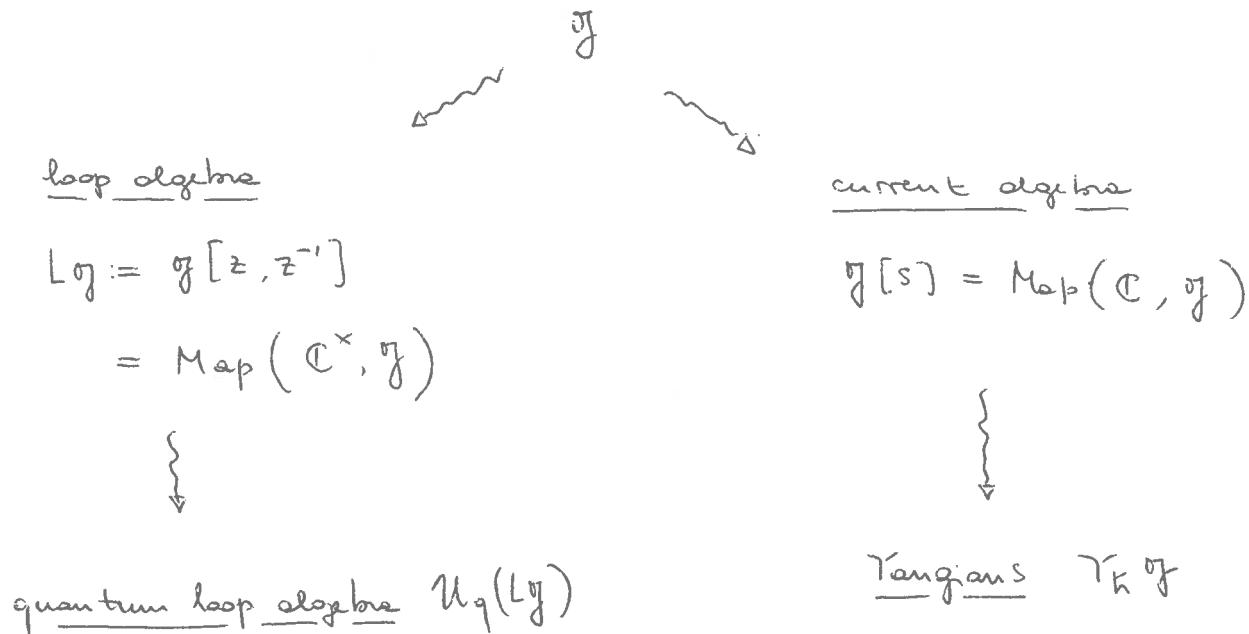


# "Tangians and Quantum loop algebras"

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Plan: to a simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  we can associate two infinite dimensional quantum groups:



- ① structure for representation theory of  $T_h \mathfrak{g}$
- ② structure for representation theory of  $U_q(L\mathfrak{g})$
- ③ explain very close similarities between  $T_h \mathfrak{g}$  and  $U_q(L\mathfrak{g})$  by constructing an equivalence of categories

$$\text{Rep}_{\text{fd}}(T_h \mathfrak{g}) \longrightarrow \text{Rep}_{\text{fd}}(U_q(L\mathfrak{g}))$$

①

Tang's ans

Notations:  $\mathfrak{h} \subset \mathfrak{g}$  Cartan subalgebra

$\{\alpha_i\} \subset \mathfrak{h}^*$  simple roots

$(\cdot, \cdot) : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$  inner product

inducing a bijection

$\{\alpha_i\} \leftrightarrow \{t_i\}$

$$\begin{matrix} \mathfrak{h} \\ \mathfrak{h}^* \end{matrix} \xrightarrow{\quad} \begin{matrix} \mathfrak{h} \\ \mathfrak{h} \end{matrix}$$

$\mathfrak{g}_{\pm \alpha_i}$  root space

$x_i^\pm$  such that  $(x_i^+, x_i^-) = 1$

$$a_{ij} := 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad b_{ij} := (\alpha_i, \alpha_j) = b_{ji}$$

Case  $\mathfrak{g} = \mathfrak{sl}_2$

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{C} \right\}$$

$$\alpha(D) = D_{11} - D_{22}$$

$$(x, y) = \text{tr}(xy)$$

$$t = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h$$

$$x^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e$$

$$x^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = f$$

Presentation of  $\mathfrak{g}$

$$\forall i, j \quad [t_i, t_j] = 0$$

$$[t_i, x_j^\pm] = \pm b_{ij} x_j^\pm$$

$$[x_i^+, x_j^-] = \delta_{ij} t_{ij}$$

$$[t, x^\pm] = \pm 2 x^\pm$$

$$[x^+, x^-] = t$$

$$\forall i \neq j \quad \text{ad}(x_i^\pm)^{1-a_{ij}} x_j^\pm = 0 \quad \text{"Serre's relations"}$$

