A Numéraire-Independent Version of the Fundamental Theorems of Asset Pricing

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Joint work (in progress) with Travis Fisher and Sergio Pulido

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Executive summary: First FTAP as a bridge between the mathematics and the finance

martingale measure	"absence of arbitrage"
expectation	pricing
valuation operator	**NFLVR for allowable strategi**

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- We aim to widen the bridge to cover cleanly the case when there are multiple financial assets, any of which may potentially lose all value relative to the others.
- To do this we shift away from having a pre-determined numéraire to a more symmetrical point of view where all assets have equal priority.

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Our own motivation

- Popular model in FX:

$$S_{1,2}(t) = S_{1,2}(0) + \int_0^t \left(aS_{1,2}(u)^2 + bS_{1,2}(u) + c\right) \mathrm{d}W(u)$$

  "Quadratic normal volatility" (stopped when hitting zero)
- Calibration usually yields strict local martingale dynamics.
- Let's assume a complete market and zero interest rate.
- Superreplication cost of $S_{1,2}(T)$ is strictly smaller than $S_{1,2}(0)$ (if we price according to risk-neutral expectations)
- This yields issues with put-call parity, which is a *market convention*.
- Possible ways out:
  - Use a different model.
  - Change the concept of pricing operator.

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New pricing operators

- Lewis: “add correction term” to risk-neutral expectation when pricing calls.
- Madan & Yor: Exchange expectations and limits.
- Cox & Hobson: Restrict class of admissible strategies.
- Carr & Fisher & Ruf:
  - Note that a change of numéraire via strict local martingale $S_{1,2}$ yields non-equivalent measure.
  - Then consider the minimal superreplication cost under both measures (the original one and the new one).
  - Yields an explicit formula for the correction term.

Issues:

- Correction term seems non-symmetric in currencies.
- What to do in an incomplete market??
- What to do with more than two currencies??

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Outline

1. Underlying objects: scenarios, price proceses, ...
2. Valuation operator: maps future random (scenario-dependent) payoffs to present deterministic prices
3. Notions of arbitrage
4. Bringing everything together: Fundamental Theorems of Asset Pricing
5. Aggregation and disaggregation in different currencies

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Relative prices are modelled by an exchange matrix

- d: number of currencies
- $s_{i,j}$: units of currency i per unit of currency j
- values 0 and ∞ for $s_{i,j}$ are allowed!
- Exchange matrix: A $d \times d$-dimensional matrix $s = (s_{i,j})_{i,j}$ taking values in $[0, \infty]^{d \times d}$ such that
  1. $s_{i,i} = 1$
  2. $s_{i,j} s_{j,k} = s_{i,k}$, whenever the product is defined.
- Note: there exists always a *strongest* currency i^* with $\sum_j s_{i*,j}(t) \leq d$.

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Underlying objects

- Filtered space $(\Omega, \mathcal{F}, \mathbb{F})$: representing possible scenarios and a flow of information.
- $S_{i,j}(t) \in [0, \infty]$ denotes the price of the j:th currency in terms of the i:th currency.
- $S(\cdot) = (S_{i,j}(\cdot))_{i,j}$ is an $\mathbb{F}$–progressive, càdlàg process such that $S(t)$ is a $d \times d$ exchange matrix:

$$S_{i,j}(t) S_{j,k}(t) = S_{i,k}(t) \quad \text{(whenever defined)}$$

- Define: $\mathfrak{A}(t) = \{i : \sum_j S_{i,j}(t) < \infty\} \neq \emptyset$.

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Value vector

- A value vector $v = (v_i)_i$ (with respect to $S(t)$) encodes the price of an asset in terms of the d currencies.
- The i:th component describes the price of an asset in terms of the i:th currency.
- v satisfies consistency condition:

$$S_{i,j}(t)v_j = v_i \quad \text{(whenever defined)}$$

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$$\mathcal{U}^t = \Big\{ C : \ \mathcal{F}(t)\text{–measurable value vector s.t.} \quad \exists K > 0 \text{ with } C_i \geq -K \sum_j S_{i,j} \text{ for all } i \Big\}.$$

$$\mathcal{D}^t = \mathcal{U}^t \cap (-\mathcal{U}^t).$$

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Valuation operator

- A *valuation operator* relates future random prices to present deterministic prices.
- Concept goes back to Harrison & Pliska (1981); see also Biagini & Cont (2006) and literature on risk measures.

