Polynomial Preserving Diffusions and Models of the Term Structure

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- Polynomial term structure models
- Polynomial preserving diffusions
    - Existence
    - Uniqueness
    - Boundary attainment
- Examples
- Moment asymptotics

Polynomial term structure models

Term structure of interest rates

- $P(t, T) =$ time t price of zero-coupon bond maturing at $T \geq t$
- Yields are defined by: $P(t, T) = e^{-(T-t)\,\mathrm{yield}(t,T)}$

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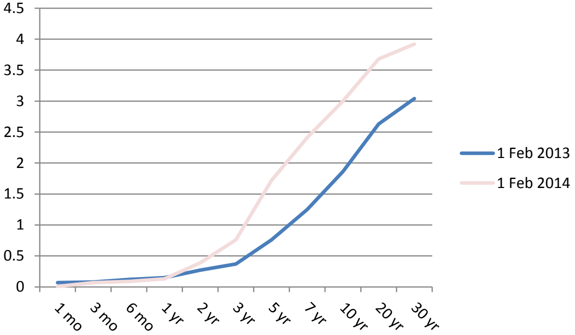
- The **term structure of interest rates** (or yield curve) at time t:

$$\tau \;\mapsto\; \text{yield}(t, t+\tau)$$



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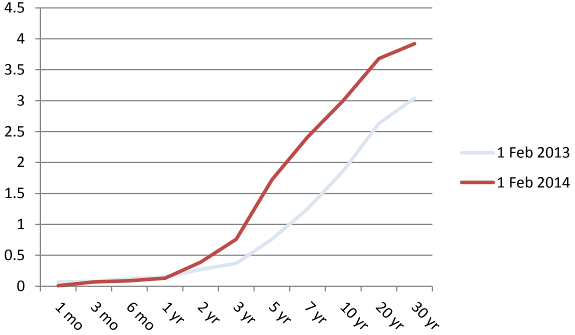

State price density models

- Filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$
- State price density: positive process ζ_t
- Model price of claim C_T maturing at T:

$$\text{price at } t = \frac{1}{\zeta_t}\mathbb{E}[\,\zeta_T C_T \mid \mathscr{F}_t]$$

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Note:

- Arbitrage-free price system is guaranteed (NUPBR)
- $\dfrac{\zeta_t}{\zeta_0} = e^{-\int_0^t r_s \mathrm{d}s} \times \left.\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t}$
- Model is under $\mathbb{P}$: Time series properties, risk management
- Risk-free zero-coupon bond: $P(t, T) = \dfrac{1}{\zeta_t}\mathbb{E}[\,\zeta_T \mid \mathscr{F}_t]$

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Previous literature:

- Constantinides (92)
- Flesaker & Hughston (96)
- Rogers (97)
- Carr, Gabaix & Wu (10)
- etc.

State price density models

Factor model:

- X_t multivariate factor process
- Postulate $\zeta_t = f(t, X_t)$ for some function $f(t, x)$
- Bond prices:

$$P(t, T) = \frac{\mathbb{E}\left[f(T, X_T) \mid \mathscr{F}_t\right]}{f(t, X_t)}$$

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- Need $f(t, x)$ and X so that $\mathbb{E}\left[f(T, X_T) \mid \mathscr{F}_t\right]$ is easy to compute

Polynomial preserving processes

Polynomial-preserving factor process

- Time-homogeneous Markov semimartingale X, state space $E \subset \mathbb{R}^d$
- Transition semigroup $T_t f(x) = \mathbb{E}_x[f(X_t)]$

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$$\mathrm{Pol}_n(\mathbb{R}^d) = \{\text{polynomials on } \mathbb{R}^d \text{ of degree} \leq n\}$$
$$\mathrm{Pol}_n(E) = \{p|_E \;:\; p \in \mathrm{Pol}_n(\mathbb{R}^d)\}$$

