The bulk-edge duality for topological insulators

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Joint work with Gian Michele Graf

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Outline

- Topological insulators
- Model Hamiltonians
- Edge and bulk indices
- The bulk-edge correspondence for $2d$ topological insulators
- Conclusions and perspectives

Topological insulators

- Topological insulators are time-reversal invariant fermionic systems that
  - behave as usual insulators in the bulk (band gap at Fermi energy)
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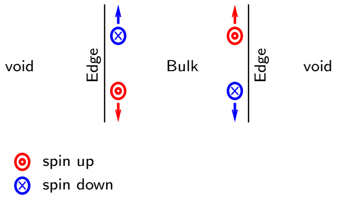

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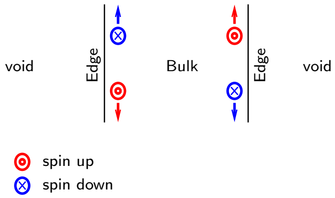

- Topology: A topological insulator can not be deformed in an ordinary one, while keeping the gap open and time-reversal invariance.

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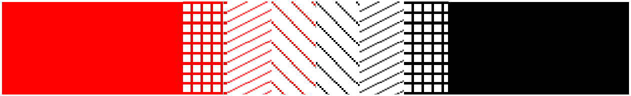
- The classification can be expressed through an index, corresponding to bulk or to edge properties.
- In what sense the two classifications are related?

Bulk-edge correspondence



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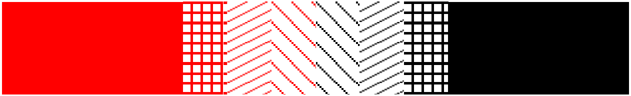

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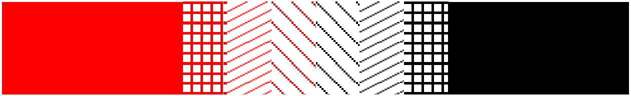

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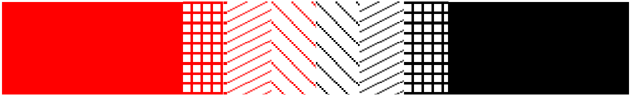

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- We introduce a new $\mathbb{Z}_2$ classification for $2d$ topological insulators, and prove the bulk-edge correspondence.

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- Bulk Hamiltonian, acting on $\psi = (\psi)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^N)$:

$$(H(k)\psi)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n(k)\psi_n$$

$A(k) =$ hopping matrix ($\in \mathrm{GL}(N)$) and $V_n(k) =$ local potential.

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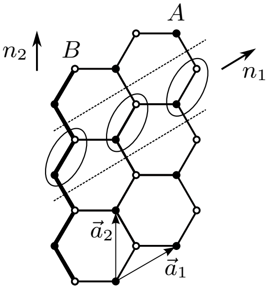
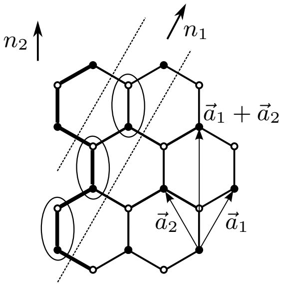
- $\sigma_{ess}(H^\sharp(k)) \subseteq \sigma_{ess}(H(k))$, but typically $\sigma_{disc}(H^\sharp(k)) \not\subseteq \sigma_{disc}(H(k))$.

Example: Graphene





Left: zigzag b.c. Right: armchair b.c.

Electron hopping on the honeycomb lattice.

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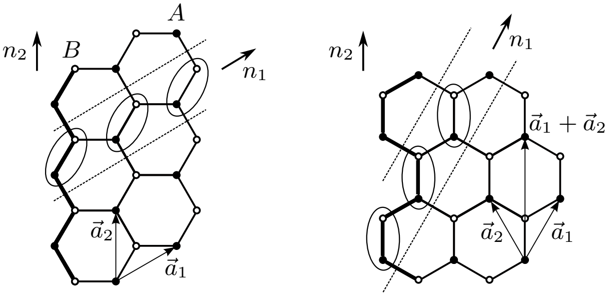

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Electron hopping on the honeycomb lattice. Armchair:

$$\text{oval} = \psi_n = \begin{pmatrix} \psi_n^A \\ \psi_n^B \end{pmatrix}\,, \quad A(k) = -t\begin{pmatrix} 0 & 1 \\ e^{ik} & 0 \end{pmatrix}\,, \quad V_n(k) = -t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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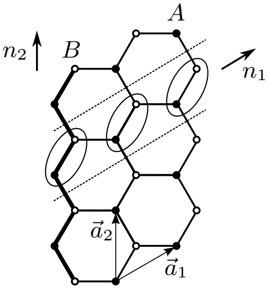

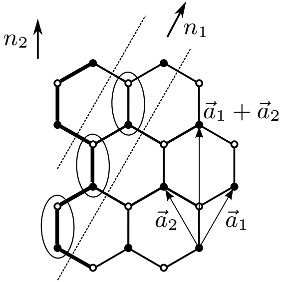

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Graphene + spin-orbit coupling: Kane-Mele model.

