Closed Differential Forms on Moduli Spaces of Sheaves

Francesco Bottacin

University of Padova

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Outline

1. Moduli Spaces of Sheaves
   - Definitions
   - Construction
   - Infinitesimal Deformations
   - More General Sheaves

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Moduli Spaces

Definition (very rough)

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- X a fixed complex manifold
- objects = holomorphic vector bundles over X
- equivalence = isomorphism of vector bundles

Moduli space $\mathcal{M}$ = set of isomorphism classes of holomorphic vector bundles over X

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Usually, this is not possible!

Construction of the Moduli Space of Vector Bundles

Differential geometric construction of the moduli space of vector bundles on a Kähler manifold.

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Notations:

- X a complex Kähler manifold
- $A^r = C^\infty$ complex r-forms over X
- $A^{p,q} = C^\infty$ complex (p,q)-forms over X
- $d : A^r \to A^{r+1}$ exterior differential

We write $d = d' + d''$, where

$$d' : A^{p,q} \to A^{p+1,q}, \qquad d'' : A^{p,q} \to A^{p,q+1}$$

Construction of the Moduli Space of Vector Bundles

- E a fixed C^∞ complex vector bundle over X
- $A^r(E) = C^\infty$ complex r-forms over X with values in E
- $A^{p,q}(E) = C^\infty$ complex (p,q)-forms over X with values in E
- $D : A^0(E) \to A^1(E)$ a connection on E

We write $D = D' + D''$, where

$$D' : A^0(E) \to A^{1,0}(E), \qquad D'' : A^0(E) \to A^{0,1}(E)$$

D'' is $\mathbb{C}$-linear and

$$D''(fs) = (d''f)s + f\,D''(s),$$

for $s \in A^0(E)$ and $f \in A^0$.

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$\mathcal{H}''(E)$ is the set of all holomorphic structures on the C^∞ vector bundle E. But...

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$\mathcal{H}''(E)$ is not the moduli space of holomorphic vector bundles. Different holomorphic structures may produce isomorphic vector bundles

Construction of the Moduli Space of Vector Bundles

$\mathrm{GL}(E)$ acts on $\mathcal{H}''(E)$:

$$D'' \mapsto f^{-1} \circ D'' \circ f = D'' + f^{-1} d'' f$$

for any $D'' \in \mathcal{H}''(E)$, $f \in \mathrm{GL}(E)$.

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Problem: This quotient is not a “nice space” (e.g., it is not Hausdorff)

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In this talk we shall consider the first approach.

Existence of Moduli Spaces

Typical existence result:

Theorem

A moduli space $\mathcal{M}$ of (semi)stable holomorphic vector bundles over X exists.

It is a complex space, usually not compact and singular.

It can be compactified by adding suitable equivalence classes of sheaves (that are not locally free)

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Remark

Usually, moduli spaces of (semi)stable sheaves on X carry information about the variety X itself.

Local Structure

Local Structure of Moduli Spaces

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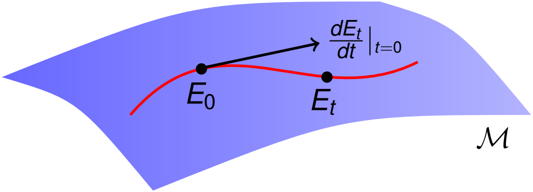

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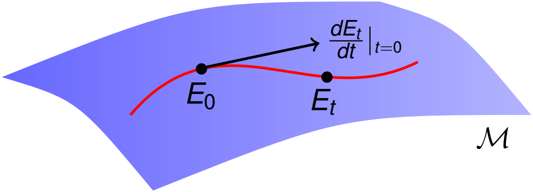

How do we make sense of $\frac{dE_t}{dt}$?

Deformations of vector bundles

Cover X by open subsets U_i, such that $E_t|_{U_i}$ is trivial

$$E_t|_{U_i} \cong U_i \times \mathbb{C}^r$$

On $U_i \cap U_j$ we get a transition function

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(r)$$

E_t is equivalent to the family $\{g_{ij}(t)\}$ of transition functions.

