

Local exclusion and Lieb-Thirring inequalities for anyons

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joint work with
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Outline of Talk

- 1 Identical particles and statistics
- 2 Fermions vs. bosons vs. anyons?
- 3 Hardy inequalities for anyons
- 4 A local exclusion principle for anyons
- 5 Lieb-Thirring inequalities for anyons
- 6 Applications

Identical particles and statistics

$$u(\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_k, \dots, \mathbf{x}_N) = \pm u(\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_j, \dots, \mathbf{x}_N)$$

M. Leinaas, J. Myrheim, *On the Theory of Identical Particles*, Il Nuovo Cimento 37B, 1977

Configuration space for N identical particles:

$$X_d^N := \left(\mathbb{R}^{dN} \setminus \mathbb{D} \right) / S_N,$$

$$\mathbb{D} := \{ (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = x \in \mathbb{R}^{dN} : \mathbf{x}_j = \mathbf{x}_k, j \neq k \}$$

$d \geq 3$:

$\pi_1(X_d^N) = S_N$, the permutation group

One-dimensional unitary representations $\rho : S_N \rightarrow U(1)$:

- $\rho(\sigma_{jk}) = +1$: **bosons** (e.g. photons), $u \in \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^d)$
- $\rho(\sigma_{jk}) = -1$: **fermions** (e.g. electrons), $u \in \bigwedge^N L^2(\mathbb{R}^d)$

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Identical particles and statistics

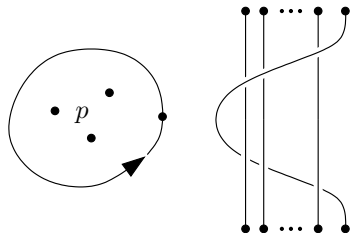
$$d = 2:$$

$\pi_1(X_2^N) = B_N$, the braid group on N strands

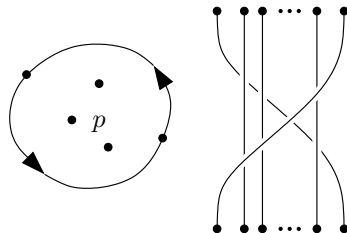
One-dimensional unitary representations $\rho : B_N \rightarrow U(1)$ characterized by a single phase $\rho(\sigma_{jk}) = e^{i\alpha\pi}$

- $\alpha = 0$: bosons
- $\alpha = 1$: fermions
- **any** phase/statistics parameter $\alpha \in [0, 2)$: **anyons**

Identical particles and statistics



$$e^{i2p\alpha\pi}$$



$$e^{i(2p+1)\alpha\pi}$$

One- resp. two-particle interchange loops with corresponding braid diagrams (where we can think of time as running upwards) and phases.

Modelling anyons

- **Anyon gauge:** square-integrable sections of a locally flat complex line bundle over X_2^N (“multivalued” wavefunctions):

$$u \in \Gamma(L), \quad L \rightarrow X_2^N$$

momenta $\nabla_j u$

- **Magnetic gauge:** bosons on \mathbb{R}^2 with Aharonov-Bohm magnetic potentials attached to each particle:

$$u \in \mathcal{H}_0^N \subseteq L^2(\mathbb{R}^{2N})$$

$$D_j u := (-i\nabla_j + \mathbf{A}_j(\mathbf{x}_j)) u, \quad \mathbf{A}_j(\mathbf{x}) := \alpha \sum_{k \neq j} (\mathbf{x} - \mathbf{x}_k)^{-1} I$$

Fermions

Pauli's exclusion principle: $u_0 \wedge u_0 = 0$, $u_0 \in L^2(\mathbb{R}^d)$

\Rightarrow Lieb-Thirring inequality:

$$\begin{aligned} \sum_{j=1}^N \int_{\mathbb{R}^{dN}} (|\nabla_j u|^2 + V(\mathbf{x}_j)|u|^2) dx \\ \geq - \sum_{k=0}^{N-1} |\lambda_k| \geq -C_d^{\text{LT}} \int_{\mathbb{R}^d} |V_-(\mathbf{x})|^{1+\frac{d}{2}} d\mathbf{x}, \end{aligned}$$

