The global gravitational anomaly of the self-dual field theory

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Motivation

Internal consistency is the only guide in our quest for a quantum theory of gravity.

There is a constraint that has been largely neglected: the cancellation of global gravitational anomalies in the effective field theory description at low energies.

It can be seen as a generalization to higher dimensions of the modular invariance constraint familiar from 2d CFTs.

No formula for the global gravitational anomaly of the self-dual field theory until recently.

Motivation

The self-dual field theory is the quantum field theory of an abelian 2ℓ-form gauge field, living on a $4\ell+2$-dimensional manifold, whose $2\ell+1$-form field strength obey a self-duality condition: $F = *F$.

Examples:

- The chiral boson in two dimensions.
- The chiral two-forms in the world volume theories of the M-theory, IIA and $E8 \times E8$ heterotic fivebranes.
- The chiral two-forms in the tensor and gravitational supermultiplets of 6d supergravities.
- The RR 4-form gauge field in type IIB supergravity.

Motivation

Witten made a proposal in 1985 for the global gravitational anomaly of the self-dual field, in the case when the self-dual field has no zero modes.

His motivation was to prove the absence of global gravitational anomalies of type IIB supergravity.

He proved the anomaly cancellation only on the 10d sphere (equivalently on 10d Minkowski spacetime).

We will get a general formula and show that it implies that type IIB supergravity is free of global anomalies on all 10d spin manifolds.

There are several other interesting potential applications in the context of string theory.

Motivation

On the mathematical side, the derivation features a nice interplay between seemingly disconnected subjects:

- The index theory for families of Dirac operators.
- Modular geometry, more precisely certain line bundles over finite coverings of the Siegel modular variety and their associated factors of automorphy.
- Algebraic topology, via certain topological invariant obtained from quadratic refinements, generalizing the Rohlin invariant.

Plan of the talk

1. Gravitational anomalies
2. The anomaly bundle
3. Derivation of the global anomaly formula
4. Type IIB supergravity
5. Outlook

Based on `1109.2904`, `1110.4639`.

See `1208.1540` for some mathematical aspects.

GRAVITATIONAL ANOMALIES

Gravitational anomalies

Consider an Euclidean QFT on a compact Riemannian manifold M.

The metric g can be seen as an external parameter, on which the partition function Z depends.

Under the action of a diffeomorphism ϕ, $Z(g)$ is not necessarily invariant. In general

$$Z((\phi^{-1})^* g) = \xi(\phi, g) Z(g) \quad \xi(\phi, g) \in \mathbb{C}$$

Consistency with the group structure requires

$$\xi(\phi_2 \circ \phi_1, g) = \xi(\phi_2, (\phi_1^{-1})^* g)\xi(\phi_1, g)$$

ξ is a 1-cocycle for the group $\mathcal{D}$ of diffeomorphisms of M.

Gravitational anomalies

A 1-cocycle such as ξ defines a line bundle $\mathscr{A}$ over $\mathcal{M}/\mathcal{D}$, the anomaly bundle.

Quotient $\mathcal{M} \times \mathbb{C}$ by $(g, z) \simeq (\phi(g), \xi(\phi, g)z)$.

The anomaly bundle carries a connection $\nabla_{\mathscr{A}}$.

In general, the partition function is not a well-defined function on $\mathcal{M}/\mathcal{D}$, but rather a section of $\mathscr{A}$.

The *local anomaly* is the curvature of $\nabla_{\mathscr{A}}$.

The *global anomaly* is the set of holonomies of $\nabla_{\mathscr{A}}$.

Gravitational anomalies

Why are we interested in anomalies?

To couple the theory to gravity, we have to integrate $Z(g)$ over $\mathcal{M}/\mathcal{D}$.

This is possible only if $Z(g)$ is an honest function over $\mathcal{M}/\mathcal{D}$.

In other words, this is possible only if the anomaly bundle is geometrically trivial, i.e. if the local and global anomalies vanish.