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Definition. X is called **Polynomial Preserving (PP)** if

$$T_t \, \mathrm{Pol}_n(E) \subset \mathrm{Pol}_n(E) \qquad \text{for all} \qquad n \in \mathbb{N}, \quad t \geq 0.$$

Polynomial preserving processes

Let $\mathscr{G}$ be the **extended generator** of X: For all $f \in \mathrm{Dom}(\mathscr{G})$,

$$f(X_t) - f(X_0) - \int_0^t \mathscr{G} f(X_s) \mathrm{d}s \quad = \quad \text{local martingale}$$

Formally: $\mathscr{G} = \frac{\mathrm{d} T_t}{\mathrm{d} t}\Big|_{t=0}$ i.e.: $T_t = e^{t\mathscr{G}}$

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Theorem (Mazet ('97), Zhou ('03), Cuchiero et al. ('10,'11), etc.)

$$X \text{ is (PP)} \qquad \Longleftrightarrow \qquad \mathscr{G}\,\mathrm{Pol}_n(E) \subset \mathrm{Pol}_n(E), \text{ all } n \in \mathbb{N}.$$

Hence: $\mathscr{G}$ restricts to an operator $\mathscr{G}|_{\mathrm{Pol}_n(E)}$ on the finite-dimensional vector space $\mathrm{Pol}_n(E)$

Polynomial preserving processes

Functions/operators	In coordinates	
$\mathscr{G}\|_{\mathrm{Pol}_{\boldsymbol{n}}(E)}$	G	$\in \mathbb{R}^{N \times N}$
p	P	$\in \mathbb{R}^{N}$
$q := T_t p = e^{t\mathscr{G}} p$	$Q = e^{tG} P$	$\in \mathbb{R}^{N}$

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Consequence:

- Finding $T_t p(x) = \mathbb{E}_x[p(X_t)]$, for $p \in \mathrm{Pol}_n(E)$, only requires computing a matrix exponential.
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Building G from $\mathscr{G}$:

- C++ implementation (with Wahid Khosrawi-Sardroudi)

Polynomial preserving processes

Examples:

- Affine processes
- Pearson diffusions (Forman, Sørensen ('08)), $E \subset \mathbb{R}$:

$$\mathrm{d}X_t = (\beta + bX_t)\mathrm{d}t + \sqrt{\alpha + aX_t + AX_t^2}\,\mathrm{d}W_t$$

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- Representation of $\mathscr{G}|_{\mathrm{Pol}_n(\mathbb{R})}$ with respect to $1, x, x^2, \ldots$:

$$G = \begin{pmatrix} 0 & \beta & 2\frac{\alpha}{2} & 0 & \ldots & 0 \\ 0 & b & 2\left(\beta + \frac{a}{2}\right) & 3 \cdot 2\frac{\alpha}{2} & 0 & \vdots \\ 0 & 0 & 2\left(b + \frac{A}{2}\right) & 3\left(\beta_0 + 2\frac{a}{2}\right) & \ddots & 0 \\ 0 & 0 & 0 & 3\left(b + 3\frac{A}{2}\right) & \ddots & n(n-1)\frac{\alpha}{2} \\ \vdots & & & 0 & \ddots & n\left(\beta + (n-1)\frac{a}{2}\right) \\ 0 & & & & 0 & n\left(b + (n-1)\frac{A}{2}\right) \end{pmatrix}$$

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Next: Option pricing, estimation, filtering, etc.

Linear-rational term structure models (Filipović-L.-Trolle, 2013)

- Factor process and state price density:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \mathrm{d}M_t$$

$$\zeta_t = e^{-\alpha t}(1 + \psi^\top X_t)$$

for some martingale M, $\kappa \in \mathbb{R}^{d\times d}$, $\theta \in \mathbb{R}^d$, $\alpha \in \mathbb{R}$, $\psi \in \mathbb{R}^d$

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- Linear-rational bond prices and short rate:

$$P(t,T) = F(T-t, X_t) = e^{-\alpha(T-t)}\frac{1 + \psi^\top X_t + \psi^\top e^{-\kappa(T-t)}(X_t - \theta)}{1 + \psi^\top X_t}$$

$$r_t = -\partial \log P(t,T)|_{T=t} = \alpha - \frac{\psi^\top \kappa(\theta - X_t)}{1 + \psi^\top X_t}$$

