

# Quantum affine algebras and categorification - I

(D. Hernandez)

Aim: To realize algebraic/combinatorial structures (e.g. cluster algebras, quantum coordinate rings with (dual) canonical basis) in terms of categories of representations of quantum affine algebras.

## §1. Quantum affine algebras and representations.

(1.1) Let  $\mathfrak{g}$  be a finite dimensional simple Lie algebra, simply-laced (A, D, E type). Let  $C = (C_{ij})_{1 \leq i, j \leq n}$  be the Cartan matrix of  $\mathfrak{g}$ .

Note:  $C_{ij} \in \{2, -1, 0\}$  and  $C_{ij} = C_{ji}$ .

Let  $q \in \mathbb{C}^\times$ , not a root of unity.

$U_q(\mathfrak{g})$  = quantum group;  $q$ -deformation of  $U(\mathfrak{g})$  (enveloping algebra of  $\mathfrak{g}$ ).

[Drinfeld-Jimbo]

$U_q(\mathfrak{g})$  is generated by  $E_i, F_i, K_i^{\pm 1}$  ( $1 \leq i \leq n$ )

Relations:  $[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$

$$\cdot \quad K_i E_j K_i^{-1} = q^{c_{ij}} E_j \quad ; \quad K_i F_j K_i^{-1} = q^{-c_{ij}} F_j$$

$$\cdot \quad \text{For } i \neq j. \quad [E_i, E_j] = 0 = [F_i, F_j] \quad \text{if } c_{ij} = 0$$

$$\left. \begin{aligned} E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 \\ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0 \end{aligned} \right\} \text{if } c_{ij} = -1$$

(1.2) Let  $\mathcal{C}_g$  be the category of finite-dimensional representations of  $U_q(g)$  (2)

- $\mathcal{C}_g$  is semisimple. The simple objects of  $\mathcal{C}_g$  are parametrized by  $P_+$  = set of dominant weights of  $g$ .
- $\lambda \in P_+ : \lambda = \sum_{i=1}^n \lambda_i \omega_i$   $\lambda_1, \dots, \lambda_n \in \mathbb{N}$   
 $\omega_1, \dots, \omega_n$  : fundamental weights
- $U_q(g)$  is a Hopf algebra.  $\Delta : U_q(g) \rightarrow U_q(g)^{\otimes 2}$  is coproduct. Hence  $\mathcal{C}_g$  is a tensor category together with a braiding:  $V \otimes V' \xrightarrow{\sim} V' \otimes V$  ('non-trivial' isomorphism)

Let  $\text{Rep}(U_q(g))$  be the Grothendieck ring of  $\mathcal{C}_g$ .  $\text{Rep}(U_q(g))$  is free group generated by  $[V]$ : isomorphism classes of objects of  $\mathcal{C}_g$  with relations:  $[V_2] = [V_1] + [V_3]$  for every short exact sequence

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

Product on  $\text{Rep}(U_q(g))$ :  $[V] \cdot [W] = [V \otimes W]$

- $\text{Rep}(U_q(g))$  is commutative ring (since  $V \otimes W \cong W \otimes V$ )
- $\text{Rep}(U_q(g)) = \bigoplus_{\lambda \in P_+} \mathbb{Z} [L(\lambda)]$  as abelian group.  
 $L(\lambda)$  simple object assoc. to  $\lambda \in P_+$
- $\simeq \mathbb{Z} [[L(\omega_1), L(\omega_2), \dots, L(\omega_n)]]$  as ring  
 $L(\omega_i)$  polynomial ring generated by  $n$  elements.

(1.3) Loop algebra of  $g$ ,  $\mathfrak{L}g := g \otimes \mathbb{C}[t^{\pm 1}]$  (ring of Laurent polynomials)

$\mathfrak{L}g$  is Lie algebra:  $[x \otimes f, y \otimes g] := [x, y] \otimes fg$

for  $x, y \in g$ ;  $f, g \in \mathbb{C}[t, \bar{t}]$

There is a  $q$ -deformation  $U_q(\mathfrak{L}g)$  of  $U(\mathfrak{L}g)$ , called quantum loop algebra. Two possible constructions of  $U_q(\mathfrak{L}g)$ : ③