$\Rightarrow$ Any low energy effective QFT obtained from a supposedly consistent quantum gravity theory must have vanishing gravitational anomalies.

This implies strong constraints on low energy effective actions in cases where anomalies can arise.

Gravitational anomalies

Gravitational anomalies typically occur in chiral fermionic theories in dimension $4\ell + 2$.

A chiral fermionic theory is associated to a chiral Dirac operator $D : \mathscr{S}^+ \otimes \mathscr{E} \to \mathscr{S}^- \otimes \mathscr{E}$ on M.

Consider the fibre bundle $\mathcal{F} := (M \times \mathcal{M})/\mathcal{D}$ over $\mathcal{M}/\mathcal{D}$ with fiber M.

We get a family of chiral Dirac operators over $\mathcal{M}/\mathcal{D}$.

The functional determinant of a chiral Dirac operator is the section of a line bundle $\mathscr{D}$ with connection $\nabla_{\mathscr{D}}$ over $\mathcal{M}/\mathcal{D}$.

The anomaly bundle $\mathscr{A}$ of the chiral fermionic theory coincides with $\mathscr{D}$, as a line bundle with connection.

Gravitational anomalies

Anomaly formulas describe the curvature and holonomies of $\nabla_{\mathscr{D}}$.

Obtained by Alvarez-Gaumé and Witten '84 (local anomaly) and Witten '85 (global anomaly), proven rigorously and generalized by Bismut and Freed '86 using index theory techniques.

Local anomaly: $\quad TM, \mathscr{E} \to \mathcal{F} \to \mathcal{M}/\mathcal{D}$

$$R_{\mathscr{D}} = \left[2\pi i \int_M \hat{A}(R_{TM}) \mathrm{ch}(R_{\mathscr{E}}) \right]^{(2)}$$

$$\hat{A}(R) = \sqrt{\det \frac{R/4\pi}{\sinh R/4\pi}} \,, \quad \mathrm{ch}(R) = \mathrm{Tr} \exp iR/2\pi \ .$$

Gravitational anomalies

For the global anomaly (holonomies):

1. Pick a loop c in $\mathcal{M}/\mathcal{D}$.
2. Construct the mapping torus $\hat{M}_c := \mathcal{F}|_c$.
3. Pick a metric g_c on c and set $g_\epsilon = g_c/\epsilon^2 \oplus g_M$, a metric on $\hat{M}_c$.
4. Consider the Dirac operator $\hat{D}_\epsilon$ on $\hat{M}_c$ twisted by $\mathscr{E}$.
5. Let η_ϵ be its eta invariant and h_ϵ the dimension of its space of zero modes.
6. The holonomy is

$$\mathrm{hol}_{\mathscr{D}}(c) = (-1)^{\mathrm{index} D} \lim_{\epsilon \to 0} \exp -\pi i(\eta_\epsilon + h_\epsilon)$$

Gravitational anomalies

Useless formula, because the eta invariant is impossible to compute explicitly.

However, if $\hat{M}_c$ bounds a spin manifold W, one can use the Atiyah-Patodi-Singer theorem to obtain a useful formula, of the form

$$\frac{1}{2\pi i} \ln \mathrm{hol}_{\mathscr{D}}(c) = \left(\mathrm{index} D_W - \int_W \hat{A}(R_{TM}) \mathrm{ch}(R_{\mathscr{E}}) \right)$$

If we know that the local anomaly vanishes, the integral terms cancel.

$\Rightarrow$ One only has to check that a sum of topological invariants is an integer.

Gravitational anomalies

This solves the problem of computing gravitational anomalies for chiral fermionic theories.

But the self-dual field theory is a chiral theory that does not fall into this framework.

We know that its local anomaly can be described with the signature Dirac operator.

What about the global anomaly?

THE ANOMALY BUNDLE

The self-dual field

Let M be a manifold of dimension $4\ell + 2$.

