Universality in Lie algebras and Chern-Simons theory

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Plan

- **Vogel's plane**

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- **Casimir operators for simple Lie algebras**

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References

R.L. Mkrtchyan, A.N. Sergeev, A.P.V. arXiv:1105.0115

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Vogel's plane

Vogel's plane is a quotient $\mathbb{P}^2/S_3$ of the projective plane with homogeneous coordinates α, β and γ. It is a moduli space of a tensor category, which is meant to be a model of the **universal simple Lie algebra [Vogel, 1999].**

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Table: Vogel's parameters for simple Lie algebras

Type	Lie algebra	α	β	γ	$t = h^\vee$
A_n	$\mathfrak{sl}_{n+1}$	-2	2	$(n+1)$	$n+1$
B_n	$\mathfrak{so}_{2n+1}$	-2	4	$2n-3$	$2n-1$
C_n	$\mathfrak{sp}_{2n}$	-2	1	$n+2$	$n+1$
D_n	$\mathfrak{so}_{2n}$	-2	4	$2n-4$	$2n-2$
G_2	$\mathfrak{g}_2$	-2	$10/3$	$8/3$	4
F_4	$\mathfrak{f}_4$	-2	5	6	9
E_6	$\mathfrak{e}_6$	-2	6	8	12
E_7	$\mathfrak{e}_7$	-2	8	12	18
E_8	$\mathfrak{e}_8$	-2	12	20	30

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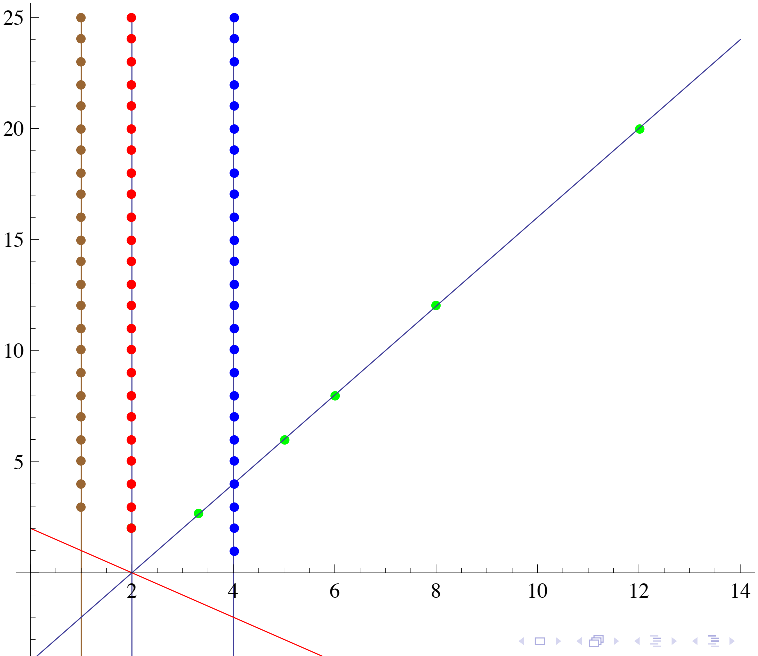
Motivations: Knot theory (**Vassiliev** invariants, **Kontsevich** integral), **Deligne**'s study of exceptional Lie algebras

Vogel's map



Universal formulae for dimension

Vogel, 1999:

$$\dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$$

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In the decomposition

$$S^2\mathfrak{g} = \mathbb{C} \oplus Y(\alpha) \oplus Y(\beta) \oplus Y(\gamma)$$

the Casimir values C_2 are respectively $4t-2\alpha, 4t-2\beta, 4t-2\gamma$ (which can be used as a definition of Vogel's parameters) and

$$dim\, Y_2(\alpha) = -\frac{(3\,\alpha-2\,t)\,(\beta-2\,t)\,(\gamma-2\,t)\,\,t\,\,(\beta+t)\,(\gamma+t)}{\alpha^2\,(\alpha-\beta)\,\beta\,(\alpha-\gamma)\,\gamma}.$$

Casimir operators

For any simple complex Lie algebra $\mathfrak{g}$ define the Casimir operators C_p as the following elements of the centre of the corresponding universal enveloping algebra $U\mathfrak{g}$ as

$$C_p = g_{\mu_1 \dots \mu_p} X^{\mu_1} \dots X^{\mu_p},\ p = 0, 1, 2, \dots$$

where X^{μ} are the generators of $\mathfrak{g}$,

$$g^{\mu_1 \cdots \mu_n} = Tr(\hat{X}^{\mu_1} \dots \hat{X}^{\mu_n}),$$

where the trace is taken in the adjoint representation of $\mathfrak{g}$ and the indices are lowered using the Cartan-Killing metric

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Consider the values of C_p on the adjoint representation. We claim that they can be expressed rationally in the terms of the universal Vogel's parameters α, β, γ.

