Multivariate B11 copula family for risk capital aggregation

a copula engineering approach

Prepared for "Talks in Financial and Insurance Mathematics"
on invitation of Prof. Paul Embrechts, ETH Zurich

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Agenda

1. Copula models for risk capital aggregation
   - Why copula models
   - Available copula families
2. Multivariate B11 copula
   - Definition & properties
   - Canonical representation
   - Relation to variance covariance method
3. Tail dependence
   - Bivariate and multivariate tail dependence
   - Calibration of canonical representation
   - Realization conditions
4. Sparse parameterization
   - Structure
   - Calibration via nonlinear programming
5. Kronecker convolution

Copulas for risk capital models

- Large insurances are expected to have "internal models" which quantify risk dependence and risk capital, cf. *McNeil, Frey, and Embrechts (2005).*

- Variance covariance aggregation method is a well known and robust approach but delivers only point estimates of the distributions

- Copulas (multivariate probability distribution on $[0,1]^N$ with uniform margins)
  - can link simulations of individual Monte Carlo models
  - fix dependence between several random variables
  - do not depend on univariate distributions
  - separate dependence structure from marginal distributions

  Sklar theorem (stylized):

$$F_{XYZ}(x,y,z) = \mathrm{C}(F_X(x), F_Y(y), F_Z(z))$$
$$\mathrm{C}(p,q,r) = F_{XYZ}(F_X^{\leftarrow}(p), F_Y^{\leftarrow}(q), F_Z^{\leftarrow}(r))$$

  - diversification emerges naturally at all quantiles

Bivariate dependence measures

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- **(Pearson) correlation ρ *depends on margins***
  - usual correlation of random variables representing losses
  - *attainable correlations can be strictly smaller than 1*

- **Rank (Spearman) correlation ρ_S**
  - correlation of random variables from copula before applying marginal distributions
  - does not depend on margin

- **Conditional quantile exceedance probability (upper tail)**

$$cqep(q) = P\left(Y \geq F_Y^{\leftarrow}(q) \middle| X \geq F_X^{\leftarrow}(q)\right)$$

- **Tail dependence coefficient (upper tail)**

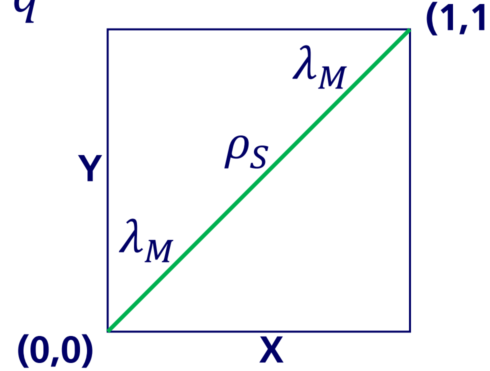
$$\lambda^u = \lim_{q \to 1^-} P\left(Y \geq F_Y^{\leftarrow}(q) \middle| X \geq F_X^{\leftarrow}(q)\right)$$

- Note:
  - above measures are symmetric in copula variables
  - tail dependence definitions can be extended to multiple variables

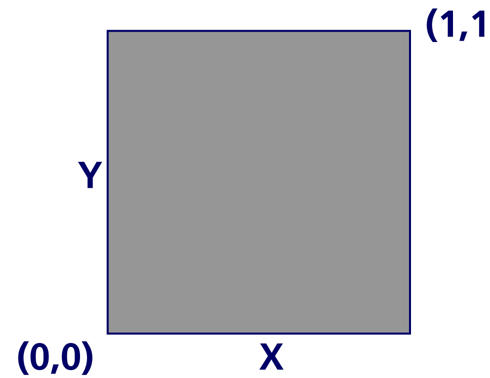
Fundamental copulas

- For all copulas below: $cqep(q)$ is a linear function of q

- Comonotonicity copula **M** (**M** stands for max)
  - **variables are perfectly dependent**
  - Spearman correlation $\rho_S = 1$
  - tail dependency coefficient $\lambda_M = 1$
  - counter monotonicity dependence coefficient $\lambda_W = 0$

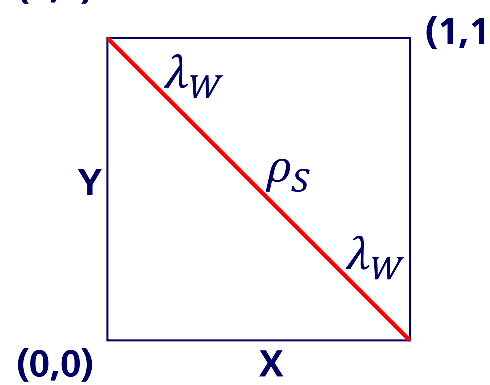


- Independence copula $\boldsymbol{\Pi}$ ($\boldsymbol{\Pi}$ stands for product)
  - **variables are perfectly independent**
  - Spearman correlation $\rho_S = 0$
  - tail dependency coefficients $\lambda_M = 0$
  - counter monotonicity dependence coefficient $\lambda_W = 0$



- Counter monotonicity copula **W** (only in 2D case)
  - **variables are perfectly counter-dependent**
  - Spearman correlation $\rho_S = -1$
  - tail dependence coefficient $\lambda_M = 0$
  - counter monotonicity dependence coefficient $\lambda_W = 1$

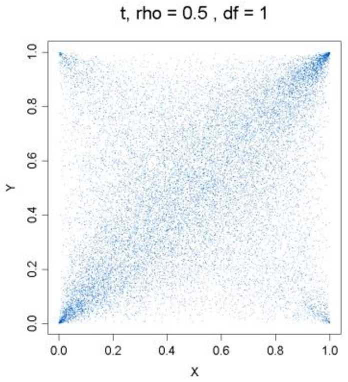
  (**W** stands for a copula opposite to **M**)