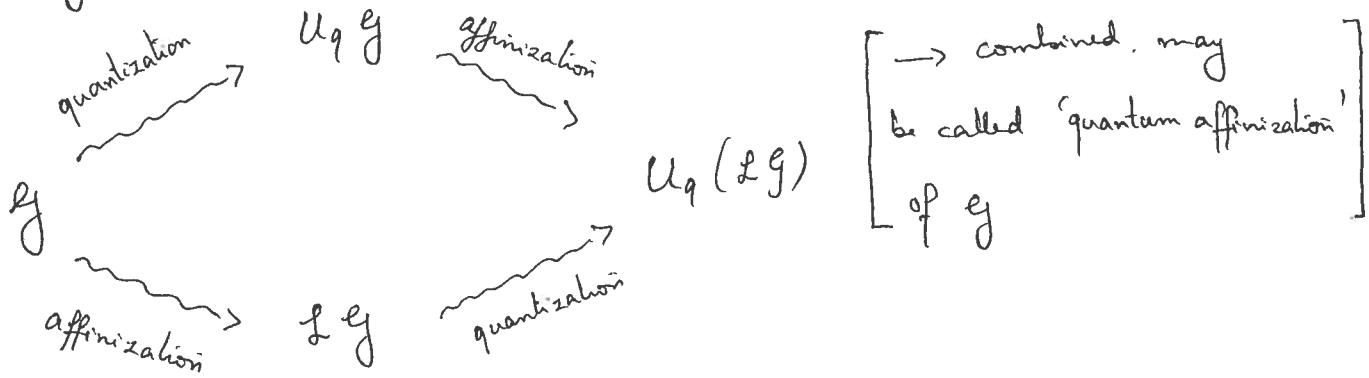
(i)  $U_q(\mathfrak{L}g)$  is a quotient of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$ .

which gives 'Drinfeld-Jimbo' presentation of  $U_q(\mathfrak{L}g)$ .

(ii) Affinization of  $U_q(g)$  (Drinfeld presentation). That is,

- Generators  $\{E_i, F_i, K_i^{\pm 1}\}$  of  $U_q(g)$  are replaced by  $\{E_{im}, F_{im}, K_{im}^{\pm 1}\}_{\substack{i=1 \dots n \\ m \in \mathbb{Z}}}$
- Explicit relations ~~can~~ be computed among these generators.

Summarizing:



(1.4) Let  $\mathcal{C}$  be the category of finite-dimensional representations of  $U_q(\mathfrak{L}g)$ .

- $\mathcal{C}$  is a tensor category ( $U_q(\mathfrak{L}g)$  is a Hopf algebra).
- $\mathcal{C}$  is not semi-simple; not braided.

- Simple objects in  $\mathcal{C}$  are parametrized by  $n$ -tuple of polynomials (Drinfeld polynomials), encoded as:

$$\text{Simple objects in } \mathcal{C} = \left\{ L^{(m)} : m \text{ is dominant monomial in } \mathbb{Z}[Y_{i,a}^{\pm 1} : \begin{array}{l} 1 \leq i \leq n \\ a \in \mathbb{C}^\times \end{array}] \right\}$$

i.e.  $m = \prod_{i,a} Y_{i,a}^{m_{i,a}}$   $m_{i,a} \in \mathbb{N}$ .

Remark.  $Y_{i,a}^{m_{i,a}}$  'represents'  $\lambda \in \mathbb{C}^*$  is root of  $i^{\text{th}}$  Drinfeld Poly.  
is of multiplicity  $m_{i,a}$ . (4)

In particular we have fundamental representations  $L(Y_{i,a}) \left[ \begin{smallmatrix} \forall i=1 \dots n \\ a \in \mathbb{C}^* \end{smallmatrix} \right]$ .

- In general,  $\dim(L(m))$  is not known.

(1.5) Let  $\text{Rep}(U_q(\mathbb{L}g))$  be the Grothendieck ring of  $\mathbb{L}$ .

$$\cdot \text{Rep}(U_q(\mathbb{L}g)) = \bigoplus_{\substack{m: \text{dominant} \\ \text{monomial in } \mathbb{Z}[Y_{i,a}]_{i=1 \dots n} \\ a \in \mathbb{C}^*}} \mathbb{Z}[L(m)] \quad \text{as abelian group.}$$

