Moment Properties of Distributions Used in Stochastic Financial Models

Jordan Stoyanov

Newcastle University (UK) & University of Ljubljana (Slovenia)

e-mail: stoyanovj@gmail.com

ETH Zürich, Department of Mathematics, 15 October 2015

PLAN:

Discussion on probability distributions and their uniqueness/non-uniqueness in terms of the moments.

Relation of these with infinite divisibility.

Non-uniqueness may appear for heavy-tailed distributions.
Most distributions used in stochastic financial models are heavy-tailed and infinitely divisible.
Uniqueness is important. Non-uniqueness is risky!

Recent results along with Classics. Open questions and Conjectures.

Joint with G.D. Lin (Taipei), C. Kleiber (Basel), S. Ostrovska (Ankara).

Standard notations and terminology

Basics: Probability space $(\Omega, \mathcal{F}, \mathbf{P})$, r.v. X, d.f. F, μ_F.

Assumption: Finite moments, $\int |x|^k \, dF(x) < \infty, k = 1, 2, \ldots$

kth order moment $m_k = \mathbf{E}[X^k]$, $\{m_k\}$ the moment sequence of X, F.

Classical Moment Problem: Two questions: existence, uniqueness.

X, or F, is either **M-determinate**, unique with the moments $\{m_k\}$, or **M-indeterminate**, non-unique, there are others, same moments.

Write: **M-det** and **M-indet**. For M-indet, **fundamental result**! Later.

Specific names depending on the support of F, the range of X:

supp(F): $[0, 1]$ (**Hausdorff**);
$\mathbb{R}^+ = [0, \infty)$ (**Stieltjes**);
$\mathbb{R}^1 = (-\infty, \infty)$ (**Hamburger**).

Why the moments? Many reasons.

Fréchet-Shohat Theorem: Sequence of d.f.s $F_1, F_2, \ldots, F_n, \ldots$ with

$m_k^{(n)} = \int x^k \, \mathrm{d}F_n(x) \ \to m_k,$ as $n \to \infty$ for $k = 1, 2, \ldots.$
Then: $\{m_k\}$ is a moment sequence of a d.f., say F.
If F is M-det, then $F_n \Rightarrow F$ as $n \to \infty$.

Remark: Chebyshev, Markov, ..., Stieltjes, many more later.
Name: Second Limit Theorem. In some areas, e.g. Random Graphs, Number Theory, Random Matrices, the only way to derive some asymptotic results is to use the above theorem, i.e. convergence of moments.
Inverse problem: Given the moments $\{m_k\}$, find the d.f. F, or density.
In Financial Models: Find bounds for derivatives prices or calculate the fair price of options, SD linear programming ...

Best known M-indet distribution?

Log-normal distribution: $Z \sim \mathcal{N}(0,1), \;\; X = \mathrm{e}^Z \sim Log\,\mathcal{N}(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\,\exp\left[-\frac{1}{2}(\ln x)^2\right], \; x > 0; \;\; m_k = \mathbf{E}[X^k] = \mathrm{e}^{k^2/2}, \; k = 1, 2, \ldots.$$

Define two infinite sets of random variables, *Stieltjes classes*:

$$\mathbf{S}_c = \{X_\varepsilon, \; \varepsilon \in [-1,1]\}, \; X_\varepsilon \sim f_\varepsilon, \; f_\varepsilon(x) = f(x)\,[1 + \varepsilon \sin(2\pi \ln x)], \; x \in \mathbb{R};$$

$$\mathbf{S}_d = \{Y_a, \; a > 0\}, \; \mathbf{P}[Y_a = a\mathrm{e}^n] = a^{-n}\,\mathrm{e}^{-n^2/2}/A, \;\; n = 0, \pm 1, \pm 2, \ldots.$$

For any $\varepsilon \in [-1,1]$ and any $a > 0$, the following relations hold:

$$\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[Y_a^k] = \mathbf{E}[X^k] = \mathrm{e}^{k^2/2}, \quad \text{for} \quad k = 1, 2, \ldots.$$

$\Rightarrow$ $Log\,\mathcal{N}$ is M-indet. Too ‘many’, same moments. Look at $\mathbf{S}_c$ and $\mathbf{S}_d$.