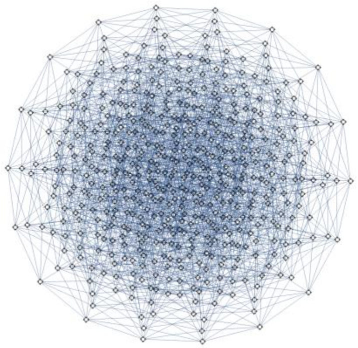
Review of popular copula families

- **Gauss copula** with correlation parameter matrix $\mathrm{P} \succ 0$:
  - fast decay of the conditional quantile exceedance probabilities as q → 1
  - no tail dependence

- **t copula** with correlation parameter matrix P and df degrees of freedom (scalar):
  - *large* df value (50 - ∞): close to Gauss copula, practically no tail dependence
  - *medium* df value (10 – 50): only minor tail dependence
  - *small* df value (0 – 10):
    - anti-dependence, underestimation of the capital
    - significant tail dependence, reduced ability to reproduce dependence matrices



- **Nested Archimedean and vine copulas**
  - offer extreme freedom in selection of the copula structure (combinatorial explosion, n!/2), creating potential artifacts
  - can be potentially difficult to calibrate using available expert judgment data



Desired copula properties for capital models in insurances

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1. Tail dependence
2. Simple structure, no artifacts
3. No counter monotonicity
4. Relationship to variance covariance method
5. Possibility to calibrate using experts judgment like tail dependence
6. Possibility to calibrate larger "product" structures

What is the appropriate copula family?

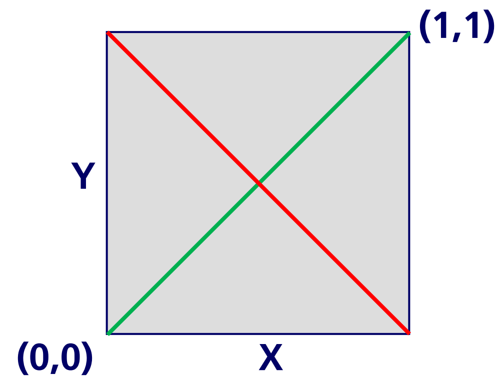
Fréchet and related 2D copula families

Suggestion of *Adrian Zweig, Head of Risk Analytics* regarding appropriate copula (2011): use mix of independence with comonotonicity

- **Fréchet copula** *Nelsen (1999)*
  convex combination of **M**, **Π**, and **W**

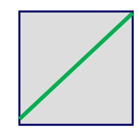
$$C_{m,w} = mM + wW + (1-m-w)\Pi$$
$$1 \geq m \geq 0, \quad 1 \geq w \geq 0, \quad 1 \geq m + w$$

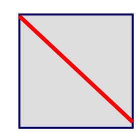
  - Spearman correlation $\rho_S = m - w$
  - tail dependence coefficients $\lambda_M = m$
  - counter monotonicity coefficients $\lambda_W = w$



- **Linear Spearman copula** *Hürlimann (2002)*
  (**Π**, and either **M** or **W**)

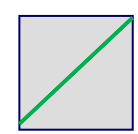
$$C_\theta = \begin{cases} \theta M + (1-\theta)\Pi & \text{if } \theta \geq 0 \\ |\theta| W + (1-|\theta|)\Pi & \text{if } \theta < 0 \end{cases}$$





- **B11 copula** *Joe (1997)*
  (**M** and **Π**)

$$C_\theta = \theta M + (1-\theta)\Pi \qquad \text{for } \theta \geq 0$$



Bivariate B11 Copula

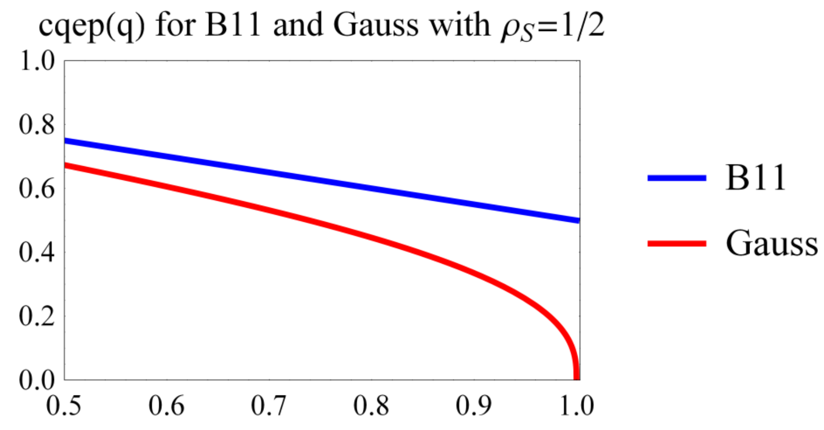
The bivariate B11 copula family is a convex combination (with mixing parameter $\theta \in [0-1]$) of bivariate fundamental copulas $\mathbf{M}$ and $\mathbf{\Pi}$

$$C_\theta(x_1, x_2) = \theta\mathbf{M}(x_1, x_2) + (1-\theta)\Pi(x_1, x_2) = \theta \min(x_1, x_2) + (1-\theta)x_1x_2$$

Property 1: conditional quantile exceedance probability functions are linear:

$$P(\mathrm{X}_2 \leq q | \mathrm{X}_1 \leq q) = \theta + (1-\theta)q$$

$$P(\mathrm{X}_2 \geq q | \mathrm{X}_1 \geq q) = \theta + (1-\theta)(1-q).$$



Property 2: Tail dependence and Spearman's ρ_S coefficients are equal to θ: $\lambda_l = \lambda_u = \rho_S = \theta$.