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Then

$$\frac{dE_t}{dt} \approx \left\{ \frac{dg_{ij}}{dt} \right\}_{ij}$$

Deformations of vector bundles

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$$g_{jk}(t)\, g_{ij}(t) = g_{ik}(t), \qquad \text{on } U_i \cap U_j \cap U_k$$

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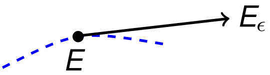
Theorem

There is a natural identification (Kodaira–Spencer map)

$$T_E\mathcal{M} \cong H^1(X, \underline{End}\, E)$$

Obstructions

Question: can a first-order deformation E_ϵ of E be extended to higher orders?



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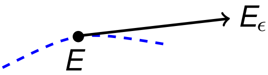

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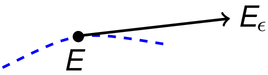

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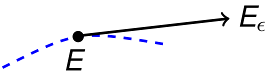

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Corollary

If $H^2(X, \underline{\mathrm{End}}\, E) = 0$ then all obstructions vanish. As a consequence, E is a non-singular point of $\mathcal{M}$

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- Obstructions actually lie in $H^2(X, \underline{End}_0\, E)$, where $\underline{End}_0\, E$ is the sub-bundle of trace-free endomorphisms of E.
- If E is a singular point of $\mathcal{M}$, a small neighborhood of E in $\mathcal{M}$ is homeomorphic to a subset of $H^1(X, \underline{End}\, E)$, which is the zero locus of a quadratic polynomial map (Kuranishi map).

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- There are notions of stability for more general sheaves (coherent torsion-free, or even torsion sheaves).
- There exist moduli spaces of (semi)stable sheaves on a projective variety. In general, they are (quasi)projective schemes.
- Technical point: if E is not locally free, all cohomology groups $H^i(X, \underline{End}\, E)$ must be replaced by $\mathrm{Ext}^i(E, E)$ ($\mathcal{E}xt^i$ is the i-th derived functor of the $\mathcal{H}om$-functor)

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- Tangent space: $T_E\mathcal{M} \cong \mathrm{Ext}^1(E, E)$
- Obstruction space: $\mathrm{Ext}^2(E, E)$
- More precisely: there is a trace map

$$\mathrm{tr} : \mathrm{Ext}^2(E, E) \to H^2(X, \mathcal{O}_X)$$

and all obstructions to deforming E lie in the kernel of this map, denoted by $\mathrm{Ext}^2_0(E, E)$

Differential Forms on $\mathcal{M}$

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Theorem

$\mathcal{M}$ is non-singular.

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The trace-free part is 0, hence there are no obstructions. □

Symplectic Structures on $\mathcal{M}$

Using smoothness of $\mathcal{M}$ and the Kodaira-Spencer isomorphism $T_E\mathcal{M} \cong \mathrm{Ext}^1(E, E)$, define a map

$$\tau : T\mathcal{M} \times T\mathcal{M} \to \mathcal{O}_{\mathcal{M}}$$

for any $E \in \mathcal{M}$, τ_E is given by composing

$$\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^2(E, E)$$

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$$\mathrm{Ext}^2(E, E) \to H^2(X, \mathcal{O}_X) = H^2(X, \omega_X) \cong \mathbb{C}$$

Symplectic Structures on $\mathcal{M}$

Theorem (Mukai)

The maps τ_E, $\forall E \in \mathcal{M}$, define a non-degenerate 2-form

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(Actually, the closedness of the 2-form on $\mathcal{M}$ was not proved in the original paper by Mukai)

Applications and generalizations

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3. Construction of Poisson structures on moduli spaces of sheaves on a Poisson surface: by choosing a Poisson structure on a surface X we can construct, in a natural way, a Poisson structure on the moduli space $\mathcal{M}$.

4. Construction of algebraically completely integrable hamiltonian systems on moduli spaces of sheaves, or related objects (e.g., Higgs bundles).