②

## Presentation of $\mathcal{Y}_{\hbar g}$

$\mathcal{Y}_{\hbar g}$  is the associative algebra over  $\mathbb{Q}[\hbar]$  with generators

$$\left\{ x_{i,r}^{\pm}, \xi_{i,r} \right\}_{i \in I, r \in \mathbb{N}}$$

and relations

$$[\xi_{i,r}, \xi_{j,s}] = 0$$

$$[\xi_{i,r}, x_{j,s}^{\pm}] = \pm b_{ij} x_{j,s}^{\pm}$$

$$[x_{i,r}^{\pm}, x_{j,s}^{\mp}] = \delta_{ij} \xi_{i,r+s}$$

$$\sum_{r \in \mathbb{G}_m} [x_{i,r_{m+1}}^{\pm} \dots [x_{i,r_{m(m)}}^{\pm}, x_{j,s}^{\pm}] \dots] = 0$$

$$\forall i \neq j, m=1-a_{ij} \in \mathbb{N}, \forall r_1 \dots r_m \in \mathbb{N}, s \in \mathbb{N}$$

$$[\xi_{i,r+1}, x_{j,s}^{\pm}] - [\xi_{i,r}, x_{j,s+1}^{\pm}] = \pm b_{ij} \frac{\hbar}{2} \{ \xi_{i,r}, x_{j,s}^{\pm} \} \quad \begin{matrix} \text{-quantum} \\ \text{relations} \end{matrix}$$

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm b_{ij} \frac{\hbar}{2} \{ x_{i,r}^{\pm}, x_{j,s}^{\pm} \}$$

$$\text{where } \{a, b\} = ab + ba.$$

We have an identification

$$\mathcal{Y}_{\hbar g} / \mathbb{K} \mathcal{Y}_{\hbar g} \cong \mathcal{U}(g[s]) \quad \text{and}$$

$$\begin{cases} x_{i,r}^{\pm} = x_i^{\pm} \otimes s^r \pmod{\hbar} \\ \xi_{i,r} = t_i \otimes s^r \pmod{\hbar} \end{cases}$$

Remark:

1.  $T_{\hbar} \mathcal{G}$  is IN-graded via  $\begin{cases} \deg(a_r) = r \text{ on loop number} \\ \deg(\hbar) = 1 \end{cases}$

2.  $U_{\hbar} \mathcal{G} \subset T_{\hbar} \mathcal{G}$

"

$\langle x_{i,0}^{\pm}, \xi_{i,0} \rangle$  on constant loop

3.  $T^{\circ} \subset T_{\hbar} \mathcal{G}$

"

$\langle \xi_{i,r} \rangle$  on maximal commutative subalgebra

"deforming"  $U(\mathfrak{h}[s])$

$\hookrightarrow$  stays commutative by reduces to that

4. The previous presentation is called "new / loop realization of the Yangian". The original presentation by Drinfeld was on  $\{x_{i,0}^{\pm}, \xi_{i,0}, x_{i,1}^{\pm}, \xi_{i,1}\}$

Fields: define the following generating series

$$\xi_i(u) := 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \in T_{\hbar} \mathcal{G}[[u^{-1}]]$$

$$x_i^{\pm}(u) := \hbar \sum_{r \geq 0} x_{i,r}^{\pm} u^{-r-1} \in T_{\hbar} \mathcal{G}[[u^{-1}]]$$

Setting  $\deg(u) = 1$ , we have  $\deg(\xi_i(u)) = \deg(x_i^{\pm}(u)) = 0$ .

The grading induces an action  $\lambda: \mathbb{C}^* \times T_{\hbar} \mathcal{G}$  s.t.

$$\lambda_z(\xi_i(u)) = \xi_i(u/z)$$

Prop: The presentation of the Yangian is equivalent to the following identities in  $\text{Ta}_n[[u^{-1}, v^{-1}][u, v]]$ :

$$[\xi_i(u), \xi_j(v)] = 0$$

$$(u-v) [x_i^+(u), x_j^-(v)] = \pm \delta_{ij} (\xi_i(v) - \xi_j(u))$$

$$\sum_{\sigma \in \mathfrak{S}_m} [x_i^\pm(u_{\sigma(1)}) \dots [x_i^\pm(u_{\sigma(m)}) , x_j^\pm(v)] \dots] = 0$$

$\forall i \neq j, m = 1 - \alpha_{ij}$

$$\left( u-v \pm b_{ij} \frac{\hbar}{2} \right) \xi_i(u) x_j^\pm(v) \xi_i(u)^{-1} =$$

$$= (u-v \pm b_{ij} \frac{\hbar}{2}) x_j^\pm(v) \mp b_{ij} \hbar x_j^\pm(u \mp b_{ij} \frac{\hbar}{2})$$

$$[x_i^\pm(u), [x_j^\pm(v) - x_j^\pm(u)]] - (v-u) [x_i^\pm(u)', x_j^\pm(v)] =$$

$$= \pm b_{ij} \frac{\hbar}{2} \{ x_i^\pm(u)', x_j^\pm(v) - x_j^\pm(u) \}$$

## Finite dimensional representation theory of $\mathcal{Y}_{\hbar}^{\mathfrak{g}}$

Fix  $\hbar \in \mathbb{C}^*$  and define the  $\mathbb{C}$ -algebra  $\mathcal{Y}_{\hbar}^{\mathfrak{g}} := \mathcal{Y}_{\hbar}^{\mathfrak{g}} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}_{\hbar}$

- $\mathcal{Y}_{\hbar}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{Y}_{\hbar''}^{\mathfrak{g}}$  by  $a_r^{\hbar} \mapsto a_r^{\hbar''} \cdot \left(\frac{\hbar}{\hbar''}\right)^r$
- $\mathcal{Y}_{\hbar}^{\mathfrak{g}}$  is now filtered by  $\deg(a_r) \leq r$ .