Black-Scholes Model, GBM = a linear SDE, constant $\mu > 0$ and $\sigma > 0$:

$dS_t = \mu S_t dt + \sigma S_t dW_t,\ S_0 > 0 \ \Rightarrow S_t \sim Log\,\mathcal{N}(\mu_t, \sigma_t^2)$, M-indet for $t > 0$.

Classics: $Log\,\mathcal{N}$ is infinitely divisible (inf.div.). Thorin (1978).

What can we say about the members of the above family

$\mathbf{S}_c = \{X_\varepsilon,\ \varepsilon \in [-1, 1]\},\ X_\varepsilon \sim f_\varepsilon,\ f_\varepsilon(x) = f(x)\,[1 + \varepsilon \sin(2\pi \ln x)],\ x \in \mathbb{R}$?

It is known that for $\varepsilon = 1$, the r.v. $X_1 \sim f_1$ is not inf.div. Same for X_{-1}.

Conjecture: All $X_\varepsilon,\ \varepsilon \neq 0$, are not inf.div.

Next, $Log\,\mathcal{N}$ is unimodal, while any $f_\varepsilon,\ \varepsilon \neq 0$, has infinitely many modes.

Open Question: Suppose that F is an absolutely continuous distribution function on $\mathbb{R}_+$ with all moments finite. Let for F two 'properties' hold:

(i) $m_k(F) = \int_0^\infty x^k \,dF(x) = e^{k^2/2},\ k = 1, 2, \ldots$; (ii) F is unimodal.

Prove that $F = Log\,\mathcal{N}$. Otherwise give a counterexample.

Remark: If F has moments $\{e^{k^2/2}\}$ and is inf.div. $\not\Rightarrow F = Log\,\mathcal{N}$. C.Berg.

Classical Conditions:

Cramér's: For a r.v. $X \sim F$ on $\mathbb{R}$, let the m.g.f. exist, i.e.

$$M(t) = \mathbf{E}[e^{tX}] < \infty \quad \text{for} \quad t \in (-t_0, t_0), \; t_0 > 0 \;\Rightarrow$$

light tails. Then: X has all moments finite, and X, i.e. F, is M-det.

If no m.g.f., **heavy tails** $\Rightarrow$ X is either M-det, or M-indet.

Hardy's: Consider a r.v. $X > 0$, $X \sim F$. Suppose $\sqrt{X}$ has a m.g.f.:

$$\mathbf{E}[e^{t\sqrt{X}}] < \infty \quad \text{for} \quad t \in [0, t_1), \; t_1 > 0.$$

Then X has all moments finite, say $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$ and more, X is M-det, i.e. F is the only d.f. with the moment sequence $\{m_k\}$.

Notice: Condition on $\sqrt{X}$, conclusion for X. Rôle of exponent $\frac{1}{2}$.

Corollary: If a r.v. $X > 0$ has a m.g.f., then its square X^2 is M-det.

Carleman's: Depending on the support, $\mathbb{R}$ or $\mathbb{R}_+$,

$$\mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_{2k})^{1/2k}}, \qquad \mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_k)^{1/2k}}.$$

Statement: $\mathrm{C} = \infty \;\Rightarrow\; F$ is M-det. Only sufficient.

Krein's: Assume density $f > 0$. For support $\mathbb{R}$ or $\mathbb{R}_+$,

$$\mathrm{K}[f] \equiv \int_{-\infty}^{\infty} \frac{-\ln f(y)}{1+y^2} \mathrm{d}y, \qquad \mathrm{K}[f] \equiv \int_{a}^{\infty} \frac{-\ln f(y^2)}{1+y^2} \mathrm{d}y, \; a \geq 0.$$

Statement: $\mathrm{K}[f] < \infty \;\Rightarrow\; F$ is M-indet. Only sufficient.

Remark: Available are converses to Carleman's and Krein's (A. Pakes, G.D. Lin) and a discrete version of Krein's (H. Pedersen).

