $f : \Sigma \to \mathbb{R}$. Moreover, one choose f in such a way that the critical points of index $k < n$ correspond to the critical value k, the critical points of index $k > n$ correspond to the critical value $k+1$, the positive critical points of index n correspond to the critical value n, while the negative critical points of index n correspond to the critical value $n+1$.*

Proof. We begin constructing a Lyapunov function $f : \Sigma \to \mathbb{R}$ from singular points of index 0. We can assume that the critical value of f at all index 0 points is 0, and that $f = \epsilon$ on the boundary of a neighborhood of index 0 locus. Arguing by induction suppose that we already constructed a Lyapunov function f on a domain Σ_k, $0 \leq k = 2n-1$, with boundary $\partial\Sigma_k$ such that $f|_{\partial\Sigma_k} = k + \epsilon$, and all singularities of X of index $\leq k$ are contained in $\mathrm{Int}\Sigma_k$, $k = 0, \dots, n-2$. Let $p_j \in \Sigma \setminus \Sigma_k, j =, \dots, n-1$, be critical points of index $k+1$ and P_j there $(k+1)$-dimensional stable manifolds. There is an extension of f to $U = \mathcal{O}p\,(\Sigma_k \cup \bigcup\limits_1^m P_j)$ as a Lyapunov function for X such that $f(p_j) = k+1$ and the regular level set $\{f = k+1+\epsilon\}$ is compact and contained in U, see Fig. **??**. To achieve this When extending the function over neighborhoods of stable manifolds of critical points of index n we first construct an extension to neighborhoods of positive index n singular points, with the critical value n, and then to neighborhoods of negative ones with the critical value $n+1$. The Morse-Smale property ensures that stable manifolds of index n negative points do not enter sufficiently small neighborhoods of index n positive points, and hence the process can go through. After that we continue the original process successively extending the function over stable manifolds of critical points of indices $k = n+1, \dots, 2n$ with the critical values $k+1$. □

3. Various flavors of contact convexity

3.1. **Contact convexity.** The notion of contact convexity was first defined in [2] and then intensively studied by Emmanuel Giroux, [3], Ko Honda, [4], and others.

A hypersurface $\Sigma \subset (M, \xi)$ is called *convex* if it admits a transverse contact vector field Z.

The set of points $S := \{x \in \Sigma; X(x) \in \xi_x\}$ is called the dividing set of Σ.

Lemma 3.1. *Suppose X is a contact vector field transverse to a hypersurface Σ and S the corresponding dividing set set. Let t be the flow coordinate such that $\Sigma = \{t = 0\}$ and $X = \frac{\partial}{\partial t}$. Then ξ on $\mathcal{O}p\,\Sigma$ can be defined by a 1-form $f(x)dt + \mu$, where $f : \Sigma \to \mathbb{R}$ is a function transversely changing sign across Σ.*

Note that the contact condition implies that $df \neq 0$ along S, and $\alpha|_S$ is a contact form. Hence, we have

Lemma 3.2. *Dividing set S is a smooth submanifold, which is transverse to the characteristic foliation, and independent of the choice of a contact vector field transverse to Σ, up to an isotopy transverse to the characteristic foliation.*

Indeed, the space of contact vector fields transverse to Σ is a convex subset of the vector space of all contact vector fields, and hence, contractible.

The dividing surface Σ divides Σ into $\Sigma_+ := \{f > 0\}$, $\Sigma_- := \{f < 0\}$. The form $\alpha = (f(x)dt + \mu)|_{\Sigma \setminus S}$ can be divided by f,

$$\frac{\alpha}{f} = dt + \frac{\mu}{f}.$$

Denote $\lambda_\pm := \left(\frac{\mu}{f}\right)\big|_{\Sigma_\pm}$. The contact condition for $\frac{\alpha}{f}$ is equivalent to $(d\lambda_\pm)^n \neq 0$. In other words, $\lambda_\pm$ are Liouville forms on $\Sigma_\pm$. Note that the corresponding Liouville fields $Z_\pm$ directs the characteristic foliation on Σ. Indeed, $\lambda_\pm \wedge \iota(Z_\pm)d\lambda_\pm = \lambda_\pm \wedge \lambda_\pm = 0$.