• Theorem [Frenkel-Reshetikhin]  $\text{Rep}(U_q(\mathbb{L}g))$  is commutative

and freely generated by fundamental representations

$$\text{Rep}(U_q(\mathbb{L}g)) \simeq \mathbb{Z} [[L(Y_{i,a}) : \substack{i=1 \dots n \\ a \in \mathbb{C}^*}]] \quad \text{polynomial ring}$$

e.g.,  $g = \mathfrak{sl}_2$ . An arbitrary simple object is a tensor product of  
 $L(m) \simeq V_1 \otimes \dots \otimes V_R$  where  $V_1, \dots, V_R$  are evaluation  
representations

•  $\exists$  evaluation homomorphisms  $U_q(\mathbb{L}\mathfrak{sl}_2) \longrightarrow U_q(\mathfrak{sl}_2)$ , or more generally  
for  $g$  of type A. But not for other types.

However an arbitrary  $\otimes$  of evaluation repns. is not simple.

→ The evaluation repns (for  $\mathfrak{sl}_2$ ) are exactly the prime repns.

( $V$  is prime if  $V \neq V' \otimes V''$  for some  $V', V''$  non-trivial)

→  $L(m)$  is evaluation repn iff  $m$  is of the form  
 $m = Y_a Y_{aq} Y_{aq^2} \dots Y_{aq^{2(k-1)}}$   $(a \in \mathbb{C}^*, k \geq 1 \text{ integer})$

Quantum affine algebras and categorification-II

(D. Hernandez)

§2. Cluster algebras and their categorification. ( $0 \leq n \leq d$ )

(2.1) Fomin-Zelevinsky cluster algebra: Let  $B = (B_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d-n}}$  be a  $d \times (d-n)$  integer matrix such that  $B = (B_{ij})_{\substack{1 \leq i, j \leq d-n}}$  is skew-sym.

$\mathbb{F} := \mathbb{C}(x_i)_{1 \leq i \leq d}$  field of rational functions of  $x_1, \dots, x_d$ .

The tuple  $((x_1, \dots, x_d); B)$  is called the initial seed. We can define new seeds from the initial one by mutations  $\mu_k$  ( $1 \leq k \leq d-n$ ).

$$(f_1, \dots, f_d; M) \xrightarrow{\mu_k} (f'_1, \dots, f'_{k-1}, f'_k, f'_{k+1}, \dots, f_d; M')$$

•  $f'_k$  and  $M'$  are given by Fomin-Zelevinsky mutation formulae

$$f'_k = \frac{1}{f_k} \left( \prod_{\substack{M_{ik} > 0 \\ i=1 \dots d}} f_i^{M_{ik}} + \prod_{\substack{M_{ik} < 0 \\ i=1 \dots d}} -f_i^{-M_{ik}} \right)$$

$$\boxed{\begin{aligned} M'_{ij} &= -M_{ij} && \text{if } j=k \text{ or } i=k \\ M'_{ij} &= M_{ij} + \frac{|M_{ik}|M_{kj} + |M_{ij}|}{2} && \text{otherwise} \end{aligned}}$$

Remark.  $\mu_k^2 = \text{Id}$ .

Cluster variables are the functions  $f_i$ 's occurring in all seeds obtained from the initial seed by successive mutations.

Cluster monomial: monomials in cluster variables occurring in some seed.

$A(B)$ : cluster algebra is the subring of  $\mathbb{F}$  generated by all cluster variables.

Thm [Fomin-Zelevinsky] Every cluster monomial is a Laurent poly.  
in the variables of an arbitrary seed.

Example

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \quad n=2, d=4.$$

$$\mathbb{F} = \mathbb{C}(x_1, x_2, \underbrace{x_3, x_4}_{\text{Frozen}})$$

$$\begin{array}{c} (x_1, x_2, x_3, x_4; B) \\ \swarrow \mu_1 \\ \left( \frac{x_2+x_3}{x_1}, x_2, x_3, x_4; \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \right) \\ \mu_2 | \\ \left( \frac{x_2+x_3}{x_1}, \frac{x_1x_3+x_2x_4+x_3x_4}{x_1x_2}, x_3, x_4; \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \right) \end{array} \dashrightarrow \begin{array}{c} \overline{\mu_2} \\ \left( x_1, \frac{x_1+x_4}{x_2}, x_3, x_4; \begin{pmatrix} 0 & +1 \\ -1 & 0 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \right) \\ | \mu_4 \\ \left( \frac{x_1x_3+x_3x_4+x_2x_4}{x_1x_2}, \frac{x_1+x_4}{x_2}, x_3, x_4; \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \right) \end{array}$$

finite number of seeds, 7 cluster variables (cluster alg. of finite type).