Stieltjes Class: Given $X \sim F$, finite moments, density f, $L_2[f]$.
Suppose we have found a function h, call it a **perturbation function**: $|h(x)| \le 1, x \in \mathbb{R}^1$ and $f(x)h(x)$, $x \in \mathbb{R}^1$, has **vanishing moments**:

$$\int x^k f(x)h(x)\,\mathrm{d}x = 0,\ k = 0, 1, 2, \dots (h \perp \mathcal{P} \ \text{in} \ L_2[f]).$$

Then the following family of functions is called **Stieltjes class:**

$$\mathbf{S} = \mathbf{S}(f, h) = \{f_\varepsilon(x) = f(x)[1 + \varepsilon\, h(x)],\ x \in \mathbb{R}^1,\ \varepsilon \in [-1, 1]\}.$$

If h is proper, for any $\varepsilon \in [-1, 1]$, f_ε is density. If $X_\varepsilon \sim F_\varepsilon$, f_ε, then

$$\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[X^k],\ k = 1, 2, \dots;\ \varepsilon \in [-1, 1];\ X_0 = X.$$

All r.v.s in $\mathbf{S}$ are M-indet. If F is M-det $\Rightarrow$ $h = 0$ and $\mathbf{S} = \{f\}$.
Index of dissimilarity in S: $\mathrm{D}(f, h) = \int |h(x)| f(x)\mathrm{d}x$, in $[0, 1]$.

Rate of growth of the moments and (in)determinacy

Stieltjes case: r.v. $X \in \mathbb{R}_+$, $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$ Define:

$$\Delta_k = \frac{m_{k+1}}{m_k}. \text{ It increases in } k \text{ and let } \Delta_k = \mathcal{O}((k+1)^\gamma) \text{ as } k \to \infty.$$

The number $\gamma =$ **rate of growth of the moments** of X.

Statement 1: If $\gamma \leq 2$, then X is M-det.

Statement 2: $\gamma = 2$ is the best possible constant for X to be M-det.
Equiv: If $\Delta_k = \mathcal{O}((k+1)^{2+\delta})$, $\delta > 0$, there is a r.v. Y which is M-indet.

Statement 3: If $\gamma > 2$, we add Lin's condition: $-x\,f'(x)/f(x)$, $x \to \infty$, is ultimately monotone and tends to infinity. Then X is M-indet.

Hamburger case: Similar statements for r.v.s on $\mathbb{R}$, with $m_{2(k+1)}/m_{2k}$.

Exp Example: $\xi \sim Exp(1)$, density e^{-x}, $x > 0$, m.g.f., Cramér, Hardy.

Result: ξ^r is M-det for $0 \le r \le 2$ and M-indet for $r > 2$. By Krein-Lin.

Now, $X = \xi^3 \sim G$ has $m_k = \mathbf{E}[X^k] = (3k)!$, fast $\nearrow$. Rate $\gamma > 2$. Density $g = G'$ is $g(x) = \frac{1}{3}\, x^{-2/3}\, e^{-x^{1/3}}$, satisfies Lin's condition $\Rightarrow \xi^3$ is M-indet.

Stieltjes class: Use $g(x)$ and perturbation $h(x) = \sin(\sqrt{3}\, x^{1/3} - \pi/3)$
$\mathbf{S}(g, h) = \{g_\varepsilon(x) = g(x)[1 + \varepsilon h(x)],\ x > 0,\ \varepsilon \in [-1, 1]\}$.

Notice, $X = \xi^3$ is both unimodal and inf.div.

Conjecture: In $\mathbf{S}(g, h)$ any g_ε, $\varepsilon \neq 0$, has infinitely many modes (easy), and is not inf.div.

Open Question: Let F on $\mathbb{R}_+$ be absolutely continuous, unimodal and with moments $\{(3k)!\}$. Is it true that $F = G$?

More Questions: Instead of 'unimodal', assume 'inf.div.', or even both. Is $F = G$?

Normal Example: $Z \sim \mathcal{N}(0,1)$, Z^2, Z^3, Z^4, $|Z|^r$.

Z is Cramér's $\Rightarrow Z$ is M-det. $|Z|$ is Cramér's $\Rightarrow Z^2$ is M-det, by Hardy's.
However, $Z^2 = \chi_1^2$ (light tail) is also Cramér's $\Rightarrow Z^4$ is M-det, by Hardy's.

Comment: To apply twice Cramér's, and twice Hardy's, is the shortest way to prove that power 4 of the normal r.v. Z, Z^4, is M-det.