Endow it with a quadratic refinement of the intersection form.

Parameterized by $\eta \in \frac{1}{2}H^{2\ell+1}_{\mathrm{free}}(M, \mathbb{Z})/H^{2\ell+1}_{\mathrm{free}}(M, \mathbb{Z})$.

Write $\mathcal{D}_\eta$ for the group of diffeomorphisms of M preserving the quadratic refinement.

The partition function of the self-dual field is defined over $\mathcal{M}/\mathcal{D}_\eta$.

The self-dual field

For manifolds of dimension $4\ell + 2$:

- The cup product pairing gives a symplectic structure ω on $\Omega^{2\ell+1}(M)$.
- The Hodge star operator squares to $-\mathbb{1}$ on $\Omega^{2\ell+1}(M)$, hence defines a complex structure.
- Both structures restrict to $H^{2\ell+1}(M, \mathbb{R})$

A complex structure on a $2n$-dimensional symplectic vector space such as $H^{2\ell+1}(M, \mathbb{R})$ can be parameterized by a complex $n \times n$ matrix τ with positive definite imaginary part.

Siegel upper-half space $\mathcal{C}$.

The anomaly bundle

Dirac operators allow to construct line bundles over $\mathcal{M}/\mathcal{D}_\eta$. We need another construction.

A metric in $\mathcal{M}$ determines a Hodge star operator, whose restriction to $H^{2\ell+1}(M,\mathbb{R})$ determines an element $\tau \in \mathcal{C}$.

The action of $\mathcal{D}_\eta$ on M induces an action on $H^{2\ell+1}(M,\mathbb{R})$, which is symplectic with respect to ω, preserves the integral cohomology and factors through $\Gamma_\eta \subset \mathrm{Sp}(2n,\mathbb{Z})$.

$\Rightarrow$ We have a map $\mathcal{M}/\mathcal{D}_\eta \to \mathcal{C}/\Gamma_\eta$.

We can use it to pull back bundles.

THE ANOMALY BUNDLE

Bundles over $\mathcal{C}/\Gamma_\eta$:

- Theta bundle $\mathscr{C}^\eta$:

  $\theta^\eta(0,\tau) : \mathcal{C} \to \mathbb{C}$ is the pull back to $\mathcal{C}$ of the section of a bundle over $\mathcal{C}/\Gamma_\eta$, the theta bundle.

- Determinant $\mathscr{K}$ of the Hodge bundle:

  The Hodge bundle $\mathscr{H}$ is defined by

  $\big(H^{2\ell+1}_{SD}(M,\mathbb{R}) \times \mathcal{C}\big)/\Gamma_\eta \to \mathcal{C}/\Gamma_\eta$.

  $\mathscr{K} := \det \mathscr{H}$.

All these bundles can be described very explicitly by means of factors of automorphy.

The anomaly bundle

Call $\mathscr{D}$ the determinant bundle of the Dirac operator coupled to chiral spinors ($\mathscr{E} = \mathscr{S}^+$), and $\mathscr{D}_s$ the determinant bundle of the signature operator ($\mathscr{E} = \mathscr{S}$).

Some facts:

- $\mathscr{D}_s = \mathscr{D}^2$
- $\mathscr{D} \simeq (\mathscr{K})^{-1}$
- $\mathscr{F}^\eta := (\mathscr{C}^\eta)^2 \otimes (\mathscr{K})^{-1}$ is a flat bundle. Its holonomies are given by a character χ^η of Γ_η (computed from the transformation formula of θ^η).

The anomaly bundle

Constructing the partition function of the self-dual field theory from first principles on an arbitrary Riemannian manifold is an open problem.

Using ideas from Belov-Moore '06, we can construct the partition function for a pair of self-dual fields by path integration:

$$\mathcal{Z} = (\theta^\eta)^2 \cdot (\text{one loop determinant})$$

The one-loop determinant vanishes *nowhere* on $\mathcal{M}/\mathcal{D}_\eta$! $\Rightarrow$ It is the section of a topologically trivial bundle.