Property 3: With probability θ $X_1 = X_2$ (use of $\mathbf{M}$ copula), otherwise X_1 and X_2 are independent (use of $\mathbf{\Pi}$ copula), hence $P(X_1 = X_2) = \theta$.

B11 Canonical Factor Decomposition

The multivariate B11 copula is defined as convex combination of **canonical copulas**
(extension of fundamental copulas $\mathbf{M}$ and $\mathbf{\Pi}$ to multiple dimensions)

- 3D example:

1	1	1	M
1	1	2	MΠ
1	2	1	MΠ
1	2	2	MΠ
1	2	3	Π

- 4D example:

1	1	1	1
1	1	1	2
1	1	2	1
1	1	2	2
1	1	2	3
1	2	1	1
1	2	1	2
1	2	1	3
1	2	2	1
1	2	2	2
1	2	2	3
1	2	3	1
1	2	3	2
1	2	3	3
1	2	3	4

B11 Canonical Copulas in 3D

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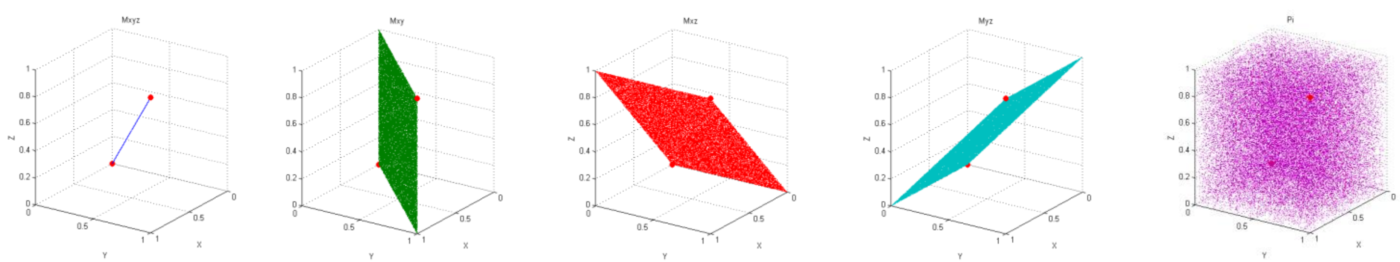
- Factors and densities

partitioning	x_1	x_2	x_3	canonical copula	cumulative copula density	probability mass
{{1,2,3}}	f_1	f_1	f_1	cc111	$\min(x_1,x_2,x_3)$	$\delta(x_1-x_2,\ x_1-x_3)$
{{1,2},{3}}	f_1	f_1	f_2	cc112	$\min(x_1,x_2) \bullet x_3$	$\delta(x_1-x_2)$
{{1,3},{2}}	f_1	f_2	f_1	cc121	$\min(x_1,x_3) \bullet x_2$	$\delta(x_1-x_3)$
{{1},{2,3}}	f_1	f_2	f_2	cc122	$x_1 \bullet \min(x_2,x_3)$	$\delta(x_2-x_3)$
{{1},{2},{3}}	f_1	f_2	f_3	cc123	$x_1 \bullet x_2 \bullet x_3$	1

- Distribution plots



- Tail dependence (= Spearman correlation)

$$\Lambda_{111} = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix},\ \Lambda_{112} = \begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix},\ \Lambda_{121} = \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix},\ \Lambda_{122} = \begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix},\ \Lambda_{123} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Multivariate B11 Copula (1)

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- Related family (MLS) first proposed by *Hürlimann (2002)*
- Multivariate B11 copula is a convex combination
  $C(x_1, \dots, x_n) = \sum_{cc \in CC} \mu_{cc} C_{cc}(x_1, \dots, x_n)$, where $\mu_{cc} \geq 0$ and $\sum_{cc \in CC} \mu_{cc} = 1$
- The family is closed under convex combination
- Conditional quantile exceedance probabilities, bivariate and multivariate tail dependence coefficients, and Spearman correlation ρ_S of a convex combination $C_\mu = \mu_1 C_1 + \dots + \mu_s C_s$ of copulas $C_1 \dots C_s$, denoted $f(C_\mu)$, can be also calculated as

$$f\left(C_\mu\right) = \mu_1 f(C_1) + \dots + \mu_s f(C_s)$$

- Consequently , tail dependence matrix of a copula is a convex combination of matrices of canonical copulas:

$$\Lambda_{2,3} = \mu_{111}\Lambda_{111} + \mu_{112}\Lambda_{112} + \mu_{121}\Lambda_{121} + \mu_{122}\Lambda_{122} + \mu_{123}\Lambda_{123} =$$

$$\begin{bmatrix} 1 & \mu_{111} + \mu_{112} & \mu_{111} + \mu_{121} \\ \mu_{111} + \mu_{112} & 1 & \mu_{111} + \mu_{122} \\ \mu_{111} + \mu_{121} & \mu_{111} + \mu_{122} & 1 \end{bmatrix}$$

Example of Canonical Parameterization

- In 3 dimensions the bivariate tail dependence matrix is

$$\alpha \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + \beta \begin{bmatrix}1 & 1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix} + \gamma \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix} + \delta \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} + \varepsilon \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