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(iii) Tractable swaption pricing:

$$\frac{1}{\zeta_t}\mathbb{E}\left[\zeta_T \left(\sum_{i=1}^n c_i P(T, T_i)\right)^+ \,\middle|\, \mathscr{F}_t\right] = \frac{1}{\zeta_t}\mathbb{E}\left[\text{polynomial}(X_T)^+ \mid \mathscr{F}_t\right]$$

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(iv) Extensive empirical analysis, estimation using 15 years of weekly swaps and swaptions data

Polynomial preserving diffusions

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Suppose X is a **diffusion**: for some $b : E \to \mathbb{R}^d$, $\sigma : E \to \mathbb{R}^{d \times d}$,

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

With $a = \sigma\sigma^\top$, we have

$$\mathscr{G}f = \frac{1}{2}\operatorname{Tr}\left(a\,\nabla^2 f\right) + b^\top \nabla f$$

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Theorem (Cuchiero et al. ('12))

$$X \text{ is (PP)} \quad\iff\quad \begin{cases} a_{ij} \in \mathrm{Pol}_2(E) & i,j = 1,\ldots,d \\ b_i \in \mathrm{Pol}_1(E) & i = 1,\ldots,d \end{cases} \qquad (*)$$

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Question: For given E, a, b, what are the conditions under which the above SDE has a unique (in law) weak solution starting from any $x \in E$?

- Want (PP), so always assume $(*)$ holds

Existence of (PP) diffusions

Challenges:

- Non-Lipschitz volatility, e.g. $\sigma(x) = a(x)^{1/2}$
- $\mathscr{G}$ not uniformly elliptic
- State space may have complicated geometry

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Why care about more general state spaces?

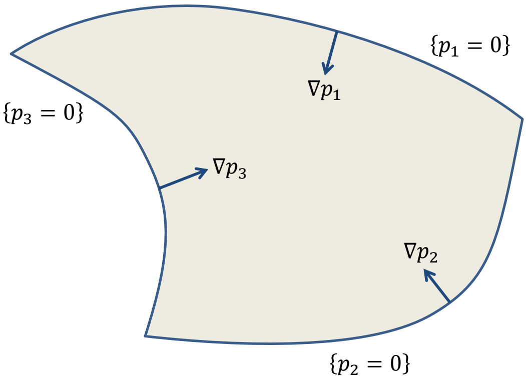
- Boundary geometry affects factor covariation structure
- Boundary attainment (important for implementation) can be analyzed in a general setting

Existence of (PP) diffusions

Class of state spaces: Let E be a basic closed semialgebraic set:

$$E = \{x \in \mathbb{R}^d : p(x) \geq 0 \text{ for all } p \in \mathscr{P}\},$$

for a finite collection $\mathscr{P} \subset \mathrm{Pol}(\mathbb{R}^d)$ of irreducible polynomials with non-constant sign.



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Lemma from real algebraic geometry on real principal ideals:
Assume $p \in \mathrm{Pol}(\mathbb{R}^d)$ is irreducible. The following are equivalent:

(i) p changes sign on $\mathbb{R}^d$

(ii) Every $q \in \mathrm{Pol}(\mathbb{R}^d)$ with $q = 0$ on $\{p = 0\}$ satisfies $q = pr$ for some $r \in \mathrm{Pol}(\mathbb{R}^d)$.

