Implied Volatilities from Strict Local Martingales

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Section 1

Strict Local Martingales

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Strict Local Martingales

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- Appear in Probability theory, e.g. in the context of Girsanov's theorem, Novikov's condition, etc.
- Interesting in financial mathematics, because they are ...
  - examples of arbitrage-free markets where market prices deviate from fundamental prices,
  - often considered as models of asset price bubbles, (cf. Heston et al. (2007), Protter, Jarrow, ...)

Fundamental Theorem of Asset Pricing

Theorem (FTAP; Delbaen & Schachermayer (1998))

Let S be a locally bounded semimartingale on a given filtered probability space. The following are equivalent:

1. *The Financial Market described by $(S, \mathbb{P})$ does not allow for arbitrage in the sense of No Free Lunch with Vanishing Risk (NFLVR).*
2. *There exists $\mathbb{Q} \sim \mathbb{P}$ such that S is a local $\mathbb{Q}$-martingale.*

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- Any 'reasonable' model for a stock price S has the local martingale property under $\mathbb{Q}$.
- If 'locally bounded' is dropped, the implication (2) $\Rightarrow$ (1) remains valid.

Pricing Bubbles (1)

Definition (Price Bubble; Heston, Loewenstein & Willard (2007))

The Financial Market $(S, \mathbb{Q})$ with time horizon T contains a price bubble, if for some $t \in [0, T)$ the current stock price S_t exceeds the fundamental price $\mathbb{E}^{\mathbb{Q}}[S_T | \mathcal{F}_t]$, i.e., if

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- Clearly, for locally bounded processes, an arbitrage-free financial market $(S, \mathbb{Q})$ contains a bubble iff S is a strict local $\mathbb{Q}$-martingale.
- If 'locally bounded' is dropped, the strict local martingale property is still sufficient for the appearance of a bubble in an arbitrage free market model.

Pricing Bubbles (2)

- In a similar way, price bubbles of Put & Call options, bond prices etc. can be studied.
- In a strict local martingale model put-call-parity may fail and other pathologies appear.
- Strict local martingales are a continuous-time phenomenon.

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Can bubbles be detected from implied volatilities?

Section 2

The Setting of Continuous Local Martingales

One-dimensional Markovian Diffusions

Local Volatility setting: Assume S given as (weak) solution of:

$$dS_t = \sigma(S_t)dW_t^{\mathbb{Q}},$$

where $\sigma(0) = 0, \sigma^{-2} \in L^1(0, \infty)$ and $S_0 > 0$.

Theorem (Delbaen & Shirakawa (2002), Blei-Engelbert-Senf (1990, 2009))

S is a strict local martingale if and only if

$$\int_1^\infty \frac{y}{\sigma(y)^2} dy < \infty.$$

Testing for Pricing Bubbles

Test for Price Bubbles (Jarrow, Kchia & Protter (2011))

- Estimate $\sigma(.)$ from historical (high-frequency) data
- Extrapolate σ to $(0, \infty)$
- Evaluate the integral criterion of Delbaen & Shirakawa

- Similar ideas can be found in Hulley & Platen (2011))
- Applied by Jarrow et al. to stock price time-series
- Claim to detect bubble in LinkedIn stock briefly after 2011 IPO.

Testing for Pricing Bubbles

Some limitations of the Jarrow-Kchia-Protter test:

- Sufficiently long time-series are needed
- Result depends on extrapolation procedure
- Test is based on local-volatility assumption
- Result is sensitive to estimation procedure

Section 3

Implied Volatility

Implied Volatility

Definition (Implied Volatility)

Given a market or model price $C(T, K)$ of a European call option with maturity T and strike K, the *implied volatility* $I(T, K)$ is the solution of

$$C(T,K) = C_{\mathrm{BS}}(T, K, I(T,K))$$

where $C_{\mathrm{BS}}(T, K, \sigma) = S_0 \mathcal{N}(d_1(T, K, \sigma)) - Ke^{-rT}\mathcal{N}(d_2(T, K, \sigma))$ is the Black-Scholes price with volatility σ.