- Rewritten as

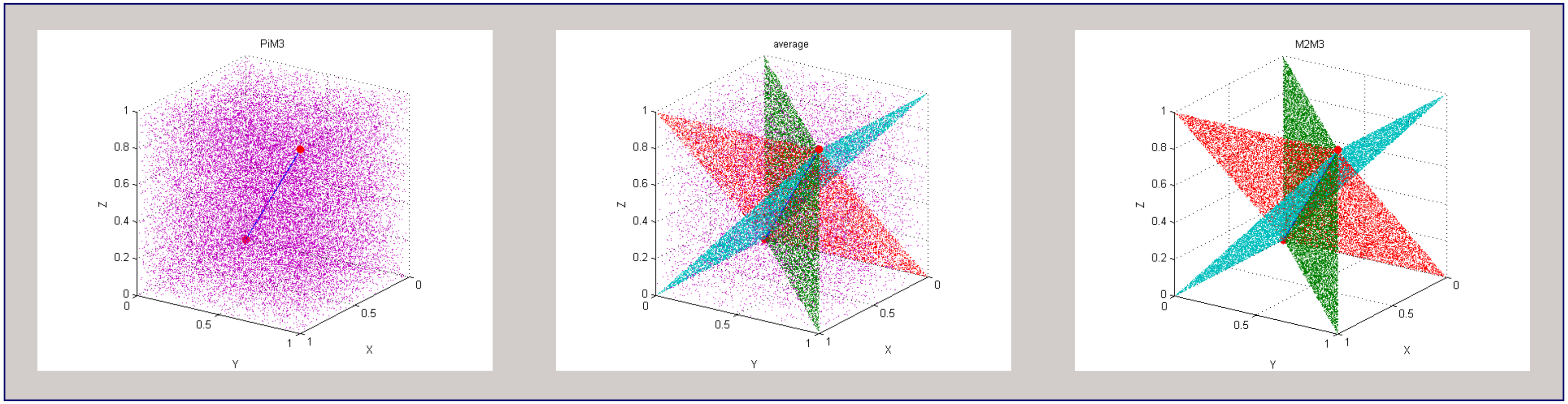
$$\Lambda = \rho_S = \begin{bmatrix}1 & \beta+\varepsilon & \gamma+\varepsilon\\ \beta+\varepsilon & 1 & \delta+\varepsilon\\ \gamma+\varepsilon & \delta+\varepsilon & 1\end{bmatrix}$$

- Let us find a realization for

$$\Lambda = \rho_S = \begin{bmatrix}1 & 0.4 & 0.4\\0.4 & 1 & 0.4\\0.4 & 0.4 & 1\end{bmatrix}$$

- There exists a *convex set* of realizations of the above matrix, including

$\alpha = 0.6,\ \beta = \gamma = \delta = 0, \varepsilon = 0.4$ $\alpha = \beta = \gamma = \delta = \varepsilon = 0.2$ $\alpha = 0, \beta = \gamma = \delta = 0.3, \varepsilon = 0.1$



- Any realization of the above matrix must have ε coefficient (full dependency of all 3 variables) in the range [0.1 – 0.4]

Multivariate B11 Copula (2)

- Note:
  - canonical parameterization is unique
  - decomposition of tail dependence matrix is usually not unique or may not even exist

- Number of canonical copulas (set partitioning ways) grows as (*Bell, 1934*) Bell number B_n:

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$$

Note that Bell polynomials appear in *Hofert (2012)*

n	B_n
1	1
2	2
3	5
4	15
5	52
6	203
7	877
8	4’140
9	21’147
10	115’975
11	678’570
12	4’213’597
13	27’644’437
14	190’899’322
15	1’382’958’545
…	…
100	4.76 x 10^115
1000	2.99 x 10^1927
4000	4.84 x 10 ^9706

Variance Covariance Equivalence (1)

- Equivalence of Variance Covariance (VC) and copula aggregations is of interest for risk capital modeling (also beyond normal distributions)

- Consider tri-variate B11 copula with weights μ_{1jk} and tail dependence matrix *P*
$$P = \mu_{123}\Lambda_{123} + \mu_{112}\Lambda_{112} + \mu_{121}\Lambda_{121} + \mu_{122}\Lambda_{122} + \mu_{111}\Lambda_{111}$$

- Aggregate of normal marginal distributions via multivariate B11 copula is mixture of normal distributions with density function as follows; cf. *Hürlimann (2002)*

$$\begin{aligned}
&f_{B11\,aggregate}(x) = \\
&\mu_{123} f_1 * f_2 * f_3\,(x) + \\
&\mu_{112}(f_1 \oplus f_2) * f_3\,(x) + \\
&\mu_{121}(f_1 \oplus f_3) * f_2\,(x) + \\
&\mu_{122}(f_2 \oplus f_3) * f_1\,(x) + \\
&\mu_{111} f_1 \oplus f_2 \oplus f_3(x) = \\
&\mu_{123} GaussPDF(0, ([\sigma_1, \sigma_2, \sigma_3]\Lambda_{123}[\sigma_1, \sigma_2, \sigma_3]^T)^{1/2}, x) + \\
&\mu_{112} GaussPDF(0, ([\sigma_1, \sigma_2, \sigma_3]\Lambda_{112}[\sigma_1, \sigma_2, \sigma_3]^T)^{1/2}, x) + \\
&\mu_{121} GaussPDF(0, ([\sigma_1, \sigma_2, \sigma_3]\Lambda_{121}[\sigma_1, \sigma_2, \sigma_3]^T)^{1/2}, x) + \\
&\mu_{122} GaussPDF(0, ([\sigma_1, \sigma_2, \sigma_3]\Lambda_{122}[\sigma_1, \sigma_2, \sigma_3]^T)^{1/2}, x) + \\
&\mu_{111} GaussPDF(0, ([\sigma_1, \sigma_2, \sigma_3]\Lambda_{111}[\sigma_1, \sigma_2, \sigma_3]^T)^{1/2}, x)
\end{aligned}$$