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Lemma (Real Nullstellensatz). Let I be the ideal generated by a family $\{r_1, \ldots, r_m\}$ of polynomials. The following are equivalent:

- The ideal I is **real**: for any $f_1, \ldots, f_k \in \mathrm{Pol}(\mathbb{R}^d)$,

$$f_1^2 + \cdots + f_k^2 \in I \quad \Longrightarrow \quad f_1, \ldots, f_k \in I.$$

- Any $f \in \mathrm{Pol}(\mathbb{R}^d)$ vanishing on $\bigcap_{i=1}^m \{r_i = 0\}$ lies in I:

$$f = f_1 r_1 + \cdots + f_m r_m \quad \text{for some} \quad f_1, \ldots, f_m \in \mathrm{Pol}(\mathbb{R}^d).$$

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Examples:

$$\begin{aligned}
&\mathbb{R}^d_+ : && \mathscr{P} = \{p_i(x) = x_i, i = 1, \dots, d\} \\
&[0,1]^d : && \mathscr{P} = \{p_i(x) = x_i,\ p_{d+i}(x) = 1 - x_i,\ i = 1, \dots, d\} \\
&\text{unit ball} : && \mathscr{P} = \{p(x) = 1 - \|x\|^2\} \\
&\mathbb{S}^m_+ : && \mathscr{P} = \{p_I(x) = \det x_{II},\ I \subset \{1, \dots, m\}\},
\end{aligned}$$

(In the last example, $\mathbb{S}^m_+ \subset \mathbb{S}^m \cong \mathbb{R}^d$, $d = m(m+1)/2$.)

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Candidate coefficients: Let a, b satisfy

$$a \in \mathbb{S}^d_+ \text{ on } E, \quad a_{ij} \in \mathrm{Pol}_2(\mathbb{R}^d), \quad b_i \in \mathrm{Pol}_1(\mathbb{R}^d).$$

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Question: Set $\sigma = a^{1/2}$ on E. For which a, b, $\mathscr{P}$ does

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \qquad X_0 = x.$$

have a unique (in law) E-valued weak solution for all $x \in E$?

Existence (necessity)

Theorem. Assume for each $x \in E$ there exists an E-valued weak solution to

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Then the condition

$$\forall p \in \mathscr{P}, \qquad \mathscr{G} p \geq 0, \quad a\nabla p = 0 \quad \text{on} \quad E \cap \{p = 0\}$$

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Reason: If X is E-valued, then $p(X_t) \geq 0$, all $p \in \mathscr{P}$. And,

$$p(X_t) = p(x) + \int_0^t \mathscr{G}p(X_s)\mathrm{d}s + \int_0^t \nabla p(X_s)^\top \sigma(X_s)\mathrm{d}W_s$$

Existence (sufficiency)

Theorem. Assume

$$\forall p \in \mathscr{P}, \qquad \mathscr{G} p > 0, \quad a\nabla p = 0 \quad \text{on} \quad \{p = 0\}.$$

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Note:

- X spends zero time at ∂E ...
- ... but it can nonetheless hit ∂E in general, e.g. BESQ(β):

$$\mathrm{d}X_t = \beta\mathrm{d}t + 2\sqrt{X_t}\mathrm{d}W_t, \qquad 0 < \beta < 2.$$

Existence (sufficiency)

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- Hence $L^0 = 0$

Existence (sufficiency)

Now apply the following lemma with $Y = p(X)$:

Lemma. Let Y be a continuous semimartingale with decomposition

$$Y_t = Y_0 + \int_0^t \mu_s \mathrm{d}s + M_t, \qquad Y_0 \geq 0,$$

where μ is continuous. If

$$\mu_t > 0 \quad \text{on} \quad \{Y_t = 0\}, \qquad L^0(Y) = 0,$$

then $Y \geq 0$ and $\{t : Y_t = 0\}$ is Lebesgue-null.

Note: For $Y = p(X)$ we have $\mu_t = \mathscr{G} p(X_t) > 0$ on $\{p(X_t) = 0\}$

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Goal: Relax $\mathscr{G}p > 0$ on $\{p = 0\}$: Allow boundary absorption

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- For each $p \in \mathscr{P}$ we have $a\nabla p = 0$ on $E \cap \{p = 0\}$.
- For any subset $\mathscr{R} \subset \mathscr{P}_{\rm abs}$, the gradients ∇r, $r \in \mathscr{R}$, are linearly independent on the set $E \cap \bigcap_{r \in \mathscr{R}} \{r = 0\}$.
- Each $p \in \mathscr{P}$ is irreducible and changes sign, and the set $E \cap \{p = 0\}$ is Zariski dense in $\{p = 0\}$.