Goal: classify all finite dimensional representations of  $\mathcal{Y}_{\hbar}^{\mathfrak{g}}$ .

Highest weight  
classification (Drinfeld)

triangular decomposition

$$\mathfrak{g}[s] = \mathfrak{n}_-[s] \oplus \mathfrak{h}[s] \oplus \mathfrak{n}_+[s]$$

$$\hookrightarrow \mathcal{Y}_{\hbar}^{\mathfrak{g}} \simeq Y^- \otimes Y^0 \otimes Y^+$$

$$\parallel \quad \langle \xi_{i,r} \rangle \parallel$$

$$\langle x_{i,r}^- \rangle \quad \langle x_{i,r}^+ \rangle$$

Def. A  $\mathcal{Y}_{\hbar}^{\mathfrak{g}}$ -module is highest weight if  $\exists v \in V, v \neq 0$

s.t.

- $V = \mathcal{Y}_{\hbar}^{\mathfrak{g}} \cdot v$
- $x_{i,r}^+ \cdot v = 0$
- $\xi_{i,r}^\pm v = d_{i,r} v \quad \forall i,r \text{ s.t. denote } \underline{d} := \{d_{i,r}\}$

Def / Prop: i) For any  $\underline{d} = \{d_{i,r}\}$  the Verma module  $M_{\underline{d}}$  is the quotient of  $\mathcal{Y}_{\hbar}^{\mathfrak{g}}$  by the left ideal generated by  $x_{i,r}^+, (\xi_{i,r} - d_{i,r})$ .

- ii)  $M_{\underline{d}}$  is highest weight and every h.w. module  $V$  with h.w.  $\underline{d}$  is a quotient of  $M_{\underline{d}}$ .
- iii)  $M_{\underline{d}}$  has a unique simple quotient  $L_{\underline{d}}$ .

## Theorem (Drinfeld)

- i) Every f.d. irreducible representation of  $T_{h\mathfrak{f}}$  is highest weight.
- ii)  $L_d$  is a finite dimensional module iff there exists a collection of monic polynomials  $P_i(u) \in \mathbb{C}[u]$ ,  $i \in I$  s.t.

$$\frac{P_i(u + h \frac{(\alpha_i, \alpha_i)}{2})}{P_i(u)} = 1 + h \sum_{r \geq 0} d_{i,r} u^{r-1}$$

Def: The collection  $\{P_i(u)\}$  is called Drinfeld polynomials of  $L_d$ .

Cot: we have a bijection

$$\left\{ \begin{array}{l} \text{finite dimensional} \\ \text{irreducible representations} \\ \text{of } T_{h\mathfrak{f}} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{I-tuple of} \\ \text{monic polynomials} \\ P_i(u) \in \mathbb{C}[u] \end{array} \right\}$$

The case of  $\mathfrak{sl}_2$ : In this case the evaluation map,  $a \in \mathbb{C}$ ,

$$ev_a: \mathfrak{g}[s] \longrightarrow \mathfrak{g} \quad | \quad f \mapsto f(a)$$

lifts up to an evaluation homomorphism

$$ev_a: T_{h\mathfrak{f}} \longrightarrow U\mathfrak{g}$$

$$x_0^\pm \longleftrightarrow e \text{ or } f$$

$$\xi_0 \longleftrightarrow h$$

$$t_1 = \left( \xi_1 - \frac{h}{2} \xi_0^2 \right) \longleftrightarrow h \left[ ah - \frac{1}{2}(ef + fe) \right]$$

Remark: if  $\mathfrak{g} \neq \mathfrak{sl}_2$ , then there is no such map.

By pull back, for any  $V \in \text{Rep}_{\text{fd}}^{\text{irr}} \mathfrak{U}_{\mathfrak{sl}_2}$  we have

$$V(a) := \text{ev}_a^* V \in \text{Rep}_{\text{fd}}^{\text{irr}} \mathcal{T}_{\mathfrak{h}, \mathfrak{sl}_2}$$