$\oplus$ quantile addition
$*$ pdf convolution

Λ_{1jk} matrices contain only 0 or 1

all identical if
$$\sigma^T P \sigma \equiv \sigma^T \Lambda_{1jk} \sigma$$

$$\sigma^T P \sigma = \mu_{123}\sigma^T\Lambda_{123}\sigma + \mu_{112}\sigma^T\Lambda_{112}\sigma + \mu_{121}\sigma^T\Lambda_{121}\sigma + \mu_{122}\sigma^T\Lambda_{122}\sigma + \mu_{111}\sigma^T\Lambda_{111}\sigma$$

Variance Covariance Equivalence (2)

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For example, VC and B11 aggregations, with P expressed as a convex combination of canonical copulas in B11, are equivalent for normal marginal distributions if

- the canonical copulas having non-zero weights are identical up to one or more permutations of variables
- random variables involved in this permutation(s) have the same standard deviations

$$P = \begin{bmatrix} 1 & a & 1-a \\ a & 1 & 0 \\ 1-a & 0 & 1 \end{bmatrix} \qquad \sigma_2 = \sigma_3,\ \sigma_1 \text{ arbitrary}$$

$$P = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix} \qquad \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$$

In general, the set of equivalence forms ***algebraic variety***

Example provided by *Markus Engeli, Zurich Insurance Company*:

$$P = \begin{bmatrix} 1 & 1 & a & 0 \\ 1 & 1 & a & 0 \\ a & a & 1 & 1-a \\ 0 & 0 & 1-a & 1 \end{bmatrix} \qquad \sigma_1, \sigma_2, \sigma_3 \text{ arbitrary}, \sigma_4 = \sigma_1 + \sigma_2$$

Also Capital Relevant: Multi-variate Tail Dependence

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Following *De Luca and Rivieccio (2012),* up to a permutation of variables, multivariate *lower* tail dependence of k-1 copula variables with respect to k-th variable can be defined as

$$\lambda^{l}{}_{1..k-1|k} = \lim_{q\to 0^+} P(\max(F_1(X_1),\dots,F_{k-1}(X_{k-1})) \le q \,|F_k(X_k) \le q) = \lim_{q\to 0^+} \frac{C(q,\dots,q)}{q}$$

Note that this coefficient can be estimated from empirical samples and is symmetric with respect to all k variables. Therefore it is possible to refer to this coefficient as to joint multivariate tail dependence and denote it $\boldsymbol{\lambda^{l}{}_{1..k}}$

Properties:

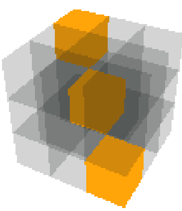
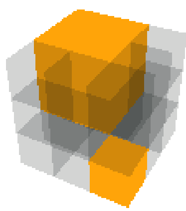
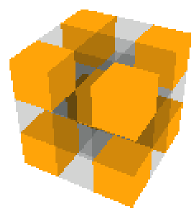
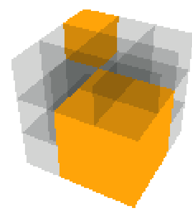
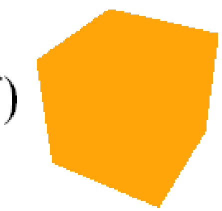
1. Direct extension of 2D case
2. Linearity $\lambda(\alpha C_1 + (1-\alpha)C_2) = \alpha\lambda(C_1) + (1-\alpha)\lambda(C_2)$
3. Measure for canonical copulas
   - 0 for $\boldsymbol{\Pi}$ and $\boldsymbol{W}$
   - 1 for $\boldsymbol{M}$
4. From 1 & 2 follows for B11 and Frechet families: the measure is the sum of coefficients of involved $\boldsymbol{M}$ copulas in the canonical decomposition
5. λ can be evaluated directly from (i) sample and (ii) copula function

Attainability of Bivariate and Multivariate Tail Dependence for Canonical Parameterization

Theorem

(Attainability of tail dependence)

All attainable k-variate tail dependence arrays of an n-variate B11 copula form a convex set (generated by the k-variate tail dependence arrays $\Lambda^{B11}{}_{k,n}$ of B11 canonical copulas)

α  $+ \ \beta$  $+ \ \gamma$  $+ \ \delta$  $+ \ (1-\alpha-\beta-\gamma-\delta)$ 

Example constrained ($\mu \geq 0, \ \sum \mu = 1$) linear / quadratic optimization problems

$$\boxed{\text{find } \min_{\mu} \left\| \Lambda_{2,n}(\mu) - \Lambda_2 \right\|_\infty}$$

$$\boxed{\text{find } \min_{\mu} \left\| \Lambda_{2,n}(\mu) - \Lambda_2 \right\|_F^{\,2}}$$

- minimum can always be found
- exact match is not always possible

Multivariate Tail Dependence Conditions for B11 Copula (1)

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- Each canonical copula except Π influences one or more tail dependence coefficients, as reflected in matrix A

A

CC	λ_{12}	λ_{13}	λ_{23}	λ_{123}	all
μ_{111}	1	1	1	1	1
μ_{112}	1	0	0	0	1
μ_{121}	0	1	0	0	1
μ_{122}	0	0	1	0	1
μ_{123}	0	0	0	0	1

A^{-T}

CC	λ_{12}	λ_{13}	λ_{23}	λ_{123}	all
μ_{111}	0	0	0	1	0
μ_{112}	1	0	0	-1	0
μ_{121}	0	1	0	-1	0
μ_{122}	0	0	1	-1	0
μ_{123}	-1	-1	-1	2	1

$$\mu^T A = \lambda^T$$
$$\lambda^T = [\lambda_{12}, \lambda_{13}, \lambda_{23}, \lambda_{123}, 1]$$
$$\sum \mu_{ijk} = 1$$

$$A^{-T}\lambda = \mu$$
$$\mu = [\mu_{111}, \mu_{112}, \dots, \mu_{123}]^T$$
$$\mu_{ijk} \geq 0$$

$$\boxed{A^{-T}\lambda \geq 0}$$

Multivariate Tail Dependence Conditions for B11 Copula (2)

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Theorem

(necessary and sufficient conditions for realization of multivariate tail dependence)

With the contribution matrix $\boldsymbol{A}$ (as defined previously) the complete vector of multivariate tail dependence coefficients, $\boldsymbol{\lambda}$, can be realized if all entries of the vector $\boldsymbol{A}^{-\boldsymbol{T}}\boldsymbol{\lambda}$ are non-negative.