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Existence: (a fairly) general case

Theorem. Suppose **Assumption** $(*)$ holds. Then for each $x \in E$ there exists an E-valued weak solution to

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \qquad X_0 = x.$$

This solution satisfies the following properties:

- for any $p \in \mathscr{P}_{\mathrm{refl}}$, the set $\{t : p(X_t) = 0\}$ is Lebesgue-null,
- for any $p \in \mathscr{P}_{\mathrm{abs}}$, the process $p(X_t)$ is absorbed at zero,

$$p(X_t) = 0 \text{ for all } t \geq \inf\{s \geq 0 : p(X_s) = 0\}.$$

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  are uniquely determined by a and b.
- Do we have **determinacy of the moment problem**?
- First answer: Not always! The log-normal distribution is indeterminate in the sense of the moment problem.

Uniqueness in law

Lemma. Assume **exponential moments exist**:

$$\text{For each } t \geq 0\text{, there is } \varepsilon > 0 \text{ with } \mathbb{E}\left[\, e^{\varepsilon \,\|X_t\|} \,\right] < \infty.$$

Then all finite-dimensional distributions $(X_{t_1}, \ldots, X_{t_m})$, and hence the law of X, are uniquely determined by a and b.

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Note: In particular, this covers

- (PP) diffusions on compact state spaces
- All affine diffusions
- Many interesting intermediate examples

An open question

- The proof of the lemma uses the following result by Petersen ('82):

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- The proof of the lemma uses the following result by Petersen ('82):

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  to deduce all finite-dimensional distributions are determinate.
- For geometric Brownian motion, the proof breaks because univariate determinacy fails. But does multivariate determinacy fail?
- **Open problem:** Find a process X, not geometric Brownian motion, such that for all $0 \leq t_1 < \ldots < t_m$, $(\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$,

  $$\mathbb{E}\left[X_{t_1}^{\alpha_1} \cdots X_{t_m}^{\alpha_m}\right] = \mathbb{E}\left[Y_{t_1}^{\alpha_1} \cdots Y_{t_m}^{\alpha_m}\right],$$

  where Y is geometric Brownian motion.
  - Can X be taken continuous?
  - Can X be taken Markovian?
  - ...

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Theorem. Suppose $p \in \mathscr{P}_{\mathrm{refl}}$, $p(X_0) > 0$, or both. Assume one of the following two conditions holds:

- $2\mathscr{G}p - h^\top \nabla p > 0$ on $E \cap \{p = 0\}$,
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Then $p(X_t) > 0$ for all $t > 0$.

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Example. BESQ(β): $\mathrm{d}X_t = \beta \mathrm{d}t + 2\sqrt{X_t}\mathrm{d}W_t$, $\mathscr{P} = \{p(x) = x\}$

- $a(x)p'(x) = 4x \cdot 1 = 4p(x)$. Hence $h = 4$
- $2\mathscr{G}p - hp' = 2\beta - 4$
- Theorem: $\beta \geq 2$ implies non-attainment (this is tight)

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Theorem. Suppose $x^* \in E \cap \{p = 0\}$ satisfies

$$2\mathscr{G}p(x^*) - h(x^*)^\top \nabla p(x^*) < 0.$$

Then $p(X_t)$ hits zero with positive probability, if X_0 is sufficiently near x^*.