\Leftrightarrow Kinetic energy inequality:

$$\begin{aligned} T := \sum_{j=1}^N \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 dx &\geq C_d^K \int_{\mathbb{R}^d} \rho(\mathbf{x})^{1+\frac{2}{d}} d\mathbf{x}, \\ \rho(\mathbf{x}) := \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |u(\mathbf{x}_1, \dots, \mathbf{x}_j = \mathbf{x}, \dots, \mathbf{x}_N)|^2 \prod_{k \neq j} dx_k \end{aligned}$$

Bosons

No Pauli principle: $u = u_0 \otimes u_0 \otimes \dots \otimes u_0 \in \mathcal{H}_0^N$

\Rightarrow **No** Lieb-Thirring inequality for $N \geq 2$:

$$\begin{aligned} \sum_{j=1}^N \int_{\mathbb{R}^{dN}} (|\nabla_j u|^2 + V(\mathbf{x}_j)|u|^2) dx \\ \geq -N|\lambda_0| \geq -C_d^{\text{LT}} N \int_{\mathbb{R}^d} |V_-(\mathbf{x})|^{1+\frac{d}{2}} d\mathbf{x}, \end{aligned}$$

Only the uncertainty principle:

$$\sum_{j=1}^N \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 dx \geq \frac{C_d^{\text{K}}}{N^{2/d}} \int_{\mathbb{R}^d} \rho(\mathbf{x})^{1+\frac{2}{d}} d\mathbf{x},$$

Anyons?

$$0 < \alpha < 1 ?$$

We prove:

Local Hardy inequality for anyons

⇒ local exclusion principle, strength depending on α

⇒ Lieb-Thirring inequality for certain α

Anyons?

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Local Hardy inequality for anyons

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\Rightarrow Lieb-Thirring inequality for certain α

Notation

$$C_0^\infty(\mathbb{R}^{2N} \setminus \mathbb{D}) \cap \mathcal{H}_0^N \subseteq \mathcal{D}_{0,\alpha}^N \subseteq \mathcal{H}_0^N$$

Space of N -anyon wavefunctions with statistics parameter α and **finite kinetic energy**

$$T := \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 dx < \infty,$$

$$D_j := -i\nabla_j + \alpha \sum_{k \neq j} (\mathbf{x}_j - \mathbf{x}_k)^{-1} I$$

Hardy inequalities for anyons

Theorem (Many-anyon Hardy inequality)

Let Ω be an open convex set in \mathbb{R}^2 and let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction. Then

$$\int_{\Omega^N} \sum_{j=1}^N |D_j u|^2 dx \geq \frac{4C_{\alpha,N}^2}{N} \int_{\Omega^N} \sum_{i<j} \frac{|u|^2}{|\mathbf{x}_i - \mathbf{x}_j|^2} \chi_{\Omega \circ \Omega}(\mathbf{x}_i, \mathbf{x}_j) dx,$$

with

$$\Omega \circ \Omega := \left\{ (\mathbf{x}_1, \mathbf{x}_2) \in \Omega^2 : \frac{1}{2} |\mathbf{x}_1 - \mathbf{x}_2| < \text{dist} \left(\frac{1}{2} (\mathbf{x}_1 + \mathbf{x}_2), \Omega^c \right) \right\},$$

and

$$C_{\alpha,N} := \min_{p \in \{0,1,\dots,N-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|.$$

Ideas of the proof

- Pairwise relative parameterization of Ω

$$\int_{\Omega^N} \sum_{j=1}^N |D_j u|^2 dx \geq \frac{1}{N} \sum_{j < k} \int_{\Omega^{N-2}} \int_{\Omega^2} |(D_j - D_k)u|^2 d\mathbf{x}_j d\mathbf{x}_k d\tilde{x}$$

$$\mathbf{R} := \frac{1}{2}(\mathbf{x}_j + \mathbf{x}_k), \quad \mathbf{r} := \frac{1}{2}(\mathbf{x}_j - \mathbf{x}_k)$$

$$v(\mathbf{R}; \mathbf{r}) := u(\mathbf{R} + \mathbf{r}, \mathbf{R} - \mathbf{r}) \in \mathcal{H}_0^2$$

$$(D_j - D_k)u = (-i\nabla_{\mathbf{r}} + \alpha\mathbf{r}^{-1}I + \mathbf{A}_{jk}(\mathbf{R} + \mathbf{r}) - \mathbf{A}_{jk}(\mathbf{R} - \mathbf{r})) v$$