Prop: i) the action of  $\mathcal{T}_{\mathfrak{h}, \mathfrak{sl}_2}$  on  $\mathbb{C}^2(a)$  is given by

$$x_r^+ \mapsto \begin{bmatrix} 0 & h^{ra^r} \\ 0 & 0 \end{bmatrix}$$

$$x_r^- \mapsto \begin{bmatrix} 0 & 0 \\ h^{ra^r} & 0 \end{bmatrix}$$

$$\xi_r \mapsto \begin{bmatrix} h^{ra^r} & 0 \\ 0 & -h^{ra^r} \end{bmatrix}$$

ii) the corresponding fields are given by

$$x^\pm(u) = \frac{h}{u-ha} e \quad \text{or} \quad \frac{h}{u-ha} f$$

$$\xi(u) = 1 + \frac{h}{u-ha} f$$

iii) the Drinfeld polynomial of  $\mathbb{C}^2(a)$  is

$$P(u) = (u - ha)$$

Theorem [Chari-Pressley] If  $a_1, \dots, a_m \in \mathbb{C}$  are s.t.  $|a_i - a_j| \neq 1 \quad \forall i \neq j$

then :

1)  $\mathbb{C}^2(a_1) \otimes \dots \otimes \mathbb{C}^2(a_m)$  is simple

2) its Drinfeld polynomial is

$$(u - ha_1) \cdots (u - ha_m)$$

## ⑪ Quantum Loop Algebras

Fix  $q \in \mathbb{C}^*$  not a root of unity.

From now on

$$g \in \mathfrak{sl}_2$$

Def [Drinfeld, Jimbo]

$\mathfrak{U}_q \mathfrak{sl}_2$  is the associative algebra with generators

$$\{K^{\pm 1}, X^\pm\}$$

and relations

$$K X^\pm K^{-1} = q^{\pm 2} X^\pm, \quad [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}$$

Remark: usually we are assuming  $q = e^{i\hbar}$ ,  $K = e^{\hbar H} = q^H$  so that

$$e^{\hbar H} X^\pm \bar{e}^{-\hbar H} = e^{\pm 2\hbar} X^\pm \quad \xrightarrow{\frac{d}{d\hbar} \Big|_{\hbar=0}} \quad [H, X^\pm] = \pm 2 X^\pm$$

$$[X^+, X^-] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} \quad \xrightarrow[\hbar \rightarrow 0]{} \quad [X^+, X^-] = H$$

$\rightsquigarrow$  we get the same relations of  $\mathfrak{sl}_2$ .

Def.  $\forall n \geq 0$ ,  $V_n$  with basis  $\{v_i, i=0 \dots n\}$  is the  $(n+1)$ -irreducible  $\mathfrak{sl}_2$ -module. The action of  $\mathfrak{sl}_2$  is given by

$$h \cdot v_i = (n-2i) v_i$$

$$x^+ \cdot v_i = i v_{i-1}$$

and  $v_0$  is highest weight  
 $v_n$  is lowest weight

$$x^- \cdot v_i = (n-i) v_{i+1}$$

$q$ -numbers: for any  $r \in \mathbb{N}$ , defines

$$[r]_q := \frac{q^r - q^{-r}}{q - q^{-1}} = q^{r-1} + q^{r-3} + \dots + q^{-(r-1)}$$

$$[r]_q! := [r]_q \cdot [r-1]_q \cdot \dots \cdot [1]_q$$

$$\left[ \begin{matrix} n \\ r \end{matrix} \right]_q := \frac{[n]_q!}{[r]_q! [n-r]_q!}$$

$$\lim_{q \rightarrow 1} [r]_q = r$$

$$[m+n]_q = q^n [m]_q + q^{-m} [n]_q$$

Prop: i) the following formulas define an action of  $U_q \mathfrak{sl}_2$  on  $V_n$ :

$$K v_i = q^{n-2i} v_i$$

$$x^+ v_i = [i]_q v_{i-1}$$

$$x^- v_i = [n-i]_q v_{i+1}$$

ii) this action is irreducible

iii) These are all the irreducible representations of  $U_q \mathfrak{sl}_2$  with finite dimension.

## The quantum loop algebra

Def:  $\mathcal{U}_q(\mathbb{L}sl_2)$  is the associative algebra with generators

$$\{K^{\pm 1}, H_k, X_e^{\pm}\}_{k \in \mathbb{Z} \setminus 0} \text{ with relations}$$

$$e \in \mathbb{Z}$$

$$[K, H_k] = 0 = [H_k, H_{k'}]$$

$$K X_e^{\pm} K^{-1} = q^{\pm 2} X_e^{\pm}$$

$$[H_k, X_e^{\pm}] = \pm \frac{[2k]}{k} X_{e+k}^{\pm}$$

$$q^{\pm 2} X_k^{\pm} X_{e+1}^{\pm} - X_{e+1}^{\pm} X_k^{\pm} = X_{k+1}^{\pm} X_e^{\pm} - q^{\pm 2} X_e^{\pm} X_{k+1}^{\pm}$$

$$[X_k^+, X_e^-] = \frac{\psi_{k+e}^+ - \psi_{k+e}^-}{q - q^{-1}}$$

where  $\psi^{\pm}(z) := K^{\pm 1} \exp \left[ \pm (q - q^{-1}) \sum_{r \geq 1} H_r z^{\mp r} \right]$  and

$$\psi^{\pm}(z) = \sum_{r \geq 0} \psi_{\pm r}^{\pm} z^{\mp r} \in \mathcal{U}_q(\mathbb{L}sl_2)[z^{\mp 1}]$$

Remark:  $\psi_0^{\pm} = K^{\pm 1}$

$$\psi_1^{\pm} = \pm K^{\pm 1} (q - q^{-1}) H_{\pm 1}$$

$$\psi_2^{\pm} = \pm K^{\pm 1} \left[ (q - q^{-1}) H_{\pm 2} \pm \frac{(q - q^{-1})^2}{2} H_{\pm 1}^2 \right]$$

and in general we can solve  $H_k, K^{\pm 1}$  in terms of  $\psi_{\pm r}^{\pm}$ 's.