Universal formula for Casimirs

Theorem (MSV, 2011)

The generating function $C(z) = \sum_{p=1}^{\infty} C_p z^p$ has the form

$$C(z) = z^2 \frac{96t^3 + 168t^3 z + 6(14t^3 + tt_2 - t_3)z^2 + (13t^3 + 3tt_2 - 4t_3)z^3}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)}$$

where

$$t_2 = \alpha^2 + \beta^2 + \gamma^2, \; t_3 = \alpha^3 + \beta^3 + \gamma^3.$$

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In particular, the first few Casimirs are

$$C_1 = 0, \; C_2 = 1, \; C_3 = -\frac{1}{4}, \; C_4 = \frac{3tt_2 - t_3}{16t^3}.$$

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Proof uses the results of **Okubo (1977)** and **Landsberg-Manivel (2004)**.

Calculations: the quartic Casimir

Cvitanovic: for the orthogonal group $SO(n)$

$$C_4 = \frac{(n-2)(n^3 - 9n^2 + 54n - 104)}{8}.$$

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The universal parameters of $SO(n)$ are

$$\alpha = -2,\ \beta = 4,\ \gamma = n - 4;\ t = n - 2$$

Assume that the numerator is a symmetric polynomial of α, β, γ :

$$n^3 - 9n^2 + 54n - 104 = At^3 + Btt_2 + Ct_3,$$

$$t_2 = \alpha^2 + \beta^2 + \gamma^2 = n^2 - 8n + 36,\ t_3 = \alpha^3 + \beta^3 + \gamma^3 = n^3 - 12n^2 + 48n - 8.$$

This gives 4 relations on three constants A, B and C, which in general should not be consistent.

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This gives 4 relations on three constants A, B and C, which in general should not be consistent.

In our case however we do have a solution: $A = 0,\ B = 3/2,\ C = -1/2$ which leads to our previous formula.

Let $\mathfrak{h}$ be Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^*$ be its dual space. The root system $R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ is defined as the set of non-zero weights of adjoint representation of $\mathfrak{g}$.

Casimir operators and root systems

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On $\mathfrak{h}$ there is a non-degenerate *canonical Cartan-Killing form*

$$< X, Y >= tr\, ad_X ad_Y, \quad X, Y \in \mathfrak{h},$$

where $ad_X : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad_X(Z) = [X, Z]$. In terms of the roots the canonical form can be written as

$$< X, Y >= \sum_{\alpha \in R} \alpha(X)\alpha(Y) = 2 \sum_{\alpha \in R_+} \alpha(X)\alpha(Y)$$

for any choice of positive roots $R_+ \subset R$.

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Harish-Chandra: the algebra of Casimir operators is isomorphic to the algebra of **shifted symmetric functions** on $\mathfrak{h}^*$ such that

$$f(w\xi - \rho) = f(\xi - \rho), \; \xi \in \mathfrak{h}^*, \; w \in W$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

Consider now the Casimir operator $\hat{C}_{2k}$ corresponding to the function

$$\hat{C}_{2k}(\lambda) = \sum_{\alpha \in R} [< \lambda + \rho, \alpha >^{2k} - < \rho, \alpha >^{2k}],$$

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Theorem (MSV, 2011).
For the adjoint representation with $\lambda = \theta$ being maximal root the generating function $\hat{C}(z) = \sum \hat{C}_{2k} z^k$ has the form

$$\hat{C}(z) = -2z \frac{d}{dz} \ln \frac{(16t^2 - (2t-\alpha)^2 z)(16t^2 - (2t-\beta)^2 z)(16t^2 - (2t-\gamma)^2 z)}{(16t^2 - \alpha^2 z)(16t^2 - \beta^2 z)(16t^2 - \gamma^2 z)}.$$

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Proof is based on **Key lemma**:

$$\prod_{\mu \in R_+} \frac{\phi((\mu, \theta + \rho))}{\phi((\mu, \rho))} = \frac{\phi((\alpha - 2t)/2)}{\phi(\alpha/2)} \frac{\phi((\beta - 2t)/2)}{\phi(\beta/2)} \frac{\phi((\gamma - 2t)/2)}{\phi(\gamma/2)}$$

for any even or odd function $\phi(x)$ (cf. **Landsberg-Manivel**).

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Chern-Simons action is

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where A is $\mathfrak{g}$-valued 1-form on M and Tr denotes some invariant bilinear form on a simple Lie algebra $\mathfrak{g}$.