Differential forms

Construction of closed differential forms
on moduli spaces of sheaves

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E holomorphic vector bundle over X. There is a natural exact sequence

$$0 \to E \otimes \Omega^1_X \to J^1(E) \to E \to 0,$$

where $J^1(E)$ is the bundle of first-order jets of sections of E.

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where $J^1(E)$ is the bundle of first-order jets of sections of E.
The corresponding extension class

$$a(E) \in \operatorname{Ext}^1(E, E \otimes \Omega^1_X)$$

is the Atiyah class of E

The Atiyah class

More generally

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$$a(E)^i \in \mathrm{Ext}^i(E, E\otimes \Omega^i_X)$$

Then, take the trace

$$\gamma^i(E) = \mathrm{tr}(a(E)^i) \in H^i(X, \Omega^i_X)$$

$\gamma^i(E)$ is a closed (i,i)-form; up to a scalar factor it coincides with the i-th component of the Chern character of E.

The construction

Idea

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A universal family of sheaves is a sheaf $\mathcal{E}$ on $X \times \mathcal{M}$, flat over $\mathcal{M}$, such that

$$\mathcal{E}|_{X\times\{E\}} \cong E,$$

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$$\mathcal{E}|_{X\times\{E\}} \cong E,$$

for any $E \in \mathcal{M}$.
Assume a universal family $\mathcal{E}$ exists. Then

$$a(\mathcal{E}) \in \operatorname{Ext}^1_{X\times\mathcal{M}}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times\mathcal{M}})$$

We obtain classes

$$\gamma^i(\mathcal{E}) \in H^i(X \times \mathcal{M}, \Omega^i_{X\times\mathcal{M}})$$

The construction

Use the Künneth decomposition

$$H^n(X \times \mathcal{M}, \Omega^n_{X \times \mathcal{M}}) \cong \bigoplus_{i,j=0}^{n} H^i(X, \Omega^j_X) \otimes H^{n-i}(\mathcal{M}, \Omega^{n-j}_{\mathcal{M}})$$

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and write

$$\gamma^n(\mathcal{E}) = \sum_{i,j} \gamma^n_{i,j}(\mathcal{E}),$$

where

$$\gamma^n_{i,j}(\mathcal{E}) \in H^i(X, \Omega^j_X) \otimes H^{n-i}(\mathcal{M}, \Omega^{n-j}_{\mathcal{M}}).$$

The construction

Now consider Serre duality

$$H^i(X, \Omega_X^j) \cong H^{n-i}(X, \Omega_X^{n-j})^*$$

and the map

$$H^{n-i}(X, \Omega_X^{n-j})^* \to H^{k+i-n}(\mathcal{M}, \Omega_{\mathcal{M}}^{k+j-n})$$

induced by

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By composition we obtain a map

$$f : H^i(X, \Omega_X^j) \to H^{k+i-n}(\mathcal{M}, \Omega_{\mathcal{M}}^{k+j-n})$$

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▸ Explicit construction

Finally, the closedness of the differential forms constructed in this way follows easily from the fact that the classes $\gamma^n(\mathcal{E})$, and all their components $\gamma^n_{i,j}(\mathcal{E})$, are d-closed (this is essentially a restatement of the fact that the Chern classes of a vector bundle are represented by closed differential forms).

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Choose a suitable open covering $\mathcal{U} = (U_i)_i$ of $\mathcal{M}$, and local universal families $\mathcal{E}_i$ over $X \times U_i$.
Over $X \times (U_i \cap U_j)$ we have two universal families, $\mathcal{E}_i$ and $\mathcal{E}_j$.
In general, they are not isomorphic (that's why we cannot glue them to obtain a global universal family).

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What is true is that

$$\mathcal{E}_j \cong \mathcal{E}_i \otimes q^* L,$$

for some line bundle L over $U_i \cap U_j$ (q is the projection $X \times (U_i \cap U_j) \to U_i \cap U_j$).