Note that

$$\frac{\alpha}{f} \wedge \left(d\left(\frac{\alpha}{f}\right)\right)^n = \frac{1}{f^{n+1}}\alpha \wedge (d\alpha)^n = dt \wedge d\lambda_\pm^n.$$

Hence, the orientation defined by $d\lambda_-$ on Σ_- differs from the given orientation of Σ_- by $(-1)^{n+1}$. Note that near zero locus of $\alpha|_\Sigma$ the form $d\alpha$ is also symplectic and near the negative points it defines the orientation opposite to the given orientation of the manifold. This agrees with the fact that this orientation differs by $(-1)^n$ from the orientation defined by the form $d\lambda_-$.

Given a Liouville manifold (Σ, λ) we say that it has a cylindrical end if the corresponding Liouville field Z is complete and there exists a compact domain Σ_0 with boundary $S := \partial\Sigma_0$, such that Z is outwardly transverse to S and each point of $\Sigma \setminus \Sigma_0$ belongs to a trajectory of Z intersecting S. There is a canonical Liouville isomorphism $(\Sigma \setminus \mathrm{Int}\Sigma_0, \lambda) \to (S \times [1, \infty), s\alpha)$, where α is the contact form $\lambda|_S$ and s the coordinate corresponding to the second factor. Note that any two hypersurfaces in the complement of Σ_0 which transverse to Z can be canonically identified via a holonomy along the trajectories of Z, and the identification preserves the contact structure $\xi := \mathrm{Ker}\alpha$ on S. We call (S, ξ) the *ideal contact boundary* of Σ.

Lemma 3.3. *Any two Liouville manifolds with contactomorphic ideal boundaries can be glued into a convex hypersurface.*

Proof. Let $(\Sigma_+, \lambda_+), (\Sigma_-, \lambda_-)$ be two Liouville manifolds with cylindrical ends and (S_+, ξ_+) and (S_-, ξ_-) be their ideal boundaries. Suppose we are given a contactomorphism $(S_+, \xi_-) \to$

(S_-,ξ_+). By choosing appropriate slices $S_\pm\subset\Sigma_\pm$ we can assume that $f^*\beta_+=\beta_-$, where $\beta_\pm:=\lambda_\pm|_{S_\pm}$. Denote by $\Sigma^0_\pm$ domains bounded by $S_\pm$ in $\Sigma_\pm$ and identify the ends $\Sigma_\pm\setminus\mathrm{Int}\Sigma^0_\pm$ with $(S_\pm\times[0,\infty),s\beta_\pm$. Define a manifold $\Sigma:=\Sigma_+\underset{f}{\cup}\Sigma_-$ by gluing $S_+\times[-1,1]$ to Σ_+ and Σ_- via diffeomorphisms $\Theta_+:S\times[-1,0)\to S_+\times[1,\infty)$ $\Theta_-:S\times(0,1]\to S_-\times[1,\infty)$ given, respectively, by the formulas

$$\Theta_+(x,u)=(x,-\frac{1}{u}),\ \Theta_-(x,u)=(f(x),\frac{1}{u}).$$

Denote $S:=S_+\times 0\subset\Sigma$.

We endow $\Sigma\setminus S$ with the form λ which is equal to $\pm\lambda_\pm$ on $\Sigma^0_\pm$, equal to $\Theta_+^*(s\beta_+)$ on $S_+\times[-1,0)$, and equal to $-\Theta_-^*(s\beta_-)=$ on $S_+\times(0,-1]$. We note that $\lambda=\frac{\beta_+}{s}$ on $S_+\times([-1,1]\setminus 0)$. Choose a non-increasing function $\tau:[-1,1]\to\mathbb{R}$ such that $\tau(u)=-u$, $\tau=\mp1$ near ±1, and define a function $\Psi:\Sigma\to\mathbb{R}$ by the formula $\Psi(x,u)=\tau(u)$ on $S_+\times[-1,1]$ and extend it to the rest of Σ as equal to ±1 on $\Sigma^0_\pm$.

Define a 1-form β on $\Sigma\setminus S$ by the formula $\beta=\Psi\lambda$. Note that near S we have $\beta=\beta_+$, and therefore, β smoothly extends to the whole Σ. Finally define a 1-form $\alpha=\beta+\Psi dt$ on $\Sigma\times\mathbb{R}$. Let us check that α is a contact form. It is sufficient to verify the property in a neighborhood of S because elsewhere α is proportional to $dt+\lambda_\pm$ which is contact. On $\mathcal{O}p\,S$ we have $\alpha=udt+\beta_+$, and hence, it is also contact.

It remains to observe that $\Sigma\times 0$ is convex in $(\Sigma\times\mathbb{R},\alpha)$ because the vector field $\frac{\partial}{\partial t}$ is contact. $\square$

3.2. **Weinstein contact convexity.** A hypersurface Σ is called *Weinstein convex* if the vector field X directing the characteristic foliation λ is of Morse-Smale type.