- Implied volatility can be equivalently defined in terms of put prices (given put-call-parity holds)
- We reparameterize by log-moneyness $x = \log(K/S_0)$

Asymptotics of Implied Volatility

Theorem (Lee's formula)

Let the underlying S be a positive $\mathbb{Q}$-martingale. Then the implied volatility satisfies

$$\limsup_{x\uparrow\infty} \frac{I(T,x)^2 T}{x} = \psi(p^* - 1) \qquad \in [0,2],$$

$$\limsup_{x\downarrow-\infty} \frac{I(T,x)^2 T}{|x|} = \psi(q^*) \qquad \in [0,2],$$

where

$$p^* = \sup\{p \geq 1 : \mathbb{E}^{\mathbb{Q}}(S_T^p) < \infty\},$$
$$q^* = \sup\{q \geq 0 : \mathbb{E}^{\mathbb{Q}}(S_T^{-q}) < \infty\} \quad \textit{and}$$
$$\psi(p) = 2 - 4(\sqrt{p(p+1)} - p).$$

Asymptotics of Implied Volatility (2)

- The heavier the right tail of $\log S_T$, the steeper the right wing of the implied volatility smile,
- The maximum possible slope of $I^2(x)T$ is 2,
- Friz & Benaim: Conditions under which limsup can be replaced by lim
- Many higher order expansions (Gulisashvili,...)
- Lee's formula holds under the assumption that
  - S is a true $\mathbb{Q}$-martingale,
  - S does not have mass at zero, i.e. $\mathbb{Q}(S_T = 0) = 0$.
- Extension to mass-at-zero: De Marco, Hillairet & Jacquier (2014).

Section 4

Implied Volatility in Strict Local Martingale Models

The Martingale Defect

We assume that S is a non-negative local $\mathbb{Q}$-martingale with $S_0 = 1$

Definition (Martingale Defect)

The quantity

$$\mathfrak{m}_T := 1 - \mathbb{E}^{\mathbb{Q}}[S_T] \qquad \in [0, 1]$$

is called the *martingale defect* of S at time T.

- $\mathfrak{m}_T = 0$: S is a true $\mathbb{Q}$-martingale,
- $\mathfrak{m}_T > 0$: S is a strict local $\mathbb{Q}$-martingale (stock price bubble).

Put- and Call-Pricing

We set

$$C_S(x) := \mathbb{E}^{\mathbb{Q}}\left[(S_T - e^x)_+\right] \quad \text{and} \quad P_S(x) := \mathbb{E}^{\mathbb{Q}}\left[(e^x - S_T)_+\right].$$

- In complete markets these are the unique minimal super-replication prices of calls resp. puts
- It holds that

  $$C_S(x) - P_S(x) = 1 - e^x - \mathfrak{m}_T$$

  and

  $$(1 - \mathfrak{m}_T - e^x)_+ \leq C_S(x) < 1 - \mathfrak{m}_T,$$

  where the lower bound is asymptotically attained as $x \to -\infty$.

Hence the following are equivalent

- S is a strict local $\mathbb{Q}$-martingale
- Put-Call parity fails
- Call prices violate the classic no-static-arbitrage bounds for small strikes
- Call-implied volatility is different from Put-implied volatility
- There exists $x^* \leq 0$ such that Call-implied volatility is undefined on $(-\infty, x^*)$.

Collateralized Calls

Cox & Hobson (2005) require that the value process V of a hedging portfolio for the Call must satisfy the collateral requirement $V_t \geq G(S_t)$ at intermediate times and show:

Theorem (Thm 5.2 in Cox & Hobson (2005))

Let G be a positive convex function satisfying $\limsup_{s\uparrow\infty} \frac{G(s)}{s} = \alpha$, and H an arbitrary payoff satisfying $H \geq G$, then under the above collateral requirement the fair price (at inception) of a European option with payoff $H(S_T)$ is equal to $\mathbb{E}^{\mathbb{Q}}(H(S_T)) + \alpha\mathfrak{m}_T$.