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$$a(\mathcal{E}_j) = a(\mathcal{E}_i) \otimes id_{q^* L} + id_{\mathcal{E}_i} \otimes a(q^* L).$$

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But not everything is lost!
On $U_i \cap U_j$ we have:

$$\begin{aligned} \underline{End}(\mathcal{E}_j) &= \mathcal{E}_j^* \otimes \mathcal{E}_j \\ &= (\mathcal{E}_i \otimes q^* L)^* \otimes (\mathcal{E}_i \otimes q^* L) \\ &= \mathcal{E}_i^* \otimes \mathcal{E}_i \\ &= \underline{End}(\mathcal{E}_i) \end{aligned}$$

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Hence, the sheaf $\underline{End}(\mathcal{E})$ is well-defined even if there is no universal family $\mathcal{E}$ (the same is true for the Ext-groups).

There is still no global Atiyah class

$$a(\mathcal{E}) \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X \times \mathcal{M}})$$

because the classes $a(\mathcal{E}_i)$ do not coincide on the intersections $U_i \cap U_j$.

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Let $p : X \times \mathcal{M} \to X$ and $q : X \times \mathcal{M} \to \mathcal{M}$ be the projections. Let $\underline{Ext}^i_q$ denote the i-th relative Ext-sheaf (the i-th derived functor of $q_* \underline{Hom}$)

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There is a natural map

$$\mathrm{Ext}^1_{X\times\mathcal{M}}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times\mathcal{M}}) \to H^0(\mathcal{M}, \underline{Ext}^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times\mathcal{M}}))$$
$$a \mapsto \tilde{a}$$

The solution

If $a(\mathcal{E})$ is the Atiyah class of a universal family $\mathcal{E}$, we define

$$\tilde{a}(\mathcal{E}) \in H^0(\mathcal{M}, \underline{Ext}^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times\mathcal{M}}))$$

to be the image of $a(\mathcal{E})$ via the previous map: $\tilde{a}(\mathcal{E})$ is the local Atiyah class of the family $\mathcal{E}$.

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The importance of the local Atiyah class is due to the following result:

Lemma

If $\mathcal{E}_j \cong \mathcal{E}_i \otimes q^* L$, then $a(\mathcal{E}_j) \neq a(\mathcal{E}_i)$ but $\tilde{a}(\mathcal{E}_j) = \tilde{a}(\mathcal{E}_i)$.

The solution

Corollary

Even if a global universal family $\mathcal{E}$ does not exist on $\mathcal{M}$, the local Atiyah class

$$\tilde{a}(\mathcal{E}) \in H^0(\mathcal{M}, \underline{Ext}^1_q(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times\mathcal{M}}))$$

is well defined (it is obtained by gluing the sections $\tilde{a}(\mathcal{E}_i)$, where $\mathcal{E}_i$ are local universal families).

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Now our original construction works, with only minor modifications!

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1. Original construction:

$$\gamma^i(\mathcal{E}) = \mathrm{tr}(a(\mathcal{E})^i) \in H^i(X \times \mathcal{M}, \Omega^i_{X\times\mathcal{M}})$$

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2. Modified version:

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Then use the analogue of Künneth decomposition for the sheaf $R^i q_*(\Omega^i_{X\times\mathcal{M}})$ to write

$$\tilde{\gamma}^n(\mathcal{E}) = \sum_{i,j} \tilde{\gamma}^n_{i,j}(\mathcal{E})$$

etc.

Applications

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Mukai's construction

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2. We also recover a construction of holomorphic symplectic structures on moduli spaces of sheaves on a holomorphic symplectic manifold of dimension > 2 (Kobayashi, 1986).

Symplectic structures

3. In some cases it is possible to construct holomorphic symplectic structures on moduli spaces of sheaves on X, when X does not possess any non-zero holomorphic 2-form (some examples due to Kuznetsov and Markushevich, 2007).

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Idea: choose a suitable i such that $H^i(X, \Omega_X^{i+2}) \neq 0$ and use the map

$$H^i(X, \Omega_X^{i+2}) \to H^0(\mathcal{M}, \Omega^2_{\mathcal{M}}).$$

Usually, the hard part is to prove that the resulting 2-form on $\mathcal{M}$ is non-degenerate.