## Presentation in terms of fields

Define  $X^\pm(z) = \sum_{k \in \mathbb{Z}} X_k^\pm z^{-k} \in \mathcal{U}_q(L\mathfrak{sl}_2)[[z, z^{-1}]]$

Prop: The defining relations of  $\mathcal{U}_q(L\mathfrak{sl}_2)$  are equivalent to the following

$$[\psi^\pm(z), \psi^\pm(w)] = 0 = [\psi^\pm(z), \psi^\mp(w)]$$

$$\psi^\pm(z) X^\pm(w) \psi^{\pm}(z)^{-1} = \frac{q^{\pm 2} z - w}{z - q^{\pm 2} w} X^\pm(w)$$

$$X^\pm(z) X^\pm(w) = \frac{q^{\pm 2} z - w}{z - q^{\pm 2} w} X^\pm(w) X^\pm(z)$$

$$[X^+(z), X^-(w)] = \frac{1}{q - q^{-1}} \delta(z/w) [\psi^+(w) - \psi^-(w)]$$

where  $\delta(\zeta) := \sum_{n \in \mathbb{Z}} \zeta^n$  is a formal Dirac delta

## Finite dimensional representations of $\mathcal{U}_q L\mathfrak{sl}_2$

Def: A representation  $V$  is highest weight if  $\exists v \neq 0, v \in V$

s.t.

$$V = \mathcal{U}_q(L\mathfrak{sl}_2)v$$

$$X_r^+ v = 0 \quad \forall r \in \mathbb{Z}$$

$$\psi_{\pm r}^\pm v = d_{\pm r}^\pm v \quad \text{for some } \underline{d} = \{d_{\pm r}^\pm\} \subset \mathbb{C}$$

Theorem [Chari-Pressley]

i) Any f.d. irreducible representations of  $\mathcal{U}_q(\mathfrak{sl}_2)$  is h.w.

ii) The unique h.w. representation  $L_d$  of h.w.  $d$  i.f.f f.d.

iff there exists a polynomial  $P \in \mathbb{C}[u]$  s.t.

$$P(0) = 1 \quad \text{and}$$



$$\sum_{r \geq 0} d_r^+ z^{-r} = q^{-\deg P} \frac{P(zq^2)}{P(z)}$$

normalized polynomial

with roots in  $\mathbb{C}^\times$

||

$$\sum_{r \geq 0} d_{-r}^- z^r$$

Remark:  $d_0^\pm$  is s.t.  $K^{\pm 1} L_d = d_0^\pm L_d$  so  $d_0^\pm = q^{\pm \deg P}$ .

Cor.: we have a bijection

$$\left\{ \begin{array}{l} \text{f.d. irreducible} \\ \text{type I\!I representations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{polynomials } P \in \mathbb{C}[u] \\ \text{s.t. } P(0) = 1 \end{array} \right\}$$

Construction of finite dimensional representations

We have an evaluation homomorphism for any  $a \in \mathbb{C}^\times$   $[g[z, z^{-1}] \rightarrow g]$

$$\text{ev}_a: \mathcal{U}_q(\mathfrak{sl}_2) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2)$$

so that to every  $V \in \text{Rep}_{\text{fd}}^{\text{irr}} \mathcal{U}_q(\mathfrak{sl}_2)$   $\mapsto V(a) := \text{ev}_a^* V \in \text{Rep}_{\text{fd}}^{\text{irr}} \mathcal{U}_q(\mathfrak{sl}_2)$

Prop [Chari-Pressley] If  $V_n$  is the type I irreducible representation of  $\mathcal{U}_q(\mathfrak{sl}_2)$  of dimension  $n+1$ , then the Annilfild polynomial of  $V_n(a)$  is

$$P(u) = (1 - q^{-n+1} a^{-1} u) (1 - q^{-n+3} a^{-1} u) \dots (1 - q^{n-1} a^{-1} u)$$

For example, if  $m=1$ , then  $P(u) = (1 - \alpha^{-1}u)$ .