Examples

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$\mathscr{G}p > 0$ **does not mean inward-pointing drift**

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- Consider the bivariate process (U, V) on $\mathbb{R} \times \mathbb{R}_+$ with dynamics

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- Set $(X, Y) = (U, V - U^2)$ on $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y \geq 0\}$:

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- For $0 < \alpha < 1$, the drift of (X, Y) points out of E:

$$b(x, y) = \begin{pmatrix} 0 \\ \alpha - 1 \end{pmatrix}$$

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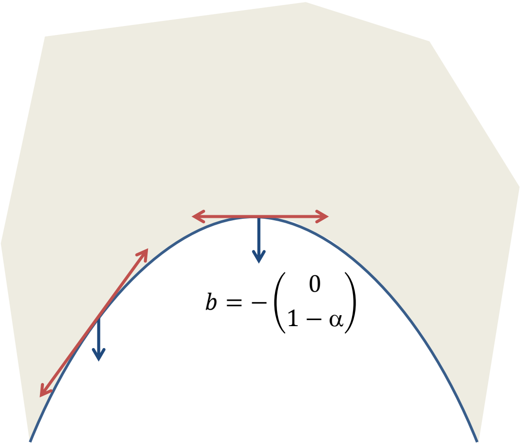

This happens for **non-convex state spaces**

Example 2

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- Let $\beta \in \mathbb{R}^d$, $B \in \mathbb{R}^{d \times d}$, $\alpha \in \mathbb{S}_+^d$. The SDE

$$\mathrm{d}X_t = (\beta + BX_t)\mathrm{d}t + \sqrt{1 - \|X_t\|^2}\,\alpha^{1/2}\,\mathrm{d}W_t$$

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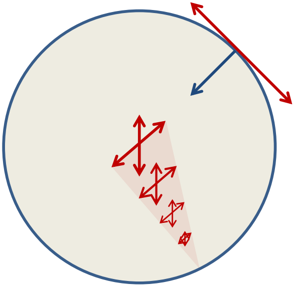
- But richer diffusion dynamics is possible:



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- Define

$$\mathscr{C} = \left\{ c : \mathbb{R}^d \to \mathbb{S}^d_+ : \begin{array}{l} c_{ij} \text{ is hom. poly. of degree 2 for all } i,j \\ c(0) = 0,\ c(x)x = 0 \text{ for all } x \in \mathbb{R}^d \end{array} \right\}$$

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- Then, with $c \in \mathscr{C}$,

$$a(x) = (1 - \|x\|^2)\,\alpha + c(x) \qquad b(x) = \beta + Bx$$

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- This is exhaustive among (PP) diffusions on E reflected at ∂E

Example 2

Examples of maps $c \in \mathscr{C}$

- Let $S_1, \ldots, S_m$ be a basis for $\mathrm{Skew}(d)$, the space of $d \times d$ skew-symmetric matrices ($m = \frac{d(d-1)}{2}$).
- Then, for any $\gamma_{kl} \in \mathbb{R}_+$,

$$c(x) = \sum_{k \leq l} \gamma_{kl} \, (S_k + S_l) \, x \, x^\top (S_k + S_l)^\top$$

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- This gives a large class of factor processes with nontrivial correlation and compact state space

Example 2'

Variations:

- The exterior of the unit ball: $E = \{x : 1 - \|x\|^2 \leq 0\}$
- Other quadratic (non-parabolic) sets: $E = \{x : 1 - x^\top H x \ \substack{\geq \\ \leq} \ 0\}$, where

$$H = \mathrm{Diag}(\pm 1, \ldots, \pm 1)$$

Conclusions

Moment asymptotics

Moments of (PP) diffusions

Recall the scalar case:

$$\mathrm{d}X_t = (\beta + bX_t)\mathrm{d}t + \sqrt{\alpha + aX_t + AX_t^2}\,\mathrm{d}W_t$$