General Result: $|Z|^r$ is M-det for $0 \le r \le 4$, and M-indet for $r > 4$.
Different proofs. Explicit Stieltjes classes.

Strange Case: $X = Z^3$ is M-indet, however $|X| = |Z|^3$ is M-det. Why?
Hint: X on $\mathbb{R}$ and $|X|$ on $\mathbb{R}_+$ have different rate of growth of moments.

Remark: The M-indet property of $X = Z^3$ can be analyzed with its unimodality and inf.div. As before: Conjectures, Open Questions.

Recent Multidimensional Results: Not much done!
KS JMVA (2013); SL, TPA (2012/2013).

Theorem: Given a random vector $X \sim F$ with arbitrary distribution in $\mathbb{R}^n$ and finite all multi-indexed moments $m_{k_1,\dots,k_n} = \mathbf{E}[X_1^{k_1} \cdots X_n^{k_n}]$.
Consider the length of X: $\|X\| = \sqrt{\|X\|^2} = \sqrt{X_1^2 + \dots + X_n^2}$.
Suppose: 1-dim. non-neg. r.v. $\|X\|$ is Cramér's: $\mathbf{E}[e^{c\|X\|}] < \infty$, $c > 0$.
Then the n-dim. Hamburger moment problem for F has a unique solution, i.e. the random vector $X \in \mathbb{R}^n$ is M-det. Equivalently: F is the only n-dim. d.f. with the set of multi-indexed moments $\{m_{k_1,\dots,k_n}\}$.

Proof: We follow two steps.
Step 1: Cramér's for $\|X\|$ $\Rightarrow$ $\|X\|^2$ is M-det, by Hardy's (Stieltjes case).
Step 2: Amazing statement by Putinar-Schmüdgen: If $\|X\|^2$ is M-det (1-dim. Stieltjes), then F is M-det (n-dim. Hamburger).

Moment determinacy of the solutions of SDEs

Stochastic process $X = (X_t,\ t \in [0, T])$, solution of Itô's type SDE:

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,\ X_0,\ t \in [0, T], \quad \text{or} \quad t \geq 0.$$

Here $W = (W_t, t \geq 0)$ is a standard BM, X_0 is constant or a r.v. Under 'general conditions' on the drift $a(\cdot)$ and the diffusion $\sigma^2(\cdot)$ this SDE has a unique weak solution, s.t. at any time t, X_t has all moments finite.

Question: When are the 1-dimensional and the n-dimensional distr. of X uniquely determined by their marginal or multi-indexed moments?

It may happen that a SDE has a unique weak solution which, however, is non-unique in terms of the moments. No surprise!

Cases: $|\sigma(\cdot)| \leq K,\ \ \sigma(x) = \sqrt{x}$ and $\sigma(x) = x$. Rôle of $\sigma^2(\cdot)$.

Now a Challenge: Given are the sequences $\{k!\}, \{(2k)!\}, \{(3k)!\}$.

Do you believe, at one glance, in what follows?

- There is only one SDE such that at any time $t > 0$, the moments of the solution X_t are $\{k!,\ k = 1, 2, \ldots\}$.
- There is only one SDE such that at any time $t > 0$, the moments of the solution X_t are $\{(2k)!,\ k = 1, 2, \ldots\}$.
- There are infinitely many SDEs such that at any time $t > 0$, all solutions have the same moments $\{(3k)!,\ k = 1, 2, \ldots\}$.

Details follow ...

Theorem 1: (Involves $\{k!\}$)

There is only one SDE with explicit coefficients such that its unique weak solution $X = (X_t, t \geq 0)$ is a stationary diffusion Markov process with correlation function $e^{-ct}, c > 0$, and at any time $t \geq 0$, the moments of X_t are $\mathbf{E}[X_t^k] = k!, \ k = 1, 2, \ldots$ More, X_t is exponentially integrable.

Theorem 2: (Involves $\{(2k)!\}$)

There is only one SDE with explicit coefficients such that its unique weak solution $X = (X_t, t \geq 0)$ is a stationary diffusion Markov process with correlation function e^{-ct} and at any time $t \geq 0$, the moments of X_t are $\mathbf{E}[X_t^k] = (2k)!, \ k = 1, 2, \ldots$ Here X_t is not exponentially integrable.