Remarks

- Number of conditions is equal to the corresponding Bell number
- Conditions can be treated as sufficient conditions for any copula (general sufficient and necessary conditions are not known)

Example: Tail Dependence of B11 Copula in 3D and 4D

- Coefficients like λ_{124} and λ_{1234} denote multivariate tail dependence
- Coefficients like λ_{12-34} denote multi-factor tail dependence coefficients, specific to canonical copulas like cc1122 (common factor for variables 1 & 2 and other one for 3 & 4) and can be treated as *free variables* restricted to [0-1] range
- Conditions of this type can be generated automatically for arbitrary copula order using a Computer Algebra System

$$\lambda_{123} \geq 0$$
$$\lambda_{12} \geq \lambda_{123}$$
$$\lambda_{13} \geq \lambda_{123}$$
$$\lambda_{23} \geq \lambda_{123}$$
$$1 + 2\,\lambda_{123} \geq \lambda_{12} + \lambda_{13} + \lambda_{23}$$

$$\lambda_{1234} \geq 0$$
$$\lambda_{123} \geq \lambda_{1234}$$
$$\lambda_{124} \geq \lambda_{1234}$$
$$\lambda_{134} \geq \lambda_{1234}$$
$$\lambda_{234} \geq \lambda_{1234}$$
$$\lambda_{12-34} \geq 0$$
$$\lambda_{13-24} \geq 0$$
$$\lambda_{14-23} \geq 0$$
$$\lambda_{12} - \lambda_{123} - \lambda_{124} + \lambda_{1234} - \lambda_{12-34} \geq 0$$
$$\lambda_{13} - \lambda_{123} - \lambda_{134} + \lambda_{1234} - \lambda_{13-24} \geq 0$$
$$\lambda_{23} - \lambda_{123} - \lambda_{234} + \lambda_{1234} - \lambda_{14-23} \geq 0$$
$$\lambda_{14} - \lambda_{124} - \lambda_{134} + \lambda_{1234} - \lambda_{14-23} \geq 0$$
$$\lambda_{24} - \lambda_{124} - \lambda_{234} + \lambda_{1234} - \lambda_{13-24} \geq 0$$
$$\lambda_{34} - \lambda_{124} - \lambda_{234} + \lambda_{1234} - \lambda_{12-34} \geq 0$$
$$\lambda_{34} - \lambda_{124} - \lambda_{234} + \lambda_{1234} - \lambda_{12-34} \geq 0$$
$$2\sum \lambda_{\mathrm{ijk}} + \sum \lambda_{\mathrm{ij-kl}} \geq \sum \lambda_{\mathrm{ij}} + 3\lambda_{1234}$$

- Selection of $\lambda_{123} = \min(\lambda_{12}, \lambda_{12}, \lambda_{23})$ in 3D case leads to Joe's condition *Joe (1997)*

Bivariate Tail Dependence Benchmark

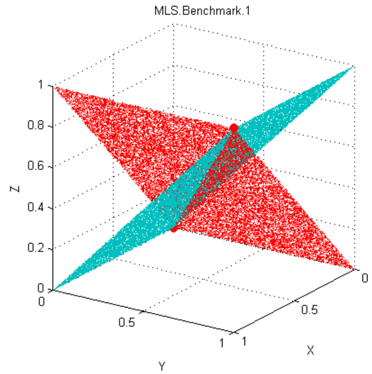
ZURICH

- **B11 copula realizes $\Lambda_{\alpha,n}$ for any $n>2$ and $\alpha \leq 1/(n-1)$**

$$\Lambda_{\alpha,n} = \begin{bmatrix} 1 & 0 & \cdots & \alpha \\ 0 & 1 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & 1 \end{bmatrix}_{n\,x\,n}$$

$$\Lambda_{\frac{1}{n-1},\, n}$$

- Examples:
  - n = 3 and α =1/2 $\quad \Lambda_{\frac{1}{2},\, 3}$
    - Joe's condition becomes sharp
    - PSD condition becomes sharp
    - also t-copula can realize it *Nikoloulopoulos et al. (2008)*



  - n = 4 and α =1/3 $\quad \Lambda_{\frac{1}{3},\, 4}$
    - t-copula can no longer realize it
    - Archimedean copulas can no longer realize it, *Hofert (2012,2013)*
    - B11 realizes it!
  - n = 4 and α = 1/2 $\quad \Lambda_{\frac{1}{2},\, 4}$
    - PSD and Joe's conditions satisfied
    - no decomposition as a convex sum of Bell factors
    - conjecture: this dependency cannot be realized by any copula