- Local magnetic Hardy inequality with symmetry:

Ideas of the proof

Lemma (Magnetic Hardy inequality with symmetry)

Let $\Omega = B_{R_2}(0) \setminus \bar{B}_{R_1}(0)$, $R_2 > R_1 \geq 0$, be an annular domain in \mathbb{R}^2 , and let there be a magnetic flux Φ inside $\bar{B}_{R_1}(0)$, determined on Ω by a vector potential $\mathbf{a} : \Omega \rightarrow \mathbb{R}^2$, s.t. $\nabla \wedge \mathbf{a} = 0$ on Ω and $\int_{\Gamma} \mathbf{a} \cdot d\mathbf{r} = \Phi$ for any simple loop Γ in Ω enclosing $\bar{B}_{R_1}(0)$.

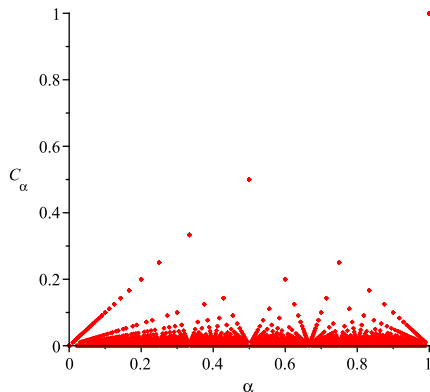
Furthermore, assume that \mathbf{a} is antipodal-**antisymmetric**, i.e.

$\mathbf{a}(-\mathbf{r}) = -\mathbf{a}(\mathbf{r})$ for all $\mathbf{r} \in \Omega$, and let $v \in C^\infty(\Omega)$ be a function on Ω with antipodal **symmetry**, $v(-\mathbf{r}) = v(\mathbf{r})$ for all $\mathbf{r} \in \Omega$. Then

$$\int_{\Omega} |D_{\mathbf{r}} v|^2 d\mathbf{r} \geq \min_{q \in \mathbb{Z}} \left| \frac{\Phi}{2\pi} - 2q \right|^2 \int_{\Omega} \frac{|v|^2}{|\mathbf{r}|^2} d\mathbf{r},$$

where $D_{\mathbf{r}} := -i\nabla_{\mathbf{r}} + \mathbf{a}(\mathbf{r})$.

The statistics-dependent constant $C_{\alpha,N}$



$$\begin{aligned} C_\alpha &:= \inf_{N \in \mathbb{N}} C_{\alpha,N} = \inf_{p,q \in \mathbb{Z}} |(2p+1)\alpha - 2q| \\ &= \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

A local exclusion principle for anyons

Lemma (Local energy / \sim Pauli principle)

Let $u \in \mathcal{D}_{0,\alpha}^n$ be a wavefunction of n anyons and let $\Omega \subseteq \mathbb{R}^2$ be either a disk or a square, with area $|\Omega|$. Then

$$\int_{\Omega^n} \sum_{j=1}^n |D_j u|^2 dx \geq (n-1) \frac{c_\Omega C_{\alpha,n}^2}{|\Omega|} \int_{\Omega^n} |u|^2 dx, \quad (1)$$

where c_Ω is a constant which satisfies $c_\Omega \geq 0.477$ for the disk, and $c_\Omega \geq 0.358$ for the square.