Collateralized Calls (2)

- We set $C_S^\alpha(x) := C_S(x) + \alpha \mathfrak{m}_T$ and call $C_S^\alpha(x)$ the α-collateralized call.
- The fully collateralized call price $C_S^1(x)$ coincides with the call prices in strict local martingale models proposed in Madan & Yor (2006), Lewis (2000) and Heston et al. (2007).
- The fully collateralized call price restores put-call-parity and respects static no-arbitrage bounds.

Collateralized Calls (3)

With respect to implied volatility we obtain the following:

Theorem (Jacquier, K.-R. (2015))

Let S be a non-negative local martingale.

(i) *The implied volatility I_S^p of the Put P_S is well defined on the whole real line;*

(ii) *The implied volatility I_S^1 of the fully collateralised Call C_S^1 is well defined on $\mathbb{R}$ and coincides with the Put-implied volatility: $I_S^1(x) = I_S^p(x)$, for all $x \in \mathbb{R}$;*

(iii) *For $\alpha \in [0, 1)$ there exists $x^*(\alpha) \leq 0$ such that the implied volatility I_S^α of the α-collateralised Call is well defined on $[x^*(\alpha), +\infty)$, but not on $(-\infty, x^*(\alpha))$.*

Theorem (Jacquier, K.-R. (2015))

Let S be a non-negative strict local martingale with martingale defect $\mathfrak{m}_T$ and suppose that $\alpha > 0$. Then, as x tends to infinity, the following expansions hold:

$$I_S^p(x) = I_S^1(x) = \sqrt{\frac{2x}{T}} + \frac{\mathcal{N}^{-1}(\mathfrak{m}_T)}{\sqrt{T}} + o(1)$$

and

$$I_S^\alpha(x) = \sqrt{\frac{2x}{T}} + \frac{\mathcal{N}^{-1}(\alpha\mathfrak{m}_T)}{\sqrt{T}} + o(1).$$

Expansion of Implied Volatility (2)

Corollary (Jacquier, K.-R. (2015))

If $\alpha = 0$ *then*

$$\lim_{x \uparrow \infty} \left(I_S^0(x) - \sqrt{\frac{2x}{T}} \right) = -\infty.$$

If $\mathfrak{m}_T = 0$*, then, for all* $\alpha \in [0, 1]$*,*

$$\lim_{x \uparrow \infty} \left(I_S^p(x) - \sqrt{\frac{2x}{T}} \right) = \lim_{x \uparrow \infty} \left(I_S^\alpha(x) - \sqrt{\frac{2x}{T}} \right) = -\infty.$$

Expansion of Implied Volatility & Testing for Bubbles

- $(I_S^p(T,x))^2 T$ always attains the maximum slope of 2 in a strict local martingale model
- At first order, IVs in a strict local martingale model look like IVs in a true martingale model with a heavy right tail
- First + Second order behavior: Necessary and sufficient condition for strict local martingale property
- Higher order expansions are possible under additional assumptions.

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Test for Price Bubbles based on implied volatility

- Fit a regression line to implied volatilities of options with large strike
- Compare slope and intercept to theoretical expansion

Testing for Bubbles

Advantages:

- Test is model-free (no assumption on dynamics of S)
- Uses implied instead of historical volatility (no time-series data necessary)

Disadvantages:

- Needs option-price data
- Also based on extrapolation ($x \to \infty$)

Section 5

Duality to martingale models with mass at zero

Models in Duality

Definition (Models in Duality)

Let $\mathbb{Q}$ and $\mathbb{P}$ be probability measures on a filtered measure space and let $T > 0$ be a fixed time horizon.