Sparse Parameterization

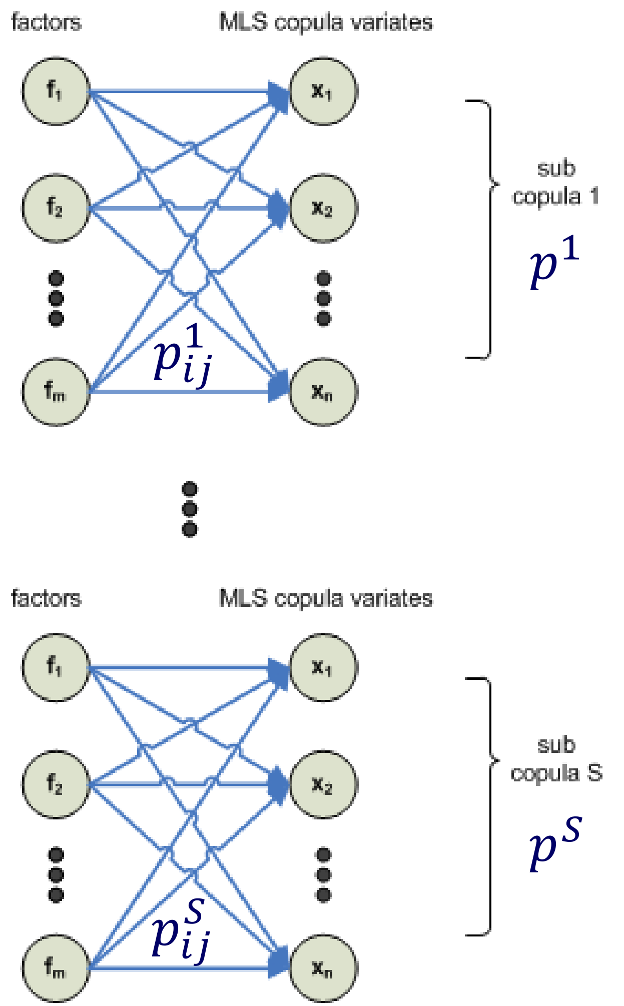
Idea: limiting the number of parameters by imposing particular structures, while retaining flexibility (parsimonious modeling principle), *Hürlimann (2012)*

- The sparse parameterization is defined as follows:
  1. there are m independent random factors $f_1, \dots, f_m$, uniformly distributed on [0-1]
  2. the copula is a convex combination of S sparse sub-copulas, each with probability $p_s, s = 1 \dots S$
  3. each x_j variable, $j = 1 \dots N$, within a single sparse sub-copula copula s is generated from i-th factor with probability p_{ij}^s (Bernoulli mixtures)

- Parameters: $p = \{p_s, p_{ij}^s\}$

  (sub copula mixture weights and factor usage probabilities)
- Bivariate tail dependence matrix:

$$\Lambda_{2,n}(p) = \left[\sum_{s=1}^{S} p_s \left(\sum_{i=1}^{m} p_{ik}^s p_{il}^s \left(1 - \delta_{kl}^{Kronecker}\right) + \delta_{kl}^{Kronecker}\right)\right]_{kl}$$



Nonlinear Optimization Tasks

- Optimization tasks with respect to parameters $p = \{p_s, p_{ij}^s\}$

  - Matching bivariate tail dependence:

    $$\min_p \left\| \Lambda_{2,n}(p) - \Lambda_2 \right\|_F^2$$

  - Matching bivariate and fixing trivariate average tail dependence:

    $$\min_p \ \left\| \Lambda_{2,n}(p) - \Lambda_2 \right\|_F^2 \quad \text{and} \quad \lambda_{min\ 3D} \leq \lambda_{avg\ 3D} \leq \ \lambda_{max\ 3D}$$

    $$\text{where } \lambda_{avg\ 3D} = \frac{1}{n(n-1)(n-2)} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(1 - \delta_{jkl}{}^{Kronecker}\right) \sum_{s=1}^{S} p_s \sum_{i=1}^{m} p_{ij}^s p_{ik}^s p_{il}^s$$

  - Matching bivariate and tri-variate tail dependence:

    $$\min_p \left[\alpha \left\| \Lambda_{2,n}(p) - \Lambda_2 \right\|_F^2 + (1-\alpha) \left\| \Lambda_{3,n}(p) - \Lambda_3 \right\|_F^2 \right]$$

  - In addition: ability to fix particular coefficient(s)

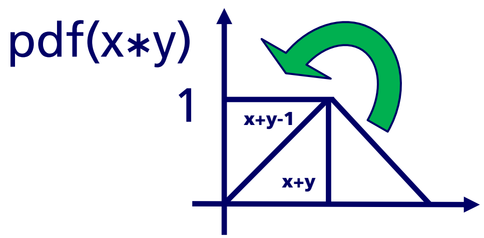
- Non-convex optimization, local minima, slow convergence possible
- Implementation in S+/R using NuOpt/RNuOpt, *Mathematical Systems Inc. (2008)*

Kronecker Structure: Circular Convolution

Practical need – risk model for several units and lines of business

Parameterization of tail dependence as a "product" of dependencies for unit location (geography) and line of business