Thm [Fomin-Zelevinsky]  $A(B)$  is cluster algebra of finite type iff  
 $\tilde{B} = (B_{ij})_{1 \leq i, j \leq d-n}$  is mutation equivalent to a matrix which is  
adjacency matrix of an ADE quiver (arbitrary orientation of Dynkin  
diagram).

(2.2) Categorification of Cluster algebra [Hernandez-Leclerc]

A monoidal categorification of a cluster algebra  $A$  is a  
monoidal category  $M$ , s.t. there is a ring isomorphism

$$\varphi: A \longrightarrow K(M) \text{ (Grothendieck ring of } M) \quad (7)$$

which induces bijection

$$\{ \text{cluster variables} \} \longleftrightarrow \begin{cases} \text{classes of simple, prime} \\ \text{real* objects in } M \end{cases}$$

\*  $V$  is real if  $V \otimes V$  is simple.

$$\{ \text{cluster monomials} \} \longleftrightarrow \{ \text{real simple objects of } M \}$$

Applications. Suppose  $M$  is a monoidal categorification of  $A$

(i) For  $A$ . • we get positivity conjecture, i.e. the coefficients of

Laurent polynomial expressing a cluster variable in terms of

cluster variables of another seed are in  $\mathbb{N}$ .

- linear independence of cluster monomials.

(ii) For  $M$ . • factorization of real simple objects in prime reprs.  
using combinatorics of cluster algebra

(2.3) Theorem [Hernandez - Leclerc; Nakajima]

Let  $\mathfrak{g}$  be a simple lie algebra of type ADE. There is a  
sub-monoidal category  $\mathcal{C}_\mathfrak{g}$  of  $\mathcal{C}$  which is a monoidal categorification  
of a cluster algebra of the same type as  $\mathfrak{g}$ .

→ This implies positivity, linear independence for all finite  
type cluster algebra.

Example  $\mathfrak{g} = \mathfrak{sl}_3$ .

$\mathcal{C}_1$  = subcategory of repn.  $V$  in  $\mathcal{C}$  such that the image  $[V]$  of  $V$

in  $\text{Rep}(U_q(\mathfrak{L}\mathfrak{g}))$  is in  $\mathbb{Z}[[L(Y_{1,1})], [L(Y_{1,q^2})], [L(Y_{2,q})], [L(Y_{2,q^3})]]$

(i)  $\mathcal{C}_1$  is monoidal

(ii) There are only 7 prime repns in  $\mathcal{C}_1$  corr. to

$$\begin{array}{ccccccccc} Y_{1,1}, & Y_{1,q^2}, & Y_{2,q}, & Y_{2,q^3}, & Y_{1,1}Y_{2,q^3}, & Y_{1,1}Y_{1,q^2}, & Y_{2,q}Y_{2,q^3} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ X_1, & \frac{x_2+x_3}{x_1}, & X_2, & \frac{x_1+x_4}{x_2}, & \frac{x_2x_4+x_3x_4+x_1x_3}{x_1x_2}, & X_3, & X_4 \end{array}$$

This bijection extends to an iso.  $K(\mathcal{C}_1) \cong A(B)$

$$\text{eg. } x_1 \left( \frac{x_2+x_3}{x_1} \right) = x_2 + x_3$$

$$\begin{aligned} [L(Y_{1,1}) \otimes L(Y_{1,q^2})] &= [L(Y_{2,q})] + [L(Y_{1,1}Y_{1,q^2})] \\ &= [L(Y_{2,q}) \oplus L(Y_{1,1}Y_{1,q^2})] \end{aligned}$$

# Quantum affine algebras and categorifications III

(D. Hernandez)

Recall:  $U_q(\mathfrak{g})$  = quantum loop algebra

$\mathcal{C}$  = category of finite dim'l repns. of  $U_q(\mathfrak{g})$ .

$$\text{Rep}(U_q(\mathfrak{g})) = K(\mathcal{C}) \simeq \mathbb{Z}[[L(Y_{i,a})]]_{1 \leq i \leq n, a \in \mathbb{C}^\times}.$$

$\mathcal{C}_1 \subset \mathcal{C}$  sub-monoidal category

$K(\mathcal{C}_1)$  has cluster algebra structure and is monoidal categorification of finite type cluster algebra of type of  $\mathfrak{g}$ .