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  c) $\Theta^{-1} H(k) \Theta = H(-k)$ for all $k \in S^1$ (likewise for $H^\sharp(k)$)

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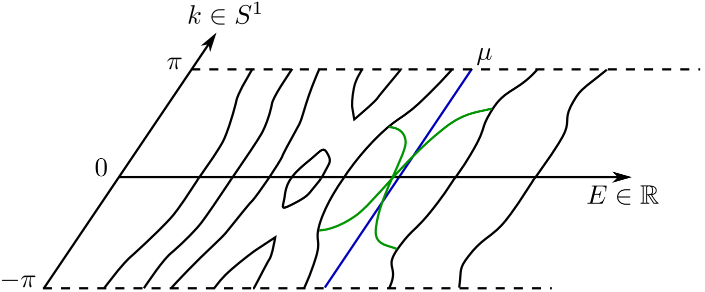
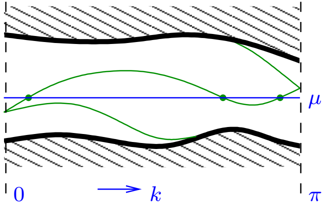


Bands, Fermi line, edge states

The edge index

Spectrum of $H^\sharp(k)$:

symmetric on $-\pi \leq k \leq 0$



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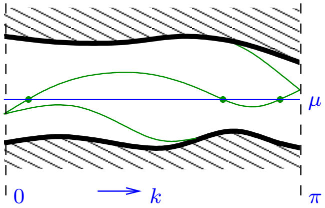

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- **Definition** (Edge Index).

$$\mathcal{I}^\sharp := \text{parity of number of eigenvalue crossings}$$

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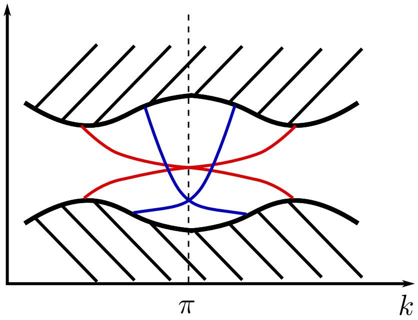

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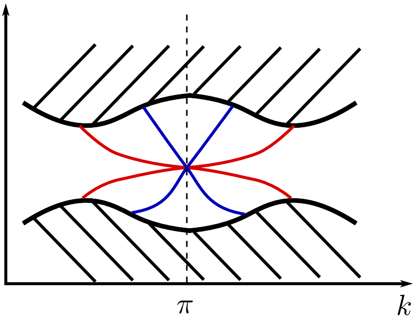

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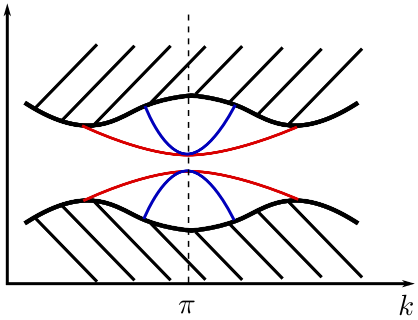

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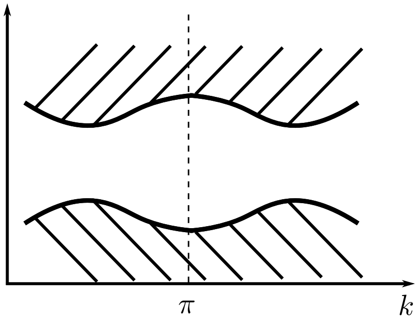

Kramers families of matrices

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- Suppose $T \in \mathrm{GL}(N)$ satisfies

$$\Theta_0 T = T^{-1} \Theta_0 \ .$$

  Then the eigenvalues of T come in pairs λ, $\bar{\lambda}^{-1}$ with equal algebraic multiplicity. In particular, their phases $z = \lambda / |\lambda|$ are even degenerate.