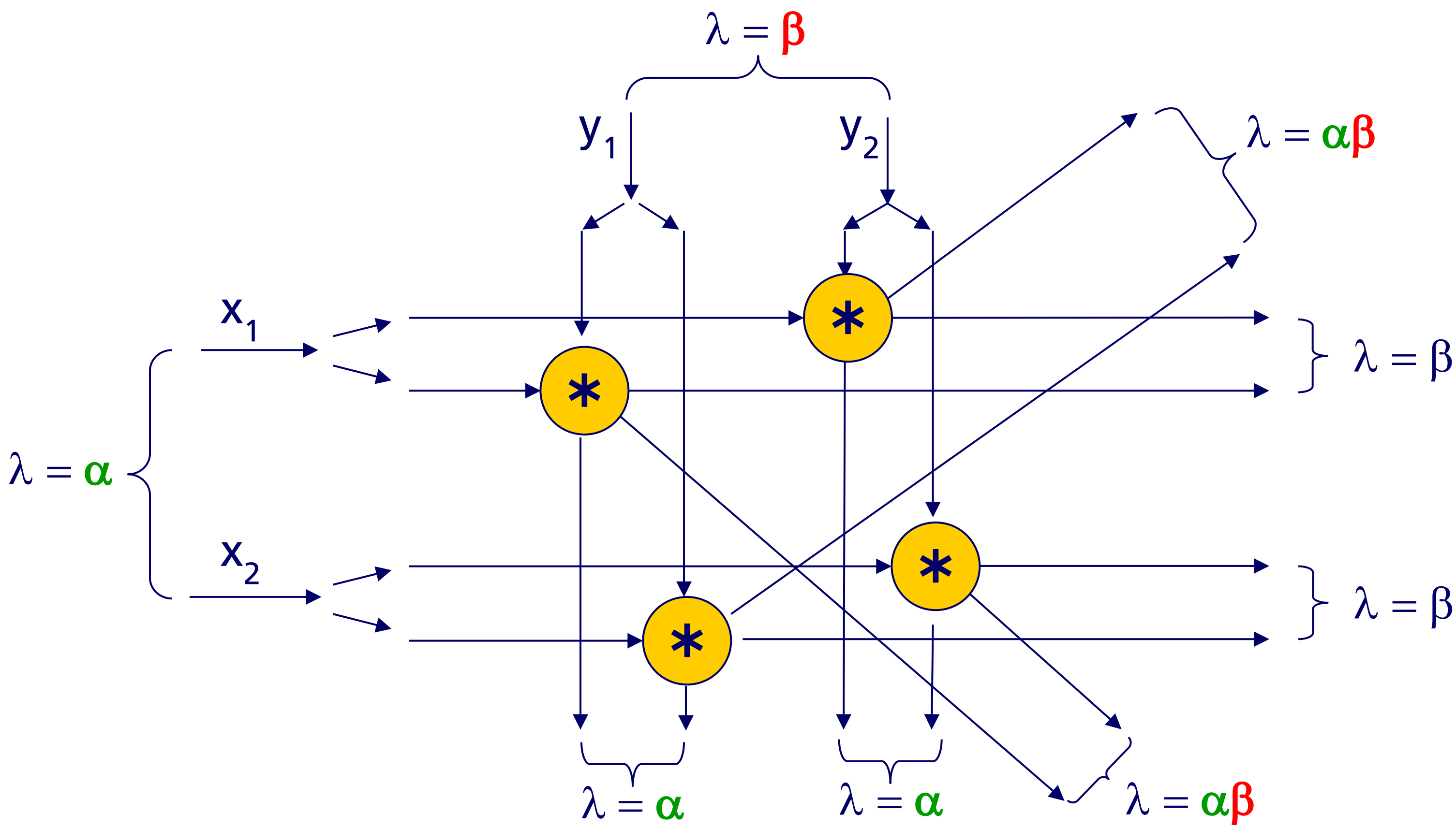
- x_i and y_j are generated from Fréchet copulas
- circular convolution : $x_i * y_j$ denotes $\begin{cases} x_i + y_j & \text{if } x_i + y_j < 1 \text{ and} \\ x_i + y_j - 1 & \text{otherwise} \end{cases}$



Note:

for x_i and y_j generated from multivariate Gauss distribution and circular time series convolution the scheme delivers product of correlations

Kronecker Structure: Tail Dependence Product Scheme (1)



- Note that $x * y_k$ and $x * y_l$ are independent if y_k and y_l are independent and identical if y_k are y_l comonotone
- λ can mean dependence or anti-dependence coefficient

Kronecker Structure: Tail Dependence Product Scheme (2)

Theorem
(Kronecker convolution of two independent multivariate B11 copulas)

Let from $x = x_1, \dots, x_q$ and $y = y_1, \dots, y_r$ be random variates from two independent multivariate B11 copulas.

Then

1. the Kronecker convolution of these copulas, $x * y$, forms a multivariate $q \times r$ B11 copula

2. the k-variate tail dependence array of $x * y$ is a Kronecker product of tail dependence arrays of *x* and *y*:

$$\Lambda^{x*y}{}_{k,\, q \times r} = \Lambda^{x}{}_{k,\, q} \otimes \Lambda^{y}{}_{k,r}$$

The proof utilizes canonical decomposition and realizations from individual canonical copulas meeting each other in $x_i * y_j$.

Conclusions

Strengths of multivariate B11 copula

- Simple bivariate dependence structure, governed by a single parameter
- Tail dependence coefficient = Spearman correlation coefficient
- Linearity of the conditional quantile exceedance probability
- Parsimony (no artifacts)
- Observed ability to reproduce results of the variance-covariance method at multiple quantiles
- Insight into the higher order tail dependence structures
- Richness of realized bivariate tail dependence structures

  → Ability to fit bivariate and multivariate tail dependence structures

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Joe Necessary Condition for Bivariate Tail Dependence Coefficients

Necessary condition for bivariate dependence coefficients from *Joe (1997)* that *must* be satisfied by bivariate tail dependence coefficient matrix $\Lambda = [\lambda_{ij}]$ (must hold for all triples):

$$\max\{0, \lambda_{ij} + \lambda_{jk} - 1\} \leq \lambda_{ik} \leq 1 - |\lambda_{ij} + \lambda_{jk}|, \qquad i < j < k$$

The condition is

- more restrictive than positive semi-definiteness
- for more than 3 dimensions: necessary but not sufficient