Let's see now how to realize the irreducible representation with Drinfeld polynomials  $P(u) = \prod_{i=1}^m (1 - \xi_i^{-1}u)$

I. partition  $\{\xi_1, \dots, \xi_m\}$  into sets of the form

$$S_i = \{\xi_i, \xi_i q^{-2}, \dots, \xi_i q^{-2m_i}\}$$

II.  $V_P \cong \bigotimes V_{m_i}(\xi_i)$

e.g. if  $\xi_i \xi_j^{-1} \neq q^{\pm 2}$   $\forall i \neq j$   $\Rightarrow S_i = \{\xi_i\}$  and

$$V_{(1-\xi_1^{-1}u) \dots (1-\xi_m^{-1}u)} \cong \bigotimes_{i=1}^m \mathbb{C}^2(\xi_i)$$

(III) The isomorphism and the functor

We have an algebra isomorphism

$$\phi: \widehat{\mathcal{U}_q(\mathcal{L}^g)} \xrightarrow{\sim} \widehat{\mathcal{T}_{\hbar}^g}$$

$\downarrow$  completion with respect to ideal  
 $z=1, q=1$

$\downarrow$  completion with respect to the  $\mathbb{N}$ -grading

Classically, this isomorphism reflects the situation:

$$\widehat{\mathcal{O}\{z, z^{-1}\}} \xrightarrow{z=1} \widehat{\mathcal{O}[S]}$$

Using  $\phi$  we have a pullback functor

$$\phi^*: \text{Rep}(\widehat{\mathcal{T}_{\hbar}^g}) \xrightarrow{\sim} \text{Rep}(\widehat{\mathcal{U}_q(\mathcal{L}^g)})$$

This isomorphism is still not satisfactory since too few representations survive.

Fix  $h \in \mathbb{C}$ ,  $\Im h > 0$  and  $q \in \mathbb{C}^\times$ ,  $|q| < 1 \Rightarrow q = e^{i\pi h}$

Then [Gautam, TL]

There is an isomorphism of categories

$$\begin{matrix} \mathcal{C} \\ \cong \\ \text{Rep}_{\text{fd}}(\mathcal{T}_{\hbar}^g) \end{matrix} \xrightarrow{\sim} \text{Rep}(\widehat{\mathcal{U}_q(\mathcal{L}^g)})$$

isomorphism is stronger than equivalence: it is surjective, not only essentially surjective

Fields revisited: recall the definitions in  $\text{Thg}[[u^{-1}]]$

$$\xi_i(u) = 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r-1}$$

$$x_i^\pm(u) = h \sum_{r \geq 0} x_{i,r}^\pm u^{-r-1}$$

Prop: on a finite dimensional representation  $V$ ,  $\xi_i(u)$  and  $x_i^\pm(u)$  are the Taylor expansions at  $u=\infty$  of rational functions with value in  $\text{End}(V)$

similarly, we have

$$\psi_{i,\pm}(z) = \sum_{r \in \mathbb{Z}_+} \psi_{i,\pm r} z^{\mp r} \in U_q(\text{Lg})[[z^{\mp 1}]]$$

$$E_{i,+}(z) = \sum_{r \geq 0} E_{i,r} z^{-r} \in U_q(\text{Lg})[[z^{-1}]]$$

$$E_{i,-}(z) = - \sum_{r \geq 0} E_{i,-r} z^r \in U_q(\text{Lg})[[z]]$$

and similarly for  $F_{i,\pm}(z)$ .

Prop [Beck-Kac, Hernandez] On a finite dimensional representation  $V$  of  $U_q(\mathbb{L})$  there exist rational functions  $\psi_i, e_i, f_i$  s.t.

$$\begin{array}{ccc} \text{expansion} & & \text{expansion} \\ \text{at zero} & \downarrow & \text{at infinity} \\ \psi_i^+(z) = \psi_i(z) = \psi_i^-(z) \\ e_i^+(z) = e_i(z) = e_i^-(z) \\ f_i^+(z) = f_i(z) = f_i^-(z) \end{array}$$

Sketch of the proof:

consider the case  $\mathfrak{g} = sl_2$  and take  $E^+(z)$ .

$$\text{Then } [H_1, E_k] = [2]_q E_{k+1} \Rightarrow [H_1, E^+(z)] = [2]_q z (E^+(z) - E_0),$$

and we have

$$\left( 1 - \frac{\text{ad}(H_1)}{[2]_q z} \right) E^+(z) = E_0$$

$$\Rightarrow E^+(z) = \left( 1 - \frac{\text{ad}(H_1)}{[2]_q z} \right)^{-1} E_0$$

# Additive difference equations (ADE's) [Birkhoff, ...]

Borodin, Kriegerer]

Consider

$$(*) \quad f(u+1) = A(u) f(u)$$

where  $A: \mathbb{C} \rightarrow GL(V)$  rational coefficient matrix -

Assumption

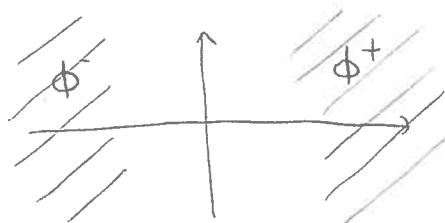
$$\left. \begin{array}{ll} A(\infty) = I & \text{regular singular} \\ A_0 = \text{res}_{u=\infty} (A(u) du) & \text{semisimple} \\ \hookrightarrow A = I + \frac{A_0}{u} + \dots \text{ near } u=\infty \end{array} \right\}$$

$A_0$  commutes with  $A(u)$

Problem: construct solution of (\*) having "good" asymptotics  
at  $u = \infty$ .