We say that a family of operators $\mathbb{V} = (\mathbb{V}^{r,t})_{0 \le r \le t \le T}$, with

$$\mathbb{V}^{r,t} : \mathcal{D}^t \to \mathcal{D}^r,$$

is a valuation operator with respect to S if it satisfies:

1. Positivity
2. Linearity
3. Continuity from below
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Valuation operator — the conditions

1. (Positivity) If $C \in \mathcal{D}^T$ and $C \geq 0$ then $\mathbb{V}^{0,T}(C) \geq 0$.
2. (Linearity) If $H \in \mathcal{L}^{\infty,r}$, and $C, C' \in \mathcal{D}^t$ then

$$\mathbb{V}^{r,t}(H\mathbf{1}_{\{H\neq 0\}}C + C') = H\mathbf{1}_{\{H\neq 0\}}\mathbb{V}^{r,t}(C) + \mathbb{V}^{r,t}(C').$$

3. (Continuity from below) If $(C_n)_{n\in\mathbb{N}} \subset \mathcal{D}^T$ is a nondecreasing sequence of nonnegative value vectors converging to $C \in \mathcal{D}^T$, then $\mathbb{V}^{0,t}(C_n)$ converges to $\mathbb{V}^{0,t}(C)$.
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Introduction of a probability measure

- Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$.
- We say $\mathbb{P}$ satisfies (PSmg) if there exists $(A_i)_i$ with $\bigcup_i A_i = \Omega$ such that for each i, $\mathbb{P}(A_i) > 0$ and S_i is a $\mathbb{P}_i$–semimartingale, where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|A_i)$ for each i.

Trading strategies and wealth processes

- Let $\mathbb{P}$ satisfy (PSmg).
- Let h denote a predictable process. Then V^h is a value vector process with $V_i^h(t) = \sum_j h_j S_{i,j}(t)$.
- h is called a $\mathbb{P}$*–trading strategy* if $h \in L(S_i, \mathbb{P}_i)$ and the *self-financing condition holds*:

$$V_i^h - V_i^h(0) = h \cdot_{\mathbb{P}_i} S_i.$$

- h is $\mathbb{P}$*–allowable* if there exists $\varepsilon > 0$ such that $V_i(t) \geq -\varepsilon \sum_j S_{i,j}(t)$.

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No-arbitrage condition

Assume that $\mathbb{P}$ satisfies (PSmg).
We say that S satisfies *NFLVR for $\mathbb{P}$–allowable strategies* if for any sequence of $\mathbb{P}$–allowable strategies (h^n) with $V^{h^n}(0) \leq 0$ and such that there exist $(\xi^n) \in L^\infty(\mathbb{R}, \mathbb{P})$ satisfying

$$V_i^{h^n}(T) \geq \xi^n \sum_j S_{i,j}(T),$$

the following conclusion holds:

$$\xi = \lim_{n\uparrow\infty} \xi^n \text{ exists and } \mathbb{P}(\xi \geq 0) = 1 \quad \Longrightarrow \quad \mathbb{P}(\xi = 0) = 1.$$

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First fundamental theorem

Write $\mathbb{P} \sim \mathbb{V}$ if for a nonnegative $C = (C_i)_i \in \mathcal{D}^T$, we have $\mathbb{V}^{0,T}(C) = 0$ if and only if $\sum_i \mathbf{1}_{\{C_i = 0\}} > 0$ $\mathbb{P}$–almost surely.

1. If $\mathbb{P}$ satisfies (PSmg) and S satisfies NFLVR for $\mathbb{P}$–allowable strategies then there exists a valuation operator $\mathbb{V} \sim \mathbb{P}$.
2. If there exists a valuation operator $\mathbb{V}$ then there exists a probability measure $\mathbb{P} \sim \mathbb{V}$ that satisfies (PSmg) and such that S satisfies NFLVR for $\mathbb{P}$–allowable strategies.

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Second fundamental theorem

Suppose that there exists a valuation operator $\mathbb{V}$ with respect to S. Then, the market is complete if and only if $\mathbb{V}$ is the unique valuation operator equivalent to $\mathbb{V}$.

Moreover, if a valuation operator exists, then

$$\inf\{V^h(0) : h \text{ super-replicates C}\} = \sup\{\widetilde{\mathbb{V}}^{0,T}(C) : \widetilde{\mathbb{V}} \sim \mathbb{V} \text{ is a valuation operator}\},$$

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Disaggregation and aggregation

A family $(\mathbb{Q}_i)_i$ of probability measures such that S_i a $\mathbb{Q}_i$–supermartingale is called *consistent* if the following change-of-numéraire formula holds

$$S_{j,i}(r)\mathbb{E}_r^{\mathbb{Q}_i}[S_{i,j}(t)X] = \mathbb{E}_r^{\mathbb{Q}_j}[X1_{\{S_{j,i}(t)>0\}}],$$

where X is a bounded, nonnegative random variable.