Lemma 3.4. *Any Weinstein convex hypersurface is convex.*

We begin the proof with the following technical lemma.

Lemma 3.5. *Let $\Sigma=S\times[0,1]$, where S is a compact manifold with boundary, be a co-oriented hypersurface in a co-oriented $(2n+1)$-dimensional contact manifold $(M,\xi=\mathrm{Ker}\alpha)$. Suppose that the characteristic foliation ℓ on Σ is formed by fibers $x\times[-1,1], x\in S$. Set $\beta=\alpha|_\Sigma$.*

a) *Suppose that $(d\beta)^n|_{Op(S\times(-1)}>0$. Then there exists a function $h:[-1,1]\to\mathbb{R}$ such that $h(u)=0$ near -1 and $h(u)=C$ near 1 such that $(d(e^h\beta))^n>0$ everywhere on Σ.*

b) *Suppose that $(d\beta)^n|_{Op(S\times(-1))}>0$ and $(d\beta)^n|_{Op(S\times1)}<0$. Consider a function $\mu:[-1,1]\to\mathbb{R}$ such that $\mu(u)=-1$ near -1, $\mu(u)=-u$ on $\left[-\frac12,\frac12\right]$, $\mu(u)=-1$ near 1 and $\mu'(u)\le 0$ everywhere on $[-1,1]$. Then there exists a function $h:[-1,1]\to\mathbb{R}$ such that $h(u)=0$ near -1 and 1 such that the 1-form $\alpha_0:=e^h\beta+\mu(u)dt$ is contact on $\Sigma\times\mathbb{R}$.*

Proof. a) Any 1-form β with the characteristic foliation ℓ can be written as $\beta=f\gamma$, where β is a contact form on S and $f:\Sigma\to\mathbb{R}$ is a non-vanishing function. Hence, $d\beta=f_udu\wedge\gamma+fd\gamma$ and $(d\beta)^n=nf^{n-1}f_udu\wedge\gamma\wedge(d\gamma)^{n-1}$. The form $\Omega:=du\wedge\gamma\wedge(d\gamma)^n$ is a positive volume form, and hence the sign of $(d\beta)^n$ coincides with the sign of the derivative f_u. Hence, multiplying by e^h where h by a function e^h with a large positive derivative has away from a neighborhood of -1 does the job.

b) As above, we have $\beta = f\gamma$. Denote $\widetilde{f} := fe^h$. We have $\alpha_0 = \widetilde{f}\gamma + \mu(u)dt$ and $d\alpha_0 = \widetilde{f}_u du \wedge \gamma + d_S\widetilde{f} \wedge \gamma\widetilde{f}d\gamma + \mu' du \wedge dt$, where d_S is the differential along S. Furthermore,

$$\begin{aligned}\alpha_0 \wedge (d\alpha_0)^n &= (\widetilde{f}\gamma + \mu dt) \wedge (\widetilde{f}_u du \wedge \gamma + d_S f \wedge \gamma + \widetilde{f}d\gamma + \mu' du \wedge dt)^n \\ &= n\widetilde{f}^{n-1}(\widetilde{f}_u\mu - \widetilde{f}\mu')dt \wedge du \wedge \gamma \wedge (d\gamma)^{n-1}\end{aligned}$$

Hence, the contact condition is equivalent to the inequality

$$\widetilde{f}_u\mu - \widetilde{f}\mu' > 0. \tag{1}$$

We have $\widetilde{f}_u\mu - \widetilde{f}\mu' = e^h f((h' + (\ln f)_u)\mu - h\mu')$, and hence, (1) can be rewritten as

$$h'\mu - h\mu' > \mu(\ln f)_u =: \mu g. \tag{2}$$

By a) near $u = -1$ we have $f_u > 0$, and hence g is negative, and the inequality is satisfied. Similarly, near $u = +1$ we have $f_u < 0$, and thus the inequality is satisfied as well. Note that $\mu' \leq 0$ while μ is negative on $[0, 1]$. Hence, by making the function h growing fast on $[-1, -\frac{1}{2}]$ and decreasing fast on $[\frac{1}{2}, 1]$ we can satisfy the inequality (2) away from $[-\frac{1}{2}, \frac{1}{2}]$ and have $h(-\frac{1}{2}) = h(\frac{1}{2}) = C > 0$ and $h'(-\frac{1}{2}) = -h(\frac{1}{2}) = C_1 > 0$. Recall that $\mu(u) = -u$ on $[-\frac{1}{2}, \frac{1}{2}]$. Hence, (2). takes the form

$$h > u(h' + g). \tag{3}$$

Suppose the constant C is chosen to satisfy the inequality $C > \max\limits_{S\times\left[-\frac{1}{2},\frac{1}{2}\right]} |g(x, u)|$. Then the inequality (3) will be satisfied as long as we extend h to $\left[-\frac{1}{2}, \frac{1}{2}\right]$ to have a unique critical point, the maximum, at 0. □