Prop:  $\exists!$  solution  $\phi^\pm(u): \mathbb{C} \rightarrow GL(V)$  s.t.

i)  $\phi^\pm$  is holomorphic if  $\operatorname{Re} u \gg 0$  (resp.  $\operatorname{Re} u \ll 0$ )



ii)  $\lim_{\substack{u \rightarrow \infty \\ \operatorname{Re} u \gg 0}} \phi^\pm(u) u^{-A_0} = I \implies \phi^\pm \sim u^{A_0}$   
 $\operatorname{Re} u \ll 0$  (resp.  $\operatorname{Re} u \gg 0$ )

Remark:  $u^{-A_0} = \exp(-A_0 \log u)$

Then we can define

$$S(u) := \phi^+(u)^{-1} \cdot \phi^-(u)$$

that is NOT constant, but periodic. So, since  $S(u+1) = S(u)$ ,

$$S = S(z), \text{ where } z = e^{2\pi u i}$$

$$S(z) \Big|_{z=\infty} = 1, \quad S(z) \Big|_{z=0} = e^{-2\pi i A_0}$$

$S$  is rational function in  $z$ .

Def:  $S(z)$  is the monodromy of (\*) called "connection matrix".

Example: consider the following ADE

$$f(u+1) = \frac{u-a}{u-b} f(u)$$

$$\Rightarrow \varphi^+(u) = \frac{\Gamma(u-a)}{\Gamma(u-b)}$$

$$\varphi^-(u) = \frac{\Gamma(1-u+b)}{\Gamma(1-u+a)} \cdot e^{\pi i(a-b)}$$

Recall: •  $\Gamma(z+1) = z \Gamma(z)$

•  $\Gamma(z)$  has poles only at  $z = 0, -1, -2, \dots$

•  $\Gamma(z) \sim e^{-z} z^{z-1/2} \sqrt{2\pi}$  [stirling]

Then the monodromy is

$$S(z) = \varphi^+(u)^{-1} \cdot \varphi^-(u)$$

$$= \frac{\Gamma(u-b) \cdot \Gamma(1-u+b)}{\Gamma(u-a) \cdot \Gamma(1-u+a)} \cdot e^{\pi i(a-b)}$$

and since

$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

we get

$$S(z) = \frac{\sin(\pi(u-a))}{\sin(\pi(u-b))} e^{\pi i(a-b)}$$

$$= \frac{z-\alpha}{z-\beta} \quad \rightsquigarrow \quad \left\{ \begin{array}{l} z = e^{2\pi i u} \\ \alpha = e^{2\pi i a} \\ \beta = e^{2\pi i b} \end{array} \right.$$

So

$$A(u) = \frac{u-a}{u-b} \quad \rightsquigarrow \quad S(z) = \frac{z-\alpha}{z-\beta}$$

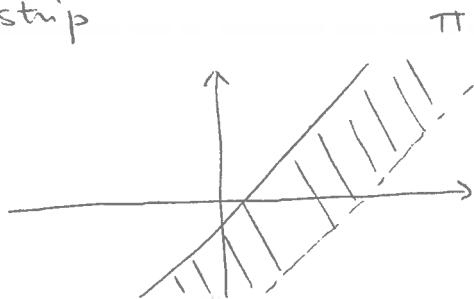


Inverse problem: given  $S(z)$ , reconstruct  $A(u)$ .

- $A$  need not to be unique: require that zeros and poles of  $A$  lie in  $e^{\pi i \mathbb{Z}}$
- $A$  need not to exist:  $A$  exists if  $[A(u), A(v)] = 0$ .

The strip category  $\mathcal{C}_\Pi \subset \text{Rep}_{\text{fd}}(\mathcal{Y}_h \otimes)$

Consider the strip



Def.  $\mathcal{C}_\Pi$  is the full subcategory of  $\text{Rep}_{\text{fd}}(\mathcal{Y}_h \otimes)$

s.t. the roots of the Drinfeld polynomials of  
their simple factors lie in  $\Pi$ .

Prop- i)  $\mathcal{C}_\Pi$  is a Serre subcategory, i.e., given a s.e.s

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0, \quad A, C \in \mathcal{C}_\Pi \text{ iff } B \in \mathcal{C}_\Pi.$$

ii)  $\mathcal{C}_\Pi$  is a tensor subcategory.

Prop:  $V \in \mathcal{C}_\Pi$  iff the poles of the fields  $x_i^\pm(u), \xi_i^\pm(u)^{\pm 1}$   
all lie in  $\Pi$

iff the poles of  $\xi_i^\pm(u)^{\pm 1}$  all lie in  $\Pi$

## Isomorphism of categories

Given  $V \in \mathcal{C}_\pi$ , define an action of  $U_q(\mathfrak{g})$  on  $V$ .

Action of  $\Psi_{i,\text{tr}}^\pm$ : start with

$$f(u+1) = A(u) f(u) \quad \text{or} \quad A(u) = \xi_i(u)$$

consider then the canonical fundamental solutions  $\phi_i^\pm(u)$  and the monodromy  $S_i(z) = (\phi_i^+)^{-1} \cdot \phi_i^-$

Define the action of  $\Psi_i$  by  $\Psi_i(z) := q^{\xi_i, \circ} S_i(z)$ .