Given a valuation operator $\mathbb{V}$ there exist a consistent family of supermartingale measures $(\mathbb{Q}_i)_i$ such that

$$\mathbb{V}_j^{r,t}(C) = \sum_i S_{j,i}(r)\mathbb{E}_r^{\mathbb{Q}_i}\left[\frac{C_i}{|\mathfrak{A}(t)|}\right] \quad (1)$$

for all $r \leq t$, $j \in \mathfrak{A}(r)$, $C \in \mathcal{D}^t$.
Conversely, given a consistent family of supermartingale measures $(\mathbb{Q}_i)_i$, (1) defines a valuation operator $\mathbb{V} \sim \sum_i \mathbb{Q}_i$.

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The appearance of strict local martingales

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The case of two assets

$d = 2$, with $C = (C_1, C_2)^{\mathrm{T}}$
E.g., $C = ((S_{1,2}(T) - K)^+, (1 - KS_{2,1}(T))^+)^{\mathrm{T}}$

$$\mathbb{V}_j^{r,t}(C) = S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1}\left[\frac{C_1}{|\mathfrak{A}(t)|}\right] + S_{j,2}(r)\mathbb{E}_r^{\mathbb{Q}_2}\left[\frac{C_2}{|\mathfrak{A}(t)|}\right]$$
$$= S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1}[C_1] + S_{j,2}(r)\mathbb{E}_r^{\mathbb{Q}_2}[C_2\mathbf{1}_{\{S_{1,2}(t)=\infty\}}]$$

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The concept of "no obvious devaluations"

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(T))$ satisfies "No Obvious Devaluations" (NOD) if

$$\mathbb{P}(i \in \mathfrak{A}(T) | \mathcal{F}(\tau)) > 0 \text{ on } \{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}$$

for all i and stopping times τ.

Aggregation without consistency

Let $(\mathbb{Q}_i)_i$ be so that S_i is a $\mathbb{Q}_i$–local martingale. Then there exists a martingale valuation operator $\mathbb{V} \sim (\sum_i \mathbb{Q}_i)$ if one of the following two conditions is satisfied:

1. S_i is a $\mathbb{Q}_i$–martingale.
2. The following three conditions hold:
   2.1 $\sum_i \mathbb{Q}_i$ satisfies (NOD).
   2.2
   $$\mathbb{Q}_k\big|_{\mathcal{F}\cap\{\sum_j S_{k,j}(T)<\infty\}} \sim \left(\sum_i \mathbb{Q}_i\right)\Bigg|_{\mathcal{F}\cap\{\sum_j S_{k,j}(T)<\infty\}}.$$
   2.3 There exist $\epsilon > 0$, $N \in \mathbb{N}$ and predictable times $T_1 \leq T_2 \leq \cdots \leq T_N$ such that
   $$\left\{(t,\omega) : \sum_j S_{k,j} \text{ jumps to } \infty\right\} \cap \left\{(t,\omega) : \sum_j S_{k,j} \leq d + \varepsilon\right\} \subset \bigcup_{n=1}^{N} [\![T_n]\!].$$

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An example for lack of aggregation

- $d = 2$; probability measure $\mathbb{P}$
- R: a three-dimensional Bessel process:

$$R(t) = 1 + \int_0^t \frac{1}{R(s)} \mathrm{d}s + W(t)$$

- Stopping time τ with $\mathbb{P}(\tau = \infty) > 0$
- $S_{1,2}(t) = 1$ for all $t < \tau$ and $S_{1,2}(t) = 1 + R(t) - R(\tau)$ for all $t \geq \tau$.
- $\mathbb{Q}_1(\cdot) = \mathbb{P}(\cdot | \tau = \infty)$ and $\mathbb{Q}_2 = \mathbb{P}$.
- Then $\mathbb{Q}_1(S_{1,2} \equiv 1) = 1$, $S_{2,1}$ is a $\mathbb{Q}_2$–local martingale, and $(\mathbb{Q}_1 + \mathbb{Q}_2) \sim \mathbb{P}$.

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Conclusion

- We consider an exchange economy with d currencies, where each currency has the possibility to complete devaluate against any other currency.
- In such an economy, we introduce the concept of a valuation operator and link it to a no-arbitrage condition.
- We interpret the lack of martingale property of an asset price as a reflection of the possibility that the numéraire currency may devalue completely.
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Many thanks for your attention!

Digression: Hyperinflation

- *Hyperinflation*: complete devaluation of the corresponding domestic numéraire and an explosion of the exchange rate with respect to any other currency.
- Examples:
  - The price of one Dollar, measured in units of the respective domestic currency, went up by a factor of over 4500 in Austria from January 1919 to August 1922 and by a factor of over 10^{10} from January 1922 to December 1923 in Germany.
  - Hungary, August 1945 to July 1946. Prices soared by a factor of over 10^{27} in that 12-month period to which the month of July contributed a staggering raise of $4 * 10^{16}$ percent of prices.
  - Bolivia, August 1984 to August 1985: Price levels increased by $20,000$ percent.
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