Theorem 3: (Involves $\{(3k)!\}$)

There are infinitely many stationary diffusion Markov processes $\{X^{(\varepsilon)},\ \varepsilon \in [-1,1]\}$ satisfying explicit SDEs, such that at any time $t \geq 0$, all $X_t^{(\varepsilon)}$ have the same moments $\mathbf{E}[(X_t^{(\varepsilon)})^k] = (3k)!,\ k = 1, 2, \ldots$.
None is exponentially integrable.

How to prove the above? We need different arguments, **A**, **B** and:

Fundamental Result (C. Berg and co.): If a d.f. F with finite moments is M-indet, then there are infinitely many absolutely continuous and infinitely many discrete distributions all with the same moments as F.

A: The numbers $k!$, $(2k)!$, $(3k)!$ are related to the r.v. $\xi \sim Exp(1)$:

$$m_k(\xi) = k!, \; m_k(\xi^2) = (2k)!, \; m_k(\xi^3) = (3k)!, \; k = 1, 2, \ldots.$$

And we know everything about their moment determinacy:

$$\xi \text{ is M-det}, \quad \xi^2 \text{ is M-det}, \quad \xi^3 \text{ is M-indet.}$$

For later we need the densities: e^{-x}, $\frac{1}{2} x^{-1/2} \mathrm{e}^{-x^{1/2}}$, $\frac{1}{3} x^{-2/3} \mathrm{e}^{-x^{1/3}}$, $x > 0$.

B: How to construct a stochastic process $X = (X_t, t \geq 0)$ with prescribed marginal distributions and correlation structure?

Regarding **B**: Start with a d.f. F, whose density $f = F'$ is continuous, bounded and strictly positive in $(a, b) := \text{supp}(F)$ (finite or infinite). Assume that the variance of F is finite; denote the mean by m_1.

For constant $c > 0$, define the function $v(x),\ x \in (a, b)$, as follows:

$$v(x) = \frac{2\,c}{f(x)} \int_a^x (m_1 - u) f(u)\,\mathrm{d}u = \frac{2\,c}{f(x)} \left(m_1\, F(x) - \int_a^x u\, f(u)\,\mathrm{d}u \right).$$

Theorem. (Bibby et al, 2005) *Suppose that F, f and v are as above, W is a standard BM indep. of r.v. X_0. Then the following SDE*

$$dX_t = -c(X_t - m_1)\,dt + \sqrt{v(X_t)}\,dW_t,\ \ X_t|_{t=0} = X_0,\ \ t \ge 0$$

has a unique weak solution $X = (X_t, t \ge 0)$ which is a diffusion Markov process. Moreover, X is ergodic with ergodic density f. If the density of X_0 is f, the process X is stationary with correlation function $e^{-ct},\ t \ge 0$.

Hint for Th. 1:

Use $\xi \sim Exp(1)$ and its density $f(x) = e^{-x}$ on $(0, \infty)$, calculate $v(x)$, the 'future' diffusion coefficient; the drift is prescribed as above.

Write explicitly a SDE with $X_0 \sim f \;\Rightarrow$ homogeneous diffusion Markov ergodic process X with invariant density f. For $t \geq 0$, the moments of X_t are $\{k!\}$, X_t is M-det, and X_t is exp. integrable. All done!

Hint for Th. 2:

Use ξ^2, its density is $f(x) = \frac{1}{2}x^{-1/2}e^{-x^{1/2}}$ on $(0, \infty)$, calculate $v(x)$, the 'future' diffusion coefficient; the drift is prescribed as before.

Write explicitly a SDE with $X_0 \sim f \;\Rightarrow$ homogeneous diffusion Markov ergodic process X with invariant density f. For $t \geq 0$, the moments of X_t are $(2k)!$, X_t is M-det, X_t is not exp. integrable. Done

Hint for Th. 3:

Step 1. Use ξ^3, density $f(x) = \frac{1}{3}\, x^{-2/3}\, e^{-x^{1/3}}$ on $(0, \infty)$. Calculate $v(x)$, write a SDE with $X_0 \sim f$ $\Rightarrow$ homog. diffusion Markov ergodic process X with invariant density f. For $t \geq 0$, the moments of X_t are $(3k)!$, X_t is M-indet, 'very' heavy tail, no exp. integrability.