$\Rightarrow$ Topologically, the anomaly bundle of a pair of self-dual fields $(\mathscr{A}^\eta)^2$ is $(\mathscr{C}^\eta)^2$.

THE ANOMALY BUNDLE

It has been known for a long time that the local anomaly of a pair of self-dual fields is described correctly by $\mathscr{D}^{-1}$.

$\Rightarrow$ The curvatures of the connections on $(\mathscr{A}^\eta)^2$ and on $\mathscr{D}^{-1}$ have the same local form.

$\Rightarrow$ $(\mathscr{A}^\eta)^2$ and $\mathscr{D}^{-1}$ coincide up to a flat bundle.

As $(\mathscr{A}^\eta)^2 \simeq (\mathscr{C}^\eta)^2$ topologically, we have

$$(\mathscr{A}^\eta)^2 = \mathscr{D}^{-1} \otimes \mathscr{F}^\eta$$

$$\Big(= \mathscr{D}^{-1} \otimes (\mathscr{K})^{-1} \otimes (\mathscr{C}^\eta)^2 \Big)$$

THE ANOMALY BUNDLE

We determined the anomaly bundle for a pair of self-dual fields and its connection.

How to compute the global anomaly?

- The holonomies of the connection on $\mathscr{D}^{-1}$ are provided by the Bismut-Freed formula.
- The holonomies of $\mathscr{F}^{\eta}$ are known from the theta transformation formula.

$\Rightarrow$ Problem solved?

No! The resulting holonomy formula is practically useless. We need to reexpress the holonomies in terms of topological invariants of a manifold bounded by the mapping torus.

The global anomaly formula

The global anomaly formula

$$(\mathscr{A}^\eta)^2 = \mathscr{D}^{-1} \otimes \mathscr{F}^\eta$$

To relate the eta invariant in the Bismut-Freed formula to data on a bounded manifold, we have to use the Atiyah-Patodi-Singer theorem.

Impossible with $D \Rightarrow$ Consider D_s instead:

$$(\mathscr{A}^\eta)^4 = \mathscr{D}_s^{-1} \otimes (\mathscr{F}^\eta)^2$$

$$\mathrm{hol}_{(\mathscr{A}^\eta)^4}(c) = \exp \pi i(\eta_0 + h) \cdot (\chi^\eta(\gamma_c))^2$$

We can rewrite this formula in terms of an "Arf invariant" of the mapping torus $\hat{M}_c$..

The global anomaly formula

The linking pairing L on $H^{2\ell+2}_{\text{tors}}(\hat{M}_c, \mathbb{Z})$ admits quadratic refinements $q : H^{2\ell+2}_{\text{tors}}(\hat{M}_c, \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$:

$$q(x+y) - q(x) - q(y) = L_G(x,y)\ , \quad q(nx) = n^2 q(x)\ .$$

The characteristic η determines such a quadratic refinement q^η.

The Arf invariant $A_\eta = A(q_\eta) \in \frac{1}{8}\mathbb{Z}/\mathbb{Z}$ is the argument of the Gauss sum

$$\text{Gauss}(q^\eta) = \sum_{g \in G} \exp 2\pi i q^\eta(g)\ .$$

Experimental fact: $(\chi^\eta(\gamma_c))^2 \exp \pi i h = \exp -\pi i 8 A_\eta(\hat{M}_c)$. No complete proof, only checks on various elements $\gamma_c \in \Gamma_\eta$.

The global anomaly formula

We obtain:

$$\mathrm{hol}_{(\mathscr{A}^{\eta})^4}(c) = \exp \pi i(\eta_0 - 8A_\eta)$$

Atiyah-Patodi-Singer says $\eta_0 = \int_W L - \sigma_W$.