Kramers families of matrices

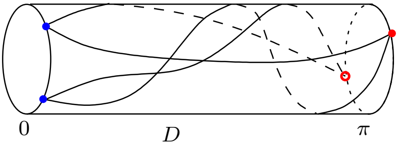
- **Definition.** A continuous family $T(k)$, $0 \le k \le \pi$, has the Kramers property if

$$\Theta_0 T(0) = T(0)^{-1}\Theta_0 \ , \qquad \Theta_0 T(\pi) = T(\pi)^{-1}\Theta_0 \ .$$

- As a result, these families of matrices admit a $\mathbb{Z}_2$ classification.

An index for Kramers families of matrices

- Let $D = (D(k))_{0 \le k \le \pi}$, with $D(k) = \{\text{phases of eigenvalues of } T(k)\}$.



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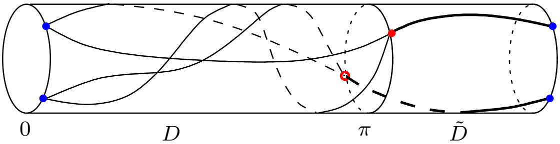

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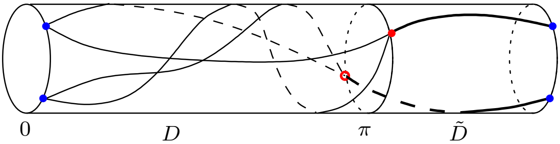

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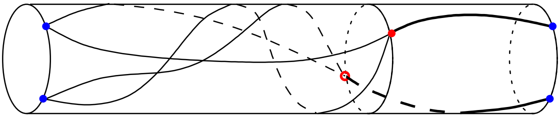

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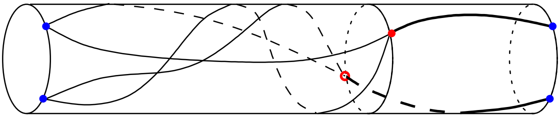

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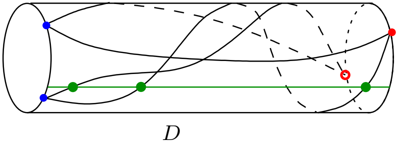
- $\mathcal{I}(D) := (-1)^{\mathcal{N}(D \# \tilde{D})}$ is a well-defined index for D.

An index for Kramers families of matrices



The index $\mathcal{I}(D)$ can be equivalently expressed as:

$$\mathcal{I}(D) = \text{parity of number of crossings of fiducial line}$$

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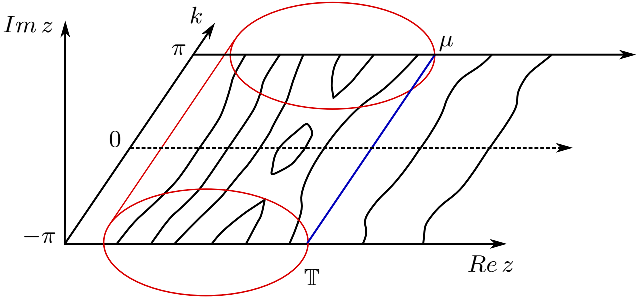
- Let $z \notin \sigma(H(k))$. Then

$$E_{z,k} := \{\psi \mid \psi_n \to 0 \text{ as } n \to +\infty\}$$

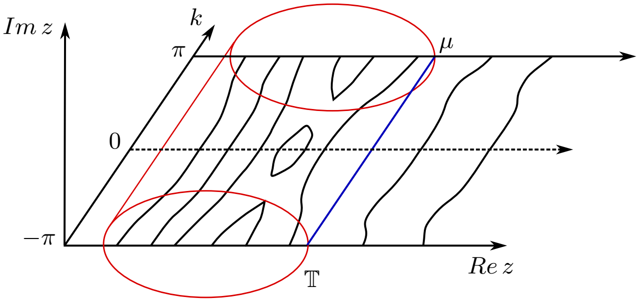
has dimension N. Moreover, $\Theta E_{z,k} = E_{\bar{z},-k}$.



Torus $\mathbb{T}$, bands, Fermi line μ.

- Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$ and involution Θ.

The bulk index



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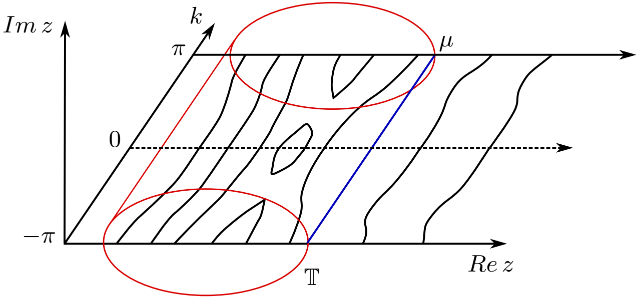

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- **Definition** (Bulk index).

$$\mathcal{I} := \mathcal{I}(E)$$

Main Result

Theorem (Bulk-edge correspondence).

