Momentum map and Reduction in Poisson geometry and deformation quantization

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Outline

Physics

1. Phase space with constraints: Poisson manifolds
2. Symmetries
3. Conserved quantities: momentum map
4. Deformation quantization

$$(C^\infty(M), \{\ ,\ \}) \longrightarrow (C^\infty_\hbar(M), [\ ,\ ]_\star)$$

Poisson action

What is a Hamiltonian action in this context?

Ingredients:

- Poisson Lie groups and Lie bialgebras
- Poisson maps

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Definition

The action of (G, π_G) on (M, π) is called Poisson action if the map $\Phi : G \times M \to M$ is Poisson, where $G \times M$ is given the product Poisson structure $\pi_G \oplus \pi$

If G carries the zero Poisson structure $\pi_G = 0$, the action is Poisson if and only if it preserves π. When $\pi_G \neq 0$, the structure π is not invariant with respect to the action.

Momentum map

Definition (Lu)

A momentum map for the Poisson action $\Phi : G \times M \to M$ is a map $\boldsymbol{\mu} : M \to G^*$ such that

$$\xi_M = \pi^\sharp(\boldsymbol{\mu}^*(\theta_\xi))$$

where θ_ξ is the left invariant 1-form on G^* defined by the element $\xi \in \mathfrak{g} = (T_e G^*)^*$ and $\boldsymbol{\mu}^*$ is the cotangent lift $T^*G^* \to T^*M$.

A Hamiltonian action is a Poisson action induced by an equivariant momentum map.

Infinitesimal momentum map

Definition

Let M be a Poisson manifold and G a Poisson Lie group. An infinitesimal momentum map is a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [\ ,\ ]) \longrightarrow (\Omega^\bullet(M), d_{DR}, [\ ,\ ]_\pi).$$

Poisson Reduction

Theorem

Let $\Phi : G \times M \to M$ be a Hamiltonian action with momentum map $\boldsymbol{\mu} : M \to G^*$ and $u \in G^*$ a regular value of $\boldsymbol{\mu}$. The Poisson reduction of (M, G) is the quotient

$$M//G = \boldsymbol{\mu}^{-1}(u)/G_u$$

$M//G$ inherits a Poisson structure from M.

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Example

Suppose that $\pi_G = 0$. Then

$$C^\infty(M//G) \simeq (C^\infty(M)/\mathcal{I})^G$$

where $\mathcal{I}$ is the ideal generated by the momentum map.

Deformation quantization approach

Poisson manifolds $\longrightarrow$ Kontsevich Theory $M \longrightarrow \mathcal{A}_\hbar = C^\infty_\hbar(M)$

Lie biagebras $\longrightarrow$ Etingof-Kazdhan Theory $\mathfrak{g} \longrightarrow \mathcal{U}_\hbar(\mathfrak{g})$

Steps:

1. Quantize a Poisson action
2. Quantize Momentum map
3. Quantize Poisson reduction

Quantum action

How can we define a quantum action of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathcal{A}_\hbar$?

- Hopf algebra action
- $\hbar \to 0$ Poisson action

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- $\hbar \to 0$ Poisson action

Definition

The quantum action is the linear map

$$\Phi_\hbar : \mathcal{U}_\hbar(\mathfrak{g}) \to End\ \mathcal{A}_\hbar : \xi \mapsto \Phi_\hbar(\xi)(f)$$

such that

1. Hopf algebra action
2. Algebra homomorphism

Quantum Hamiltonian action

1. Quantum momentum map which, as in the classical case, factorizes the quantum action
2. $\hbar \to 0$ classical momentum map

Non commutative forms

The non-commutative analogue of the de Rham complex is $(\Omega(\mathcal{A}_\hbar), d)$ with the universal derivation

$$d : \mathcal{A}_\hbar \to \Omega(\mathcal{A}_\hbar)$$

Quantum momentum map

The map

$$adb \longmapsto a[b, \cdot]_*$$

induces a non commutative product on $\Omega(\mathcal{A}_\hbar)$ and natural morphism of differential graded algebras

$$\Omega^1(\mathcal{A}_\hbar) \longrightarrow C^1(\mathcal{A}_\hbar, \mathcal{A}_\hbar)$$

This induces a first definition of the momentum map.

Quantum momentum map

Definition

A quantum momentum map is defined to be a linear map

$$\boldsymbol{\mu}_\hbar : \mathcal{U}_\hbar(\mathfrak{g}) \to \Omega(\mathcal{A}_\hbar) : \xi \mapsto \sum_i a^i_\xi d b^i_\xi .$$

such that

$$\Phi_\hbar(\xi) = \frac{1}{\hbar} \sum_i a^i_\xi \left[b^i_\xi , \cdot \right]_\star$$

is a quantized action.

Definition

A quantum momentum map is defined to be a linear map

$$\boldsymbol{\mu}_\hbar : T(\mathcal{U}_\hbar(\mathfrak{g})[1]) \to \Omega^\bullet(\mathcal{A}_\hbar) : \xi_1 \otimes \cdots \otimes \xi_n \mapsto a_1 db_1 \otimes \cdots \otimes a_n db_n$$

such that

$$\Phi_\hbar(\xi_1 \otimes \cdots \otimes \xi_n)(f_1, \ldots, f_n) = \frac{1}{\hbar^n} a_1[b_1, f_1] \ldots a_n[b_n, f_n]$$

Quantum Reduction

Definition

Le $\mathcal{I}_\hbar$ be the left ideal of $\mathcal{A}_\hbar$ generated by $\boldsymbol{\mu}_\hbar$. The action of $\mathcal{U}_\hbar(\mathfrak{g})$ descends to an action on $\mathcal{A}_\hbar/\mathcal{I}_\hbar$ and we define the reduced algebra by

$$\mathcal{A}_\hbar^{red} = (\mathcal{A}_\hbar/\mathcal{I}_\hbar)^{\mathcal{U}_\hbar(\mathfrak{g})}$$

Hopf algebra action condition

Assume that ξ acts by

$$\Phi_\hbar(\xi) = \frac{1}{\hbar} a[b, \cdot\,]$$

for some $a, b \in C^\infty_\hbar(M)$. Note that $a \neq 0$ as soon as ξ is not killed by the cocycle δ.