Let S be a strictly positive local $\mathbb{Q}$-martingale and M be a non-negative true $\mathbb{P}$-martingale on $[0, T]$. Denote by $\tau := \inf\{t > 0 : M_t = 0\}$ the first hitting time (of M) of zero and assume that τ is predictable and $\tau > 0$, $\mathbb{P}$-a.s.

We say that the pair $(S, \mathbb{Q})$ is in duality to $(M, \mathbb{P})$ if $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_T$, with

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = M_T \qquad \text{and} \qquad S_t = \frac{1}{M_t} \quad \mathbb{P}\text{-a.s. on } \{t < \tau \wedge T\}.$$

Models in Duality (2)

- In financial modelling, $\mathbb{Q}$ can be interpreted as the 'share measure' corresponding to the stock price M under $\mathbb{P}$ or—in the context of currency models—as the 'foreign measure' corresponding to the domestic measure $\mathbb{P}$ and the exchange rate process M.
- The martingale defect of S (under $\mathbb{Q}$) equals the mass at zero of M (under $\mathbb{P}$)
- Hence, strict local martingale models are dual to true martingale models with mass at zero.
- Existence of a dual model (to a given strict local martingale model) is shown in Kardaras et al. (2015) under very general conditions.
- Relations between call and put-prices under $\mathbb{Q}$ and $\mathbb{P}$ are known as 'put-call-duality' or 'put-call-symmetry'.

Implied Volatilties in dual models

Theorem (Jacquier, K.-R. (2015))

Let S be a strictly positive strict local $\mathbb{Q}$-martingale in duality with the true $\mathbb{P}$-martingale M with mass at zero.

Denote by $I_M(x)$ the implied volatility under $\mathbb{P}$ for log-strike x and underlying M.

Then, for all $x \in \mathbb{R}$,

$$I_S^p(x) = I_S^1(x) = I_M(-x).$$

- Implied volatility in martingale models with mass at zero has been studied in De Marco, Hillairet & Jacquier (2014).
- 'Dualizing' their results to the strict local martingale case we obtain higher order expansions of implied volaility...

Implied Volatilties in dual models (2)

Corollary (Jacquier, K.-R. (2015))

Let S be a strictly positive strict local $\mathbb{Q}$-martingale, $T > 0$ and $\mathfrak{m}_T$ the martingale defect of S. Set $G(x) := \mathbb{E}^{\mathbb{Q}}(S_T \mathbf{1}_{\{S_T \geq e^x\}})$ and $\mathfrak{n}_T := \mathcal{N}^{-1}(\mathfrak{m}_T)$.

- *If $G(x) = o(x^{-1/2})$ as x tends to infinity, then*

$$I_S^p(x) = I_S^1(x) = \sqrt{\frac{2x}{T}} + \frac{\mathfrak{n}_T}{\sqrt{T}} + \frac{\mathfrak{n}_T^2}{2\sqrt{2Tx}} + \frac{\exp(\frac{1}{2}\mathfrak{n}_T^2)}{\sqrt{2Tx}}\Psi(x),$$

as x tends to infinity, where the function Ψ is such that $0 \leq \limsup_{x\uparrow\infty} \Psi(x) \leq 1$.

...

Implied Volatilties in dual models (3)

Corollary

...

- *If* $G(x) = \mathcal{O}(e^{-\varepsilon x})$ *as* x *tends to infinity, for some* $\varepsilon > 0$, *then*

$$I_S^p(x) = I_S^1(x) = \sqrt{\frac{2x}{T}} + \frac{\mathfrak{n}_T}{\sqrt{T}} + \frac{\mathfrak{n}_T^2}{2\sqrt{2Tx}} + \Phi(x),$$

as x *tends to infinity ,where the function* Φ *satisfies* $\limsup_{x\uparrow\infty} \sqrt{2Tx}|\Phi(x)| \le 1$.

Thank you for your attention!

A. Jacquier, M. Keller-Ressel. **Implied Volatility in Strict Local Martingale Models** *(2015). arXiv:1508.04351.*