Proof of Lemma 3.4. We call a $2n$-form on Σ positive (resp. negative) if it defines the given (resp. opposite) orientation. Choose a contact form α for the contact structure ξ and set $\beta := \alpha|_\Sigma$. Note that $(d\beta)^n$ is positive in a neighborhood of the positive critical point locus of ℓ, and negative in a neighborhood of the negative one.

Let $\Phi : \Sigma \to \mathbb{R}$ be a good Lyapunov function for the characteristic foliation ℓ constructed in Lemma 2.1, and X be a vector field directing ℓ. Let N_1 be the union of stable manifolds of all critical points of index 1. Denote $N_1^\epsilon := N_1 \cap \{\epsilon \leq \Phi \leq 1 - \epsilon\}$. For a sufficiently small $\epsilon > 0$ denote $F_{k,\pm} := \{\Phi = k \pm \epsilon\}$, $k = 0, \dots, 2n + 1\}$,

Consider a closed tubular neighborhood Σ_1 of N_1^ϵ in $\{\epsilon \leq f \leq 1-\epsilon\}$ foliated by trajectories of ℓ. We can present Σ_1 as a product $S_1 \times [0, 1]$, where the fibers $x \times [0, 1]$, $x \in \Sigma_1$ are arcs of leaves of ℓ connecting points in $F_{0,+}$ and $F_{1,-}$. Let us apply Lemma 3.5a) to find a function $h : \Sigma_1 \to \mathbb{R}$ which is equal to 0 on $\mathcal{O}p\,(\Sigma_1 \cap F_{0,+})$ and to a constant C on $\mathcal{O}p\,(\Sigma_1 \cap F_{1,-})$ and such that the form $d(e^h\beta)^n > 0$ on Σ_1. We extend h in any way to the rest of M and set $\alpha_1 := e^h\alpha$ and $\beta_1 := e^h\beta = \alpha_1|_\Sigma$. To simplify the notation we rename β_1 back to β. By flowing the level sets of the function Φ with the flow of $-X$ we can arrange without changing the critical values that $\{\Phi \leq 1 + \epsilon\} \subset \{f \leq \epsilon\} \cup \Sigma_1$. In particular the form $(d\beta)^n$ is positive in $\Phi \leq 1+\epsilon\}$. Continuing inductively the process for positive critical points of index $2, \dots, n$

we similarly arrange that $(d\beta)^n$ is positive in $\Phi \le n + \epsilon\}$. Similarly, by arguing downward from index $2n$ negative points we arrange that $(d\beta)^n$ is negative in $\Phi \ge n + 1 - \epsilon\}$.

Let $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(u) = u$ if $-\epsilon \le u \le \epsilon$, equal to 2ϵ for $u > 2\epsilon$, to -2ϵ for $u < -2\epsilon$ and satisfies the condition $\theta'(u) \ge 0$. Set $\Psi := \theta(\Phi - n - \frac{1}{2})$. Using Lemma 3.5b) we can scale the form β over $\widetilde{\Sigma} := \{n + \epsilon \le \Phi \le n + 1 - \epsilon\}$ to ensure that the 1-form $\alpha_0 := \beta + \Psi dt$ on $\Sigma \times \mathbb{R}$ is contact. Note that the form β is invariant with respect to translations along the t-axis, and hence, $\frac{\partial}{\partial t}$ is contact. But germs of contact forms along a hypersurface Σ which have the same restriction to Σ are diffeomorphic via a diffeomorphism fixed on Σ. □

References

[1] K. Cieliebak and Y. Eliashberg, From Stein to Weinstein and back, *Colloquium Publications*, AMS, **59**(2012).

[2] Y. Eliashberg and M. Gromov, Convex symplectic manifolds, *Proceedings of Symposia in Pure Mathematics*, vol. 52(1991), Part 2, 135–162.

[3] E. Giroux, Convexité en topologie de contact, *Comm. Math. Helvet.*, **66**(1991), 637677. 615–689.

[4] K. Honda, On the classification of tight contact structures. I, *Geom. Topol.* 4 (2000), 309368.