(cp. F. J. Dyson, A. Lenard, *Stability of Matter. I*, JMP 1967)

Idea of proof: use part of the kinetic energy together with $|D_j u| \geq |\nabla_j |u||$, and then bound the corresp. pairwise bosonic Neumann operator $H := -\Delta_{\mathbf{x}_j} - \Delta_{\mathbf{x}_k} + C_{\alpha,n}^2 |\mathbf{r}|^{-2} \chi_{\Omega \circ \Omega}$ on Ω^2

Lieb-Thirring inequalities for anyons

Theorem (Kinetic energy inequality for anyons)

Let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction on \mathbb{R}^2 . Then

$$\sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 dx \geq C_K C_{\alpha,N}^2 \int_{\mathbb{R}^2} \rho(\mathbf{x})^2 dx,$$

for some positive constant C_K .

Corollary (Lieb-Thirring inequality for anyons)

Let V be a real-valued potential on \mathbb{R}^2 . Then

$$\sum_{j=1}^N \int_{\mathbb{R}^{2N}} (|D_j u|^2 + V(\mathbf{x}_j) |u|^2) dx \geq -C_{\text{LT}} C_{\alpha,N}^{-2} \int_{\mathbb{R}^2} |V_-(\mathbf{x})|^2 dx,$$

for some positive constant C_{LT} .

Ingredients of the proof

Lemma (Local Pauli principle)

Let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction on \mathbb{R}^2 and $\Omega \subseteq \mathbb{R}^2$ a simply connected domain on which the local energy bound (1) holds for some constant c_Ω . Then

$$T_\Omega := \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 \chi_\Omega(\mathbf{x}_j) dx \geq \frac{c_\Omega C_{\alpha,N}^2}{|\Omega|} \left(\int_\Omega \rho - 1 \right).$$

Lemma (Local uncertainty principle)

Let $u \in \mathcal{D}_{0,\alpha}^N$ be an N -anyon wavefunction on \mathbb{R}^2 , and Q a square with area $|Q|$. Then

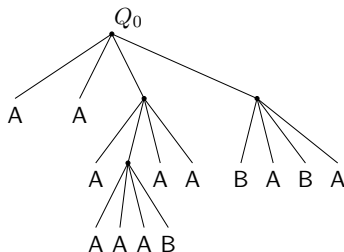
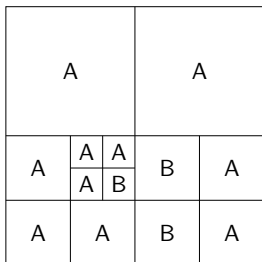
$$T_Q \geq C'_2 (\int_Q \rho)^{-1} \int_Q \left[\rho(\mathbf{x})^{\frac{1}{2}} - \left(\frac{\int_Q \rho}{|Q|} \right)^{\frac{1}{2}} \right]_+^4 dx.$$

Splitting algorithm

Start with a square $Q = Q_0 \subseteq \mathbb{R}^2$ s.t. $\text{supp } u \subseteq Q_0$

- split $Q \rightarrow \bigcup_{j=1}^4 Q'_j$ s.t. $|Q'_j| = |Q|/4$
- if $\int_{Q'_j} \rho < 2$: don't split Q'_j further, mark it A
- if all Q'_j marked A: back up to Q , mark it B (discard Q'_j 's)
- iterate splitting algorithm for each unmarked $Q'_j =: Q$

Splitting algorithm



Example of a splitting of Q_0 and a corresponding tree \mathbb{T} of subsquares. For the B-square at level 3 in the tree, the set $\mathcal{A}(Q)$ of associated A-squares consists of 8 elements, while for the two B-squares at level 2, $\mathcal{A}(Q)$ coincide and has 4 elements.

Three classes of subsquares:

$$\mathcal{A}_1 : \quad \text{A-squares } Q \in \mathbb{T} \text{ s.t. } \int_Q \rho \leq c \frac{(\int_Q \rho)^2}{|Q|}, \quad (0 \leq \int_Q \rho < 2)$$

$$\mathcal{A}_2 : \quad \text{A-squares } Q \in \mathbb{T} \text{ s.t. } \int_Q \rho > c \frac{(\int_Q \rho)^2}{|Q|}, \quad (0 \leq \int_Q \rho < 2)$$

$$\mathcal{B} : \quad \text{B-squares } Q \in \mathbb{T}, \quad (2 \leq \int_Q \rho < 8)$$

Local energy bounds:

\mathcal{B} : local Pauli energy + local uncertainty principle \Rightarrow

$$T_{Q_B} \geq C_{\alpha, N}^2 \left(c_1 \int_{Q_B} \rho^2 + c_2 \frac{(\int_{Q_B} \rho)^2}{|Q_B|} \right)$$

Proof cont.