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The universal Chern-Simons theory depends on 4 parameters $\alpha, \beta, \gamma, \kappa$ defined up to a common multiple, where α, β, γ are Vogel's parameters. In fact it is more convenient to replace κ by

$$\delta = \kappa + t = \kappa + \alpha + \beta + \gamma.$$

Chern-Simons partition function of S^3

The Chern-Simons partition function

$$Z(M) = \int DA exp\left(\frac{ik}{4\pi}\int_M Tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)\right)$$

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Let $\mathfrak{h}$ be Cartan subalgebra of the Lie algebra $\mathfrak{g}$, r be its rank, $Q \subset \mathfrak{h}^*$, $Q^\vee \subset \mathfrak{h}$ be the root and coroot lattices and $(,)$ be the minimal invariant bilinear form on $\mathfrak{g}$, then

$$Z = Z(S^3) = Vol(Q^\vee)^{-1}(k+h^\vee)^{-r/2}\prod_{\mu\in R_+} 2\sin\frac{\pi(\mu,\rho)}{(k+h^\vee)}.$$

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For an arbitrary invariant form we have

$$Z = Z(S^3) = Vol(Q^\vee)^{-1}\delta^{-r/2}\prod_{\mu\in R_+} 2\sin\frac{\pi(\mu,\rho)}{\delta}.$$

Splitting the partition function

Rewrite Z as the product $Z = Z_1 Z_2$, where

$$Z_1 = Vol(Q^\vee)^{-1}\delta^{-r/2} \prod_{\mu \in R_+} \frac{2\pi(\mu,\rho)}{\delta}$$

and

$$Z_2 = \prod_{\mu \in R_+} \sin \frac{\pi(\mu,\rho)}{\delta} \Big/ \frac{\pi(\mu,\rho)}{\delta}.$$

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The first factor (non-perturbatve part) has a clear geometric meaning (**Ooguri, Vafa**):

$$Z_1 = \frac{(2\pi\delta^{-1/2})^{dim\,\mathfrak{g}}}{Vol(G)},$$

where $Vol(G)$ is the volume of the corresponding compact simply connected group G.

Perturbative part

Consider the corresponding free energy $F_2 = -\ln Z_2$. Using

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} (1 - (\frac{x}{n})^2)$$

we have

$$\ln \frac{\sin \pi x}{\pi x} = \sum_{n=1}^{\infty} \ln(1 - (\frac{x}{n})^2) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \frac{x^{2m}}{n^{2m}} = \sum_{m=1}^{\infty} \frac{\zeta(2m)}{m} x^{2m},$$

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Thus the perturbative part of free energy is

$$F_2 = \sum_{m=1}^{\infty} \frac{\zeta(2m)}{m} \sum_{\mu \in R_+} \left(\frac{(\mu, \rho)}{\delta} \right)^{2m}.$$

To show its universality we should express the sums $\sum_{\mu \in R} (\mu, \rho)^{2m}$ in terms of Vogel's parameters.

Weyl formula and universality

Consider the exponential generating function of $p_k = \sum_{\mu \in R} (\mu, \rho)^k$:

$$F(x) = \sum_{k=1}^{\infty} \frac{p_k}{k!} x^k = \sum_{\mu \in R} (e^{x(\mu,\rho)} - 1).$$

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Theorem (MV, 2012).

$$F(x) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(\frac{x\alpha}{4})} \frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})} - \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}$$

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Idea of the proof: use Weyl's character formula for the adjoint representation

$$\chi_\theta(x\rho) = \prod_{\mu \in R_+} \frac{\sinh(x(\mu, \theta+\rho)/2)}{\sinh(x(\mu,\rho)/2)}$$

and Key lemma.

Corollary: Freudenthal-de Vries strange formulae

Expanding the previous formula in x we have in the leading order

$$\sum_{\mu \in R_+} (\mu, \rho)^2 = \frac{t^2}{12} dim\, \mathfrak{g},$$

which is a homogeneous form of the **Freudenthal-de Vries strange formula**

$$< \rho, \rho >= \frac{1}{24} dim\, \mathfrak{g}.$$

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In the next orders we have

$$\sum_{\mu\in R_+} (\mu,\rho)^4 = \frac{t(18t^3 - 3tt_2 + t_3)}{480} dim\, \mathfrak{g},$$

where $t_2 = \alpha^2 + \beta^2 + \gamma^2$, $t_3 = \alpha^3 + \beta^3 + \gamma^3$, and

$$\sum_{\mu\in R_+} (\mu,\rho)^6 = \frac{t(396t^5 - 157t^3 t_2 + 15tt_2^2 + 39t^2 t_3 - 5t_2 t_3)}{16128} dim\, \mathfrak{g}.$$

Expectation value of the unknotted Wilson loop C in S^3

$$< W(C) >= \frac{1}{Z} \int dA e^{iS(A)} W(C), \quad W(C) = TrP(\exp \int A_\mu dx^\mu)$$

with A_μ taken in adjoint representation of $\mathfrak{g}$ can be given as

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Central charge c can be expressed universally as

$$c = \frac{\kappa(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma(\kappa+\alpha+\beta+\gamma)} = \frac{(\delta-t)(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma\delta}.$$

More universal formulae in Chern-Simons theory

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Proof is based on explicit formulae given by **Witten**.

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Which sectors of Chern-Simons theory are universal ? In particular, is the volume of G a universal quantity ?

What does this all mean for other values of parameters?

Vogel's map