## §3. Quantum Grothendieck rings.

$K(\mathcal{C})$  has quantizations ( $t$ -deformation) as follows:

- (3.1)  $K(\mathcal{C})$  has quantizations ( $t$ -deformation) as follows:
  - [Nakajima; Varagnolo-Vasserot] obtained from a convolution diagram of quiver varieties.  $R_t(\mathcal{C})$
  - [Hernandez] obtained from Frenkel-Reshetikhin  $q$ -characters.  $K_t(\mathcal{C})$

Remark. These two deformations are only slightly different.

Construction of  $K_t(\mathcal{C})$ .  $K_t(\mathcal{C})$  is a subalgebra of quantum torus.

Define quantum Cartan matrix  $C(z)_{ij} := \begin{cases} z + \bar{z} & \text{if } c_{ij} = 2 \\ 0 & \text{if } c_{ij} = 0 \\ -1 & \text{if } c_{ij} = -1 \end{cases}$

$$\tilde{C}(z) = \sum_{m \geq 0} z^m \tilde{C}_m \quad \text{expansion of the inverse of } C(z) \text{ in } z$$

$$\rightarrow \text{Replace the product } Y_{i,q^p a} Y_{j,q^p a} = Y_{j,q^p a} Y_{i,q^p a} \quad \begin{cases} 1 \leq i, j \leq n \\ p, D \in \mathbb{Z} \\ a \in \mathbb{C}^\times \end{cases}$$

$$\text{by } Y_{i,q^p a} *_{t^D} Y_{j,q^p a} = t^{N(i,p;j,D)} Y_{j,q^p a} *_{t^D} Y_{i,q^p a} \quad \begin{cases} Y_{i,a} *_{t^D} Y_{j,b} = Y_{j,b} *_{t^D} Y_{i,a} \\ \text{if } \frac{a}{b} \notin q^{\mathbb{Z}}. \end{cases}$$

$$\mathcal{N}(i, p; j, d) := \tilde{c}_{ij}^{(p-d-1)} - \tilde{c}_{ij}^{(p-d+1)} - \tilde{c}_{ij}^{(d-p-1)} + \tilde{c}_{ij}^{(d-p+1)} \quad (10)$$

Thus we get a quantum torus  $\mathbb{Y}_t$ .  $K_t(\mathcal{C})$  is a subalgebra of  $\mathbb{Y}_t$  generated by  $[L(m)]_t$ , classes of simple objects ( $m$ : dominant mon.)

$$K_t(\mathcal{C}) = \bigoplus_{\substack{m: \text{dominant} \\ \text{monomial}}} \mathbb{C}(t^{\frac{1}{2}}) \cdot [L(m)]_t$$

(3.2) Structure of  $K_t(\mathcal{C})$ . Let  $Q$  be an orientation of the Dynkin diagram of  $\mathfrak{g}$ .

Then [Hernandez-Leclerc] There is a sub-monoidal category  $\mathcal{C}_Q$  of

$\mathcal{C}$  such that its quantum Grothendieck ring  $K_t(\mathcal{C}_Q) = \bigoplus_{L(m) \in \mathcal{C}_Q} \mathbb{C}(t^{\frac{1}{2}}) [L(m)]_t$  is isomorphic to  $U_t(n)$ , where  $n$  is the positive part of  $\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus n_+$ .

Example.  $A_2 \xrightarrow[1]{\quad} \xrightarrow[2]{\quad} \mathcal{C}_Q \subset \mathcal{C}_1$

(see example on page ⑧).  $K(\mathcal{C}_1) = \mathbb{Z}[[L(Y_{1,1}), [L(Y_{1,q^2})], [L(Y_{2,q})], [L(Y_{2,q^3})]]$

$$K(\mathcal{C}_Q) = \mathbb{Z}[[L(Y_{1,1}), [L(Y_{1,q^2})], [L(Y_{2,q})]]$$

Recall that in classical case we have  
 $[L(Y_{1,1})] \cdot [L(Y_{1,q^2})] = [L(Y_{2,q})] + [L(Y_{1,1}Y_{1,q^2})]$  (classical T-system)