\mathcal{A}_2 : local uncertainty principle, $c := 24 \Rightarrow$

$$T_{Q_A} \geq \frac{C'_2}{8} \int_{Q_A} \rho^2$$

\mathcal{A}_1 : negligible contribution to the energy compared to \mathcal{B} !

For each B-square Q_B (say at level k in \mathbb{T}):

$$\frac{(\int_{Q_B} \rho)^2}{|Q_B|} \geq \frac{4}{4^{-k}|Q_0|},$$

while for all \mathcal{A}_1 -squares associated with Q_B :

$$\sum_{Q \in \mathcal{A}_1(Q_B)} \int_Q \rho^2 \leq \sum_{j=1}^k \sum_{\substack{Q \in \mathcal{A}_1(Q_B) \\ \text{at level } j}} c \frac{(\int_Q \rho)^2}{|Q|} \leq \sum_{j=1}^k 3c \frac{4}{4^{-j}|Q_0|} \leq 96 \frac{4^{k+1}}{|Q_0|}.$$

Proof cont.

Hence

$$T_{Q_B} \geq C_{\alpha, N}^2 \left(c_1 \int_{Q_B} \rho^2 + \frac{c_2}{96} \sum_{Q \in \mathcal{A}_1(Q_B)} \int_Q \rho^2 \right),$$

and

$$\begin{aligned} T &= \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 \left(\sum_{Q_A \in \mathcal{A}} \chi_{Q_A}(\mathbf{x}_j) + \sum_{Q_B \in \mathcal{B}} \chi_{Q_B}(\mathbf{x}_j) \right) dx \\ &\geq \sum_{Q_A \in \mathcal{A}_2} T_{Q_A} + \sum_{Q_B \in \mathcal{B}} T_{Q_B} \geq C_K C_{\alpha, N}^2 \int_{Q_0} \rho^2. \end{aligned}$$



Application 1: Ground state energy of the free anyon gas

Energy per unit area of a gas of “free” anyons with odd-fractional statistics parameter $\alpha = \frac{\ell}{\nu}$, confined to an area $|\Omega|$, is bounded below by

$$\frac{T}{|\Omega|} \geq C_K \frac{\bar{\rho}^2}{\nu^2},$$

with density $\bar{\rho} := N/|\Omega|$.

Application 2: Anyons in a harmonic oscillator

Harmonic oscillator Hamiltonian:

$$H = \sum_{j=1}^N D_j^2 + \omega^2 |x|^2.$$

By minimization of the corresponding quadratic form, we then have

$$H \geq \frac{4}{3} \sqrt{\frac{C_K}{\pi}} C_{\alpha, N} \omega N^{\frac{3}{2}}.$$

Application 3: Stability of 'anyonic matter'

Anyons interacting by 3D Coulomb interactions:

$$H(N, Z, R) := \sum_{j=1}^N D_j^2 + \sum_{1 \leq j < k \leq N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|} - \sum_{j=1}^N \sum_{k=1}^K \frac{Z}{|\mathbf{x}_j - \mathbf{R}_k|} + \sum_{1 \leq j < k \leq K} \frac{Z^2}{|\mathbf{R}_j - \mathbf{R}_k|}.$$

By Baxter's electrostatic inequality in 3D, and Lieb-Thirring...:

Theorem (Stability of 'anyonic matter' for odd-fractional statistics)

For a normalized N -anyon wavefunction $u \in \mathcal{D}_{0,\alpha}^N$ with $\alpha = \frac{\mu}{\nu}$ an odd numerator fraction, we have the lower bound

$$\langle u, H(N, Z, R)u \rangle \geq -C \nu^2 Z^2 (K + N),$$

for some positive constant C .