Drift of $(1, X_t, X_t^2, \ldots, X_t^n)$:

$$\begin{pmatrix} 0 & \beta & 2\frac{\alpha}{2} & 0 & \dots & 0 \\ 0 & b & 2\left(\beta + \frac{a}{2}\right) & 3\cdot 2\frac{\alpha}{2} & 0 & \vdots \\ 0 & 0 & 2\left(b + \frac{A}{2}\right) & 3\left(\beta_0 + 2\frac{a}{2}\right) & \ddots & 0 \\ 0 & 0 & 0 & 3\left(b + 3\frac{A}{2}\right) & \ddots & n(n-1)\frac{\alpha}{2} \\ \vdots & & & 0 & \ddots & n\left(\beta + (n-1)\frac{a}{2}\right) \\ 0 & & & & 0 & n\left(b + (n-1)\frac{A}{2}\right) \end{pmatrix}^\top \begin{pmatrix} 1 \\ X_t \\ X_t^2 \\ X_t^3 \\ \vdots \\ X_t^n \end{pmatrix} \mathrm{d}t$$

Note: Moment asymptotics as $t \to \infty$ depends on diagonal entries

Goal: Look for something similar in the general case

Moments of (PP) diffusions

- Derive dynamics of $(1, X_t, X_t \otimes X_t, X_t^{\otimes 3}, \cdots, X_t^{\otimes m})$ in

$$\bigoplus_{n \geq 0} (\mathbb{R}^d)^{\otimes n}$$

Moments of (PP) diffusions

- Derive dynamics of $(1, X_t, X_t \otimes X_t, X_t^{\otimes 3}, \cdots, X_t^{\otimes m})$ in

$$\bigoplus_{n \geq 0} (\mathbb{R}^d)^{\otimes n}$$

- Think of $X_t \otimes X_t$ as

$$X_t X_t^\top = \begin{pmatrix} X_{1t}^2 & X_{1t}X_{2t} & \cdots & X_{1t}X_{dt} \\ X_{2t}X_{1t} & X_{2t}^2 & & \\ \vdots & & \ddots & \\ X_{dt}X_{1t} & & & X_{dt}^2 \end{pmatrix}$$

- Think of $X_t^{\otimes 3}$ as a 3-dimensional array with entries $X_{it}X_{jt}X_{kt}$
- Etc.

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- Consider the **shuffle product** ш:

$$\underbrace{(x_{i_1} \otimes \cdots \otimes x_{i_m})}_{\in (\mathbb{R}^d)^{\otimes m}} \ \text{ш} \ \underbrace{(x_{j_1} \otimes \cdots \otimes x_{j_n})}_{\in (\mathbb{R}^d)^{\otimes n}} = \sum_{k_1, \ldots, k_{m+n}} x_{k_1} \otimes \cdots \otimes x_{k_{m+n}}$$

where the sum runs over all **shuffles** of $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_n)$

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- Example:

$$\begin{aligned}(x \otimes y) \text{ ш } (u \otimes v) &= x \otimes y \otimes u \otimes v + x \otimes u \otimes y \otimes v \\ &+ x \otimes u \otimes v \otimes y + u \otimes x \otimes y \otimes v \\ &+ u \otimes x \otimes v \otimes y + u \otimes v \otimes x \otimes y\end{aligned}$$

Moments of (PP) diffusions

Lemma (formal Itô product rule) For $n \geq 2$,

$$\mathrm{d}(X_t^{\otimes n}) \;=\; X_t^{\otimes(n-1)} \shuffle \mathrm{d}X_t \;+\; X_t^{\otimes(n-2)} \shuffle (\mathrm{d}X_t)^{\otimes 2}$$

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In our case:

- $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \qquad (\mathrm{d}X_t)^{\otimes 2} = a(X_t)$

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- $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \qquad (\mathrm{d}X_t)^{\otimes 2} = a(X_t)$
- Drift $b(x) = \beta + Bx$, where

$$\beta \in \mathbb{R}^d, \qquad B : \mathbb{R}^d \to \mathbb{R}^d \quad \text{(linear)}$$

- Diffusion $a(x) = \alpha + A(x) + \aleph(x, x)$, where

$$\alpha \in (\mathbb{R}^d)^{\otimes 2}, \qquad A : \mathbb{R}^d \to (\mathbb{R}^d)^{\otimes 2}, \qquad \aleph : (\mathbb{R}^d)^{\otimes 2} \to (\mathbb{R}^d)^{\otimes 2}$$