In  $K_t(\mathcal{C})$  we have

$$\begin{cases} [L(Y_{1,1})]_t * [L(Y_{1,q^2})]_t = t^{\frac{1}{2}} [L(Y_{2,q})]_t + t^{-\frac{1}{2}} [L(Y_{1,1}Y_{1,q^2})]_t & \text{and} \\ [L(Y_{1,q^2})]_t * [L(Y_{1,1})]_t = t^{-\frac{1}{2}} [L(Y_{2,q})]_t + t^{\frac{1}{2}} [L(Y_{1,1}Y_{1,q^2})]_t \end{cases}$$

Let us denote  $e_1 = [L(Y_{1,1})]_t$ ,  $e_2 = [L(Y_{1,q^2})]_t$ . By the two relations in  $K_t(e)$  we have:  $K_t(e_Q)$  is generated by  $e_1$  and  $e_2$ . We have:

$$\begin{aligned} e_1 * e_1 * e_2 &= \bar{t}' [V]_t + t [W]_t && \text{where } V = L(Y_{1,1}^2 Y_{1,q^2}) \\ e_1 * e_2 * e_1 &= [V]_t + [W]_t && W = L(Y_{1,1} Y_{2,q}) \\ e_2 * e_1 * e_1 &= t [V]_t + \bar{t}' [W]_t \end{aligned}$$

$$\Rightarrow \boxed{e_1 * e_1 * e_2 - (t + \bar{t}') e_1 * e_2 * e_1 + e_2 * e_1 * e_1 = 0}$$

quantum Serre Relations.

A side remark. In order to do computations given above, one

writes the  $q$ -character, e.g.

$$[L(Y_{1,1})] = Y_{1,1} + Y_{1,q^2}^{-1} Y_{2,q} + Y_{2,q^3}^{-1}$$

and uses the commutation relations between  $Y$ 's.

(3.3) Remarks. (i)  $U_t(n)$  has a structure of quantum cluster algebra [Geiss-Leclerc-Schütz]. Hence we get a  $q$ -cluster algebra structure on  $K_t(e_Q)$ . The specialization at  $t=1$  is compatible with the cluster alg. str. on  $K_t(e_Q) \cap K_t(e_1)$ .

(ii)  $U_t(n)$  has a (dual) canonical basis (Lusztig). One has Prop.  $\{[L(m)]_t\}$  basis of  $K_t(e_Q)$  and (dual) canonical basis of  $U_t(n)$  (up to a factor of  $t^{1/n}$ )

Theorem.  $\mathcal{C}_Q$  is a categorification\* of  $\mathbb{C}[N]$  with its dual canonical basis. That is,

\*  $K(\mathcal{C}_Q) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[N]$  such that

$$\left\{ \begin{array}{l} \text{basis of simple} \\ \text{objects} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{dual canonical} \\ \text{basis } \mathcal{B}^* \end{array} \right\}$$

(3.4) Application. Theorem (Hernandez). Let  $S_1, \dots, S_n$  be simple objects in  $\mathcal{C}$ . Then  $S_1 \otimes \dots \otimes S_m$  is simple if, and only if  $\forall i \neq j, S_i \otimes S_j$  is simple.

This theorem, together with Theorem (3.3) implies the following:

for  $b_1, \dots, b_n \in \mathcal{B}^*$  we have:  $b_1 \cdots b_n \in \mathcal{B}^* \Leftrightarrow b_i b_j \in \mathcal{B}^* (\forall i \neq j)$

Example.  $A_2$ .  $N = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}$

$$\mathbb{C}[N] = \mathbb{C}[x, y, z].$$

$$\mathcal{B}^* = \left\{ x^a z^b (xy - z)^c \mid (a, b, c) \in \mathbb{N}^3 \right\} \cup \left\{ y^a z^b (xy - z)^c \mid a, b, c \in \mathbb{N} \right\}$$

$$K(\mathcal{C}_Q) = \mathbb{Z} \left[ [L(Y_{1,1})], [L(Y_{1,q^2})], [L(Y_{2,q})] \right]$$

$$\varphi : K(\mathcal{C}_Q) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[N]$$

$$L(Y_{1,1}) \longleftrightarrow x$$

$$L(Y_{1,q^2}) \longleftrightarrow y$$

$$L(Y_{2,q}) \longleftrightarrow z$$

$$L(Y_{1,1} Y_{1,q^2}) \longleftrightarrow xy - z$$

$$\text{and } \boxed{\{[V] : V \text{ simple}\} \leftrightarrow \mathcal{B}^*}$$