Step 2. **Stieltjes class:** Use the above f and $h(x) = \sin(\sqrt{3}x^{2/3} - \pi/3)$:

$$\mathbf{S} = \{f_\varepsilon(x) = f(x)[1 + \varepsilon h(x)],\ x \in \mathbb{R}^1,\ \varepsilon \in [-1, 1]\}$$

Use density f_ε, calculate v_ε, write SDE$^{(\varepsilon)}$, same drift as before, $X_0 \sim f_\varepsilon$, and diffusion coeff. v_ε $\Rightarrow$ homogeneous diffusion ergodic Markov process $X^{(\varepsilon)} = (X_t^{(\varepsilon)},\ t \geq 0)$ with invariant density f_ε.

Thus obtain an infinite family $\{X^{(\varepsilon)},\ \varepsilon \in [-1, 1]\}$. Clearly, all $X_t^{(\varepsilon)},\ t \geq 0,\ \varepsilon \in [-1, 1]$, have moments $\{(3k)!\}$. No exp. integrability.

Question: Why chosen the moments $k!,\ (2k)!,\ (3k)!,\ k = 1, 2, \ldots$?

Answer: Not just for curiosity. It was good to use our findings on *Exp*. More important is that these numbers show where are the boundaries between the following three groups of SDEs:

SDEs with M-determinacy and exponential integrability;
SDEs with M-determinacy but no exponential integrability;
SDEs with M-indeterminacy and hence no exponential integrability.

General result: We can find explicitly a SDE such that its solution has the moment sequence $\{m_k\}$ coming from an arbitrary absolutely continuous distribution F. Details ... the next time!

More topics: Random sums and ruin times, volatility processes, ...

Two Final Questions:

Open Question 1: For $\xi \sim Exp$, we know that the cube ξ^3 has moments $\{(3k)!\}$ and is M-indet. Since ξ^3 is absolutely continuous, in view of the above Fundamental theorem, we arrive at this question: How to find a discrete r.v. whose moment sequence is $\{(3k)!\}$?

Open Question 2: Given $X \sim F$, $L_2[f]$ = Hilbert space, 'weight' $f = F'$.
Known: If X is Cramér's, $\mathbf{E}[e^{tX}] < \infty$, $t \in (-t_0, t_0)$, then the set of polynomials $\mathcal{P}$ is dense in $L_2[f]$.
In this spirit, Engelbert & Di Tella TPA (2015) showed that the monomials are dense in an appropriate L_2 Hilbert space of r.v.s.
Essential in their proof is the M-det of X, implied by Cramér's.

Now, Hardy's is a weaker condition for M-uniqueness. Hence a question:
If a r.v. $X \sim F$, $X > 0$ is Hardy's, $\mathbf{E}[e^{t\sqrt{X}}] < \infty$, $t \in [0, t_1)$, is it true that the polynomials $\mathcal{P}$ are dense in the Hilbert space $L_2[f]$?
What about the monomials?

References: Books: AK(1938), ST(1943), A(1961/65)

Hardy GH (1917/1918): *The Mathematical Messenger* **46/47**

Pedersen HL (1988): *J. Approx. Theory* **95** 90–100

Lin GD (1997): *Statist. Probab. Letters* **35** 85–90

Bibby BM, Skovgaard IM, Sorensen, M (2005): *Bernoulli* **11** 191–220

Putinar M, Schmüdgen K (2008): *Indiana Univ. Math. J.* **57** 2931–68

Kleiber C, Stoyanov J (2011/2013): *J. Multivar. Analysis* **113** 7–18

Stoyanov J, Lin GD (2012): *Theory Probab. Appl.* **57**, no. 4, 811–820

Lin GD, Stoyanov J (2013): *JOTP* doi 10.1007/s10959-014-0546-z

Stoyanov J, Lin GD, DasGupta A (2014) *JSPI* **154** 166–177

Lin GD, Stoyanov J (2015): *Probab & Math Statist.* In Press.
Online: arXiv:1406.1654v2 [math.PR] 12 Nov 2014

Stoyanov J (2013): **Counterexamples in Probability. 3rd edn.**
Dover Publications, New York. (1st ed, 2nd ed: Wiley 1987, 1997)