Hopf algebra action $\Longrightarrow \Phi_\hbar(\eta) = \frac{1}{\hbar} a[a^{-1}, \cdot]$

$$\Delta_\hbar(\xi) = \xi \otimes 1 - \hbar\, \eta \otimes \xi + 1 \otimes \xi$$
$$\Delta_\hbar(\eta) = \eta \otimes 1 - \hbar\, \eta \otimes \eta + 1 \otimes \eta$$

Algebra homomorphism condition

We calculate the bracket of generators to get the deformed algebra structure of $\mathfrak{g}$:

$$
\begin{aligned}
[\Phi_\hbar(\xi), \Phi_\hbar(\eta)]\, f &= \frac{1}{\hbar^2}(a[b, a[a^{-1}, f]] - a[a^{-1}, a[b, f]]) \\
&= a[b, a][a^{-1}, f] + a^2[[b, a^{-1}], f].
\end{aligned}
$$

Imposing that $\Phi_\hbar$ is a Lie algebra homomorphism we obtain different algebra structures that we discuss case by case.

Two dimensions: $[a, b] = 0$

Consider the Lie bialgebra $\mathfrak{g} = \mathbb{R}^2$ with generators ξ, η and a deformation quantization $C^\infty_\hbar(M)$ of a Poisson manifold M.

Algebra homomorphism $\Longrightarrow \mathcal{U}_\hbar(\mathbb{R}^2)$ generated by $[\xi, \eta] = 0$

deformation quantization of

Abelian Lie bialgebra $\mathfrak{g} = \mathbb{R}^2$ with cobracket

$$\delta(\xi) = -\frac{1}{2}\eta \wedge \xi$$
$$\delta(\eta) = 0$$

Two dimensions: $[a, b] = 0$

Classical action

$$\Phi(\xi) = a_0\{b_0, \cdot\}$$
$$\Phi(\eta) = a_0\{a_0^{-1}, \cdot\}.$$

Quantum reduction

$$(C_\hbar^\infty(M)/\mathcal{I}_\hbar)^{\mathcal{U}_\hbar(\mathbb{R}^2)} = \{a = \lambda, b = \mu\}^{\mathcal{U}_\hbar(\mathbb{R}^2)}$$

Quantization of the Poisson reduced algebra

$$(C^\infty(M)/\mathcal{I})^{\mathbb{R}^2} = \{a_0 = \lambda, b_0 = \mu\}^{\mathbb{R}^2}$$

Three dimensions: $\mathfrak{su}(2)$

Consider $a, b, c \in C^\infty_\hbar(M)$ satisfying

$$\begin{aligned} aba^{-1} &= e^{2\hbar} b \\ aca^{-1} &= e^{-2\hbar} c \\ [b, c] &= \frac{\hbar^2}{e^{-\hbar} - e^{\hbar}} a^{-2} - (1 - e^{2\hbar}) cb \end{aligned}$$

and the generators ξ, η, ζ acting respectively by

$$\begin{aligned} \Phi_\hbar(\xi) f &= \frac{1}{\hbar} a[b, f] \\ \Phi_\hbar(\eta) f &= \frac{1}{\hbar} [c, f] a \\ \Phi_\hbar(\zeta) f &= afa^{-1}. \end{aligned}$$

Three dimensions: $\mathfrak{su}(2)$

1. Lie algebra homomorphism

$$\zeta\xi\zeta^{-1} = e^{2\hbar}\xi$$
$$\zeta\eta\zeta^{-1} = e^{-2\hbar}\eta$$
$$[\xi, \eta] = \frac{\zeta^{-1} - \zeta}{e^{-\hbar} - e^{\hbar}}$$

2. Hopf algebra action

$$\Delta_\hbar(\zeta) = \zeta \otimes \zeta$$
$$\Delta_\hbar(\xi) = \xi \otimes 1 + \zeta \otimes \xi$$
$$\Delta_\hbar(\eta) = 1 \otimes \eta + \eta \otimes \zeta^{-1}.$$

Three dimensions: $\mathfrak{su}(2)$

Let

$$\Lambda = a^{-2} - e^{\hbar}\frac{(1-e^{2\hbar})^2}{\hbar^2}cb$$

The ideal $\mathcal{I}_\hbar$ generated by Λ in $\mathcal{A}_\hbar$ is $\mathcal{U}_\hbar(\mathfrak{su}(2))$-invariant, and

$$(C^\infty_\hbar(M)/\mathcal{I}_\hbar)^{\mathcal{U}_\hbar(\mathfrak{su}(2))}$$

is the deformation quantization of the Poisson reduction

$$M//SU(2)$$

corresponding to the symplectic leaf $a_0^{-2} - 4b_0c_0 = 0$ in $SU(2)^* = SB(2,(C))$.

Open questions

1. Lifting to symplectic groupoids
2. Classification