Take indeed  $V$  an irreducible representation with h.w.  $\Omega$  and

Drinfeld polynomials

$$P_i(u) = (u - \alpha_1^i) \cdots (u - \alpha_{N_i}^i)$$

Then

$$\xi_i(u) \Omega = \frac{P_i(u + \hbar d_i)}{P_i(u)} \Omega \quad \text{or} \quad d_i = \frac{(\alpha_i, \alpha_i)}{2}$$

$$= \prod_{j=1}^{N_i} \frac{(u - \alpha_j^i + \hbar d_i)}{(u - \alpha_j^i)}$$

$$\Rightarrow S_i(z) \Omega = \prod_{j=1}^{N_i} \frac{z - \alpha_j^i q_i^{-2}}{z - \alpha_j^i} \quad \text{where} \quad \begin{cases} \alpha_j^i = e^{2\pi i \alpha_j^i} \\ q_i = q^{d_i} \end{cases}$$

then

$$\Psi_i(z) \Omega = \frac{P_i(q_i^2 z)}{P_i(z)} q^{-\deg P_i} \sim P_i(z) = \prod_{j=1}^{N_i} (z - \alpha_j)$$

Action of  $E_{i,k}, F_{i,k}$ :

$$\begin{cases} g_i^+(u) = c_i^+ \phi_i^+(u+1)^{-1} & \text{on } \phi_i^+ \text{ fundamental} \\ g_i^-(u) = c_i^- \cdot q^{\xi_{i,0}} \phi_i^-(u) & \text{solutions of} \\ & f(u+1) = \xi_i(u) f(u) \end{cases}$$

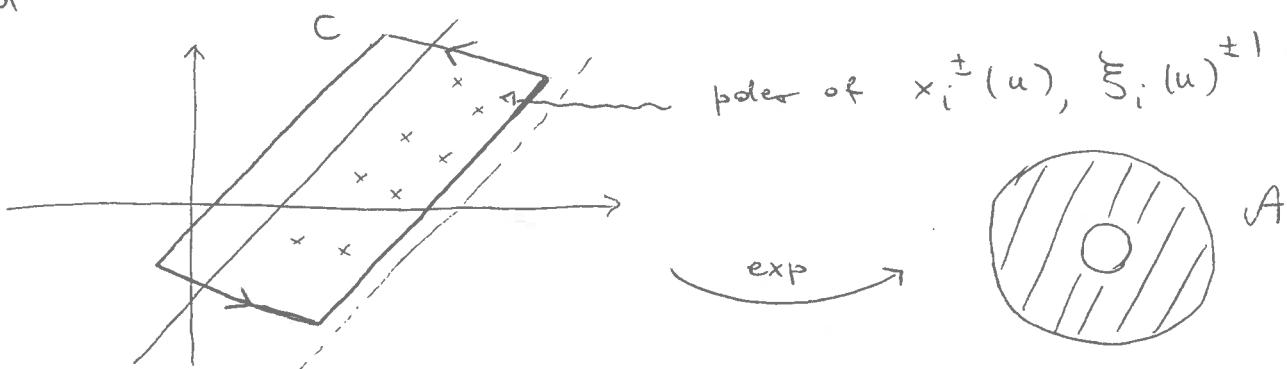
$$\text{where } c_i^\pm \in \mathbb{C}^\times \text{ and } c_i^+ \cdot c_i^- - d_i h^2 = \Gamma(1 - h d_i)^2$$

Then

$$E_{i,k} = \frac{1}{2\pi i} \oint_C e^{2\pi i k u} g_i^+(u) x_i^+(u) du$$

$$F_{i,k} = \frac{1}{2\pi i} \oint_C e^{2\pi i k u} g_i^-(u) x_i^-(u) du$$

and



then

$$e_i(z) = \frac{1}{2\pi i} \oint_C \frac{z}{z - e^{2\pi i u}} g_i^-(u) x_i^+(u) du \quad \left. \right\} z \text{ outside of } A$$

$$f_i(z) = \frac{1}{2\pi i} \oint_C \frac{z}{z - e^{2\pi i u}} g_i^+(u) x_i^-(u) du \quad \left. \right\} z \text{ outside of } A$$

## Sketch of the inverse functor

Let  $V \in \text{Rep}_{\text{fd}}(\mathcal{U}_q(\mathfrak{g}))$ .

Action of  $\xi_i(u)$ : given by the unique coefficient matrix

$A_i$  such that the monodromy is

$\Psi_i(z)$  and the poles of  $A_i(u)$  are in  $\Pi$ .

Action of  $x_i^\pm(u)$ : given by the formulae

$$x_i^+(u) = \frac{1}{2\pi i} \oint_C \frac{1}{u-v} g_i^-(v)^{-1} e_i(e^{2\pi i v}) dv$$

$$x_i^-(u) = \frac{1}{2\pi i} \oint_C \frac{1}{u-v} g_i^+(v)^{-1} f_i(e^{2\pi i v}) dv$$