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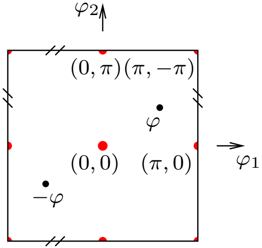
- Bulk and edge descriptions agree.
- $\mathcal{I} = -1$: topological insulator. $\mathcal{I} = 1$: trivial insulator.

Time reversal invariant bundles

$\mathbb{T}$



- $\mathbb{T} \ni \varphi = (\varphi_1, \varphi_2)$
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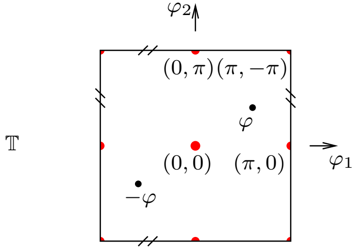

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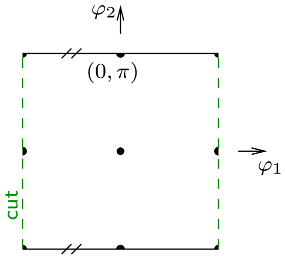
- Frame bundle $F(E)$ has fibers $F(E)_\varphi \ni v = (v_1, \ldots, v_N)$ consisting of bases v of E_φ.

Classification of time reversal invariant bundles

Consider the cut torus:



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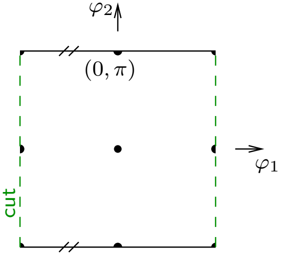

Lemma. On the cut torus the frame bundle admits a section $\varphi \mapsto v(\varphi) \in F(E)_\varphi$ which is time-reversal invariant:

$$v(-\varphi) = (\Theta v(\varphi))\varepsilon$$

with ε the block diagonal matrix with blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

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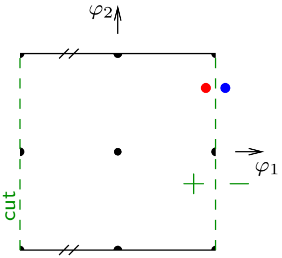

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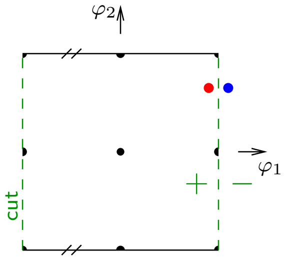

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Eq. (1) implies a relation between $T(\varphi_2)$ and $T(-\varphi_2)$:

$$\boxed{\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0 \qquad (\Theta_0 = \varepsilon C)}$$

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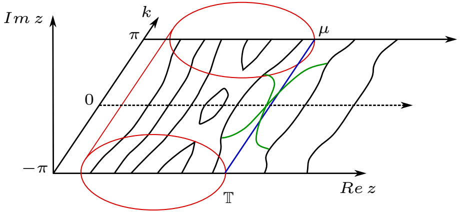
- It remains to prove that the bulk index $\mathcal{I} = \mathcal{I}(E)$ is equal to $\mathcal{I}^\sharp$.

Sketch of the proof of the bulk-edge correspondence



Fermi line
edge states
torus

- ψ, $\psi^\sharp$ solutions (bulk, edge) at z, k decaying at $n \to +\infty$
- Bijective map $\psi \mapsto \psi^\sharp$, so that $\psi_n = \psi^\sharp_n$ $(n > n_0)$
- $\exists \psi \neq 0 \mid \psi^\sharp_{n=0} = 0 \Leftrightarrow z \in \sigma(H^\sharp(k))$
- There is a section of the frame bundle $F(E)$, global on $\mathbb{T}$, except at edge eigenvalue crossings
- Cut the torus along the Fermi line; let $T(k)$ be the transition matrix
- There $T(k) = \mathbb{I}_N$, except near eigenvalue crossings
- As k traverses one of them, $T(k)$ has eigenvalues 1 (multiplicity $N-1$) and $\lambda(k)$ making one turn of S^1
- Hence indices are equal.

Further results and final remarks

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- Interpretation of the link between bulk and edge through scattering theory (*Levinson theorem*).

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Perspectives:

- $3d$ topological insulators (how many invariants?)
- No periodicity (*e.g.* disordered case)
- QFT/effective actions approach (so to consider interactions)