(A and $\aleph$ linear)

This gives:

$$\mathrm{d}\begin{pmatrix} 1 \\ X_t \\ X_t^{\otimes 2} \\ \vdots \\ X_t^{\otimes m} \end{pmatrix} = \begin{pmatrix} 0 & 0 & & & \cdots & 0 \\ \psi_1 & \pi_1 & 0 & & & \\ \phi_2 & \psi_2 & \pi_2 & 0 & & \\ 0 & \phi_3 & \psi_3 & \pi_3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \phi_m & \psi_m & \pi_m \end{pmatrix} \begin{pmatrix} 1 \\ X_t \\ X_t^{\otimes 2} \\ \vdots \\ X_t^{\otimes m} \end{pmatrix} \mathrm{d}t + \text{(martingale)}$$

where

$$\begin{aligned} \phi_n &: (\mathbb{R}^d)^{\otimes(n-2)} &\longrightarrow& (\mathbb{R}^d)^{\otimes n} \\ \psi_n &: (\mathbb{R}^d)^{\otimes(n-1)} &\longrightarrow& (\mathbb{R}^d)^{\otimes n} \\ \pi_n &: (\mathbb{R}^d)^{\otimes n} &\longrightarrow& (\mathbb{R}^d)^{\otimes n} \end{aligned}$$

are linear maps given by

$$\begin{aligned} \phi_n(x_1 \otimes \cdots \otimes x_{n-2}) &= (x_1 \otimes \cdots \otimes x_{n-2}) ш \alpha \\ \psi_n(x_1 \otimes \cdots \otimes x_{n-1}) &= (x_1 \otimes \cdots \otimes x_{n-2}) ш A(x_{n-1}) + (x_1 \otimes \cdots \otimes x_{n-1}) ш \beta \\ \pi_n(x_1 \otimes \cdots \otimes x_n) &= (x_1 \otimes \cdots \otimes x_{n-2}) ш \aleph(x_{n-1}, x_n) + (x_1 \otimes \cdots \otimes x_{n-1}) ш Bx_n \end{aligned}$$

This gives:

$$
\mathrm{d}\begin{pmatrix} 1 \\ X_t \\ X_t^{\otimes 2} \\ \vdots \\ X_t^{\otimes m} \end{pmatrix} = \begin{pmatrix} 0 & 0 & & & \cdots & 0 \\ \psi_1 & \pi_1 & 0 & & & \\ \phi_2 & \psi_2 & \pi_2 & 0 & & \\ 0 & \phi_3 & \psi_3 & \pi_3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \phi_m & \psi_m & \pi_m \end{pmatrix} \begin{pmatrix} 1 \\ X_t \\ X_t^{\otimes 2} \\ \vdots \\ X_t^{\otimes m} \end{pmatrix} \mathrm{d}t + (\text{martingale})
$$

- Taking expectations yields linear ODE for $(1, \mathbb{E}[X_t], \ldots, \mathbb{E}[X_t^{\otimes n}])$
- Convergence of moments as $t \to \infty$ depends on eigenvalues of π_n:

$$
\begin{aligned}
\pi_n(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{n-1} \sqcup\!\sqcup\, Bx_n \\
+ x_1 \otimes \cdots \otimes x_{n-2} \sqcup\!\sqcup\, \aleph(x_{n-1}, x_n),
\end{aligned}
$$

  which depend only on B and $\aleph$.

- Affine case: all moments converge $\iff \mathrm{Re}(\sigma(B)) \subset (-\infty, 0)$

Conclusion

Summary:

- Polynomial term structure models provide a flexible, yet tractable framework for term structure modeling
- Existence and uniqueness of (PP) diffusions on a large class of state spaces
- Boundary attainment
- Moment asymptotics
- The semialgebraic structure of the state spaces connects the study of (PP) processes to real algebraic geometry
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Going forward:

- Uniqueness in full generality
- Transition densities? Further large time properties? Jumps?
- Further applications: Equities, commodities, ...

Thank You!