Moreover, an old result of Brumfiel and Morgan shows that $A_\eta = \frac{1}{8}\int_W \lambda_\eta^2 - \sigma_W \bmod 1$, for λ a certain "integral lift of the Wu class".

$$\mathrm{hol}_{(\mathscr{A}^{\eta})^4}(c) = \exp \pi i \int_W (L - \lambda_\eta^2)$$

We need to take a fourth root of this formula. Non-trivial operation!

THE GLOBAL ANOMALY FORMULA

Most naive way of taking the fourth root:

$$\mathrm{hol}_{\mathscr{A}^\eta}(c) = \exp \frac{2\pi i}{8} \int_W (L - \lambda_\eta^2)$$

Consistency checks:

- Reproduces the correct relative anomaly for $\mathscr{A}^{\eta'} \otimes (\mathscr{A}^\eta)^{-1}$ (Lee-Miller-Weintraub '88).
- If the formula above defines the holonomies of a well-defined bundle, then it is the correct one, by an argument on $\mathcal{C}/\Gamma_\eta$.
- This is the case when $\lambda = 0$.
- The formula appeared in the mathematical work of Hopkins and Singer '02.

Type IIB supergravity

Type IIB supergravity

The anomaly formula is useful only if we can determine λ_η for the physically relevant choice of QR.

In the case of 10-dimensional type IIB supergravity, for the relevant choice of QR, λ_η vanishes.

We will check the cancellation of global anomalies for "cohomological" IIB supergravity: the topological sectors of the Ramond-Ramond fields are labelled by cohomology, not K-theory.

This is not the low energy limit of the type IIB superstring!

Type IIB supergravity

Global anomaly cancellation in Type IIB has been already studied by Witten in his original paper on global gravitational anomalies.

He used:

$$\mathrm{hol}_{\mathscr{A}}(c) = \exp \frac{2\pi i}{8} \left(\int_W L - \sigma_W \right)$$

and obtained $\exp -\frac{2\pi i}{8}\sigma_W$ for the total global anomaly.

Would type IIB suffer from a global anomaly?

Witten showed that it's not the case in 10d Minkowsky space-time. He also mentionned that his result should be trusted only when $H^{2\ell+1}(M,\mathbb{Z}) = 0$.

Indeed in this case $(\mathcal{A})^4 = (\mathcal{D}_s)^{-1}$.

TYPE IIB SUPERGRAVITY

As $\lambda = 0$, our anomaly formula predicts

$$\mathrm{hol}_{\mathscr{A}}(c) = \exp \frac{2\pi i}{8} \int_W L$$

$\Rightarrow$ No gravitational anomaly.

This is compatible with Witten's result, because σ_W is a multiple of 8 whenever $H^{2\ell+1}(M, \mathbb{Z}) = 0$ (Brumfiel-Morgan '73).

In conclusion:

- This check shows that there is no global gravitational anomaly in the "cohomological" version of type IIB supergravity.
- Witten's formula is valid when $H^{2\ell+1}(M, \mathbb{Z}) = 0$, but does not make the anomaly cancellation manifest.

Outlook

We derived an expression the global gravitational anomaly of the self-dual field.

The anomaly bundle $\mathscr{A}^\eta$ is a “theta bundle” on the space of metrics modulo diffeomorphisms pulled back from the space of complex structures of the intermediate Jacobian.

It carries a connection modeled on the Bismut-Freed connection living on determinant bundles of Dirac operators.

The holonomy formula should compute the holonomies of this connection.

Outlook

There are many interesting physical applications to be explored.

- Check anomaly cancellation in IIB taking into account K-theory.
- Derive an anomaly formula for the five-branes.
- Check anomaly cancellation in 6d supergravities.

The main difficulty is to determine the class λ in these cases. Recent progresses in this direction.

Given all we already learned from anomalies about supergravities, string theory and M-theory, we can hope global anomalies will provide new insights.