K3 fibrations

4. X smooth 3-fold, K3-fibration over $\mathbb{P}^1$:

$$\pi : X \to \mathbb{P}^1$$

such that $X_t = \pi^{-1}(t)$ is a K3 surface, $\forall t \in \mathbb{P}^1$.

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$\mathcal{M}$ moduli space of stable sheaves on X, supported on the fibers of π. Then $\mathcal{M}$ is a fibration over $\mathbb{P}^1$ and, for any $t \in \mathbb{P}^1$, $\mathcal{M}_t$ is a moduli space of sheaves over X_t.

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The proof that the 3-form is non-degenerate uses the fact that $\mathcal{M}_t$ is a moduli space of sheaves over a K3 surface, and the Mukai 2-form on $\mathcal{M}_t$ is non-degenerate (results by Thomas, Bridgeland, Maciocia).

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If we choose moduli data so that $\mathcal{M}$ has (a component of) dimension n, is $\mathcal{M}$ (or some component of it) another Calabi-Yau manifold?
True if $n = 2$ (Mukai). In this case both X and a component of $\mathcal{M}$ are K3 surfaces.

Hilbert schemes of points

5. X smooth projective variety (of dimension n).
$X^{[d]} = \mathrm{Hilb}^d(X)$ Hilbert scheme parametrizing 0-dimensional subschemes of X of length d (i.e., d points on X, not necessarily distinct).

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If $Z \in X^{[d]}$, $Z = \{P_1, \ldots, P_d\}$, $P_i \neq P_j$, then

$$T_Z X^{[d]} = \bigoplus_{i=1}^{d} T_{P_i} X$$

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When the d points are not distinct, things become more complicated.
Let $U \subset X^{[d]}$ be the open subset parametrizing d-tuples of distinct points of X. Any differential form σ on X defines a corresponding differential form $\tilde{\sigma}$ on U.

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Using our construction of differential forms on moduli spaces of sheaves, we can construct a differential form on $X^{[d]}$, starting with a differential form σ on X.

(If X is a K3 surface, $X^{[d]}$ is an example of an irreducible symplectic manifold)

Thank you!

Stability of Vector Bundles

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X = compact Kähler manifold of dimension n
g = Kähler metric on X
Φ = associated Kähler form ($\Phi = \sqrt{-1} \sum g_{ij} \, dz^i \wedge d\bar{z}^j$)
E = holomorphic vector bundle over X
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E = holomorphic vector bundle over X
$c_1(E)$ = first Chern class of E (it is represented by a closed $(1,1)$-form on X)

Define the degree of E:

$$\deg(E) = \int_X c_1(E) \wedge \Phi^{n-1}$$

Define the slope of E:

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

Definition of Stability

Remark

Similar definitions can be given for a torsion-free sheaf of $\mathcal{O}_X$-modules (such a sheaf is locally free on a dense open subset of X, the locus where it is not locally free has codimension ≥ 2)

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Definition

E is stable (resp. semistable) if, for every coherent torsion-free subsheaf $F \subset E$, with $0 < \operatorname{rk} F < \operatorname{rk} E$,

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(This is “Mumford–Takemoto” or “slope” stability. There is a more general notion called “Gieseker” stability)

Explicit construction

Given $\sigma \in H^i(X, \Omega_X^{i+p})$ we can give an alternative, more explicit, construction of a p-form ω on $\mathcal{M}$.

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For any $E \in \mathcal{M}$, ω_E is a map

$$\omega_E : T_E\mathcal{M} \times \cdots \times T_E\mathcal{M} \to \mathbb{C}$$

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- $H^p(X, \mathcal{O}_X) \to H^{i+p}(X, \Omega_X^{i+p})$ (cup-product with σ)

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- $H^p(X, \mathcal{O}_X) \to H^{i+p}(X, \Omega_X^{i+p})$ (cup-product with σ)
- $H^{i+p}(X, \Omega_X^{i+p}) \to H^n(X, \Omega_X^n) \cong \mathbb{C}$ (cup-product with $c_1(E)^{n-i-p}$)