Singular foliations and their holonomy

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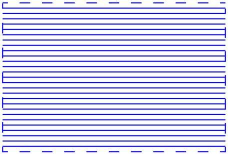
Summary

1.1 Definition: Foliation (regular)

Viewpoint 1:

Partition to connected submanifolds. Local picture: 

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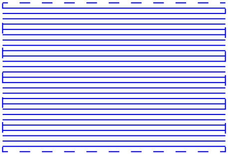

In other words: There is an open cover of M by foliation charts of the form $\Omega = U \times T$, where $U \subseteq \mathbb{R}^p$ and $T \subseteq \mathbb{R}^q$.

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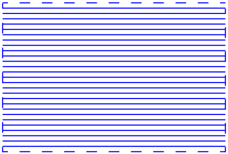
Partition to connected submanifolds. Local picture: 

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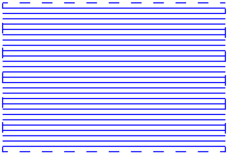

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Viewpoint 2:

Frobenius theorem

Consider the unique $C^\infty(M)$-module $\mathcal{F}$ of vector fields tangent to leaves.

Fact: $\mathcal{F} = C^\infty_c(M, F)$ and $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

Holonomy groupoid of a regular foliation
Holonomy

We wish to put a smooth structure on the equivalence relation

$$\{(x, y) \in M^2 : L_x = L_y\}$$

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A neighborhood of (x, x') where $x \in W = U \times T$ and $x' \in W' = U' \times T'$ should be of the form $U \times U' \times T$: we need an identification of T with T'. (Here T, T' are local transversals.)

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Definition

A holonomy of $(M, \mathcal{F})$ is a diffeomorphism $h : T \to T'$ such that $t, h(t)$ are in the same leaf for all $t \in T$.

Examples of holonomies

- Take $W = U \times T$. Then id_T is a holonomy.
- If h is a holonomy, h^{-1} is a holonomy.
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- Let $\gamma : [0,1] \to M$ be a smooth path in a leaf.
  Cover γ by foliation charts $W_i = U_i \times T_i (1 \leqslant i \leqslant n)$. Consider the composition

$$h(\gamma) = h_{W_n, W_{n-1}} \circ \ldots \circ h_{W_2, W_1}$$

Definition

The holonomy of the path γ is the germ of $h(\gamma)$.

Fact: Path holonomy depends only on the homotopy class of the path!

The holonomy groupoid

Definition

The holonomy groupoid is $H(F) = \{(x, y, h(\gamma))\}$ where γ is a path in a leaf joining x to y.

- Manifold structure. If $W = U \times T$ and $W' = U' \times T'$ are charts and $h : T \to T'$ path-holonomy, get chart

$$\Omega_h = U' \times U \times T$$

- Groupoid structure. $t(x, y, h) = x$, $s(x, y, h) = y$ and $(x, y, h)(y, z, k) = (x, z, h \circ k)$.

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$H(F)$ is a Lie groupoid. Its Lie algebroid is F. Its orbits are the leaves.

$H(F)$ is the smallest possible smooth groupoid over $\mathcal{F}$.

Holonomy revisited

Starting from the projective module of vector fields $\mathcal{F}$ the notion of holonomy in the regular case is:

- Pick a path $\gamma : [0,1] \to L$ and $S_{\gamma(0)}, S_{\gamma(1)}$ small transversals of L.

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Linearizes to a representation

$$dh : \pi_1(L, x) \to GL(N_x L)$$

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Debord showed that a projective $\mathcal{F}$ always has a smooth holonomy groupoid.

Stability for regular foliations

Local Reeb stability theorem

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This is equal to

$$\frac{H_x \times N_x L}{H_x^x}$$

The action of H_x^x on $N_x L$ is the one that integrates the Bott connection

$$\nabla : F \to CDO(N), \quad (X, \langle Y \rangle) \to \langle [X, Y] \rangle$$

The singular case

- What is the notion of holonomy in the singular case?
- Is there any sense in which the holonomy groupoid of a singular foliation is smooth?
- When is a singular foliation isomorphic to its linearization?

Stefan-Sussmann foliations

Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module $\mathcal{F}$ of $C_c^\infty(M; TM)$, stable under brackets.

No longer projective. Fiber $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$: upper semi-continuous dimension.

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Let L be a leaf and $x \in L$. There is a short exact sequence of vector spaces

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where ev_x is evaluation at x. Get a transitive Lie algebroid

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"Regular" leaves = leaves of maximal dimension.

On regular leaves $\mathfrak{g}_x = 0$.

Examples

Actually: Different foliations may yield same partition to leaves

1. $\mathbb{R}$ foliated by 3 leaves: $(-\infty, 0), \{0\}, (0, +\infty)$.
   $\mathcal{F}$ generated by $x^n \frac{\partial}{\partial x}$. Different module $\mathcal{F}$ for every n.
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3. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.
   No obvious best choice. $\mathcal{F}$ given by the action of a Lie group

$$GL(2, \mathbb{R}), SL(2, \mathbb{R}), \mathbb{C}^*$$

Extra difficulty: Keep track of the choice of $\mathcal{F}$!

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Indeed (U, t, s) keeps track of path holonomies near the identity:

bisections of $(U, t, s) \rightsquigarrow$ path holonomies

Passing to germs

Cover M with a family $\{(U_i, t_i, s_i)\}_{i\in I}$. Let $\mathcal{U}$ be the family of all finite products of $\{(U_i, t_i, s_i)\}_{i\in I}$ and of their inverses.

Holonomy groupoid (A-Skandalis)

The holonomy groupoid is

$$H(\mathcal{F}) = \coprod_{U\in\mathcal{U}} U/\sim$$

where $U \ni u \sim u' \in U'$ iff there is a morphism of bi-submersions $f : U \to U'$ (defined near u) such that $f(u) = u'$.

$H(\mathcal{F})$ is a topological groupoid over M, usually not smooth.

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5. action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$:

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\})^2 \cup SL(2, \mathbb{R}) \times \{0\}$$

topology: Let $x \in \mathbb{R}^2 \setminus \{0\}$. Then $(\frac{x}{n}, \frac{x}{n}) \in H(\mathcal{F})$ converges to every g in stabilizer group of x... namely to every point of $\mathbb{R}$!

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To answer this, let G_x the connected and simply connected Lie group integrating $\mathfrak{g}_x$. Near the identity, consider the map

$$\widetilde{\varepsilon}_x : G_x \to U_x^x, \quad exp_{\mathfrak{g}_x}(\sum_{i=1}^{n} \xi_i [X_i]) \mapsto exp(\sum_{i=1}^{n} \xi_i Y_i)$$

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Composing with $\sharp : \mathcal{U}_x^x \to H_x^x$ we get a morphism

$$\varepsilon_x : G_x \to H_x^x$$

$\ker \varepsilon_x$ is the essential isotropy group of the leaf L_x.

Theorem (A-Zambon)

The transitive Lie groupoid H_L is smooth and integrates A_L if and only if the essential isotropy group of L is discrete.

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The C-F obstruction is the "monodromy group" $\mathcal{N}_x(A_L) = \ker(G_x \to \Gamma_x^x)$ induced by $\mathfrak{g}_x \to A_L$.

Relation with monodromy

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We do not yet know how the isotropy and monodromy groups are related in general...

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Theorem (A-Zambon)

Assume that for any time-dependent vector field $\{X_t\}_{t\in[0,1]} \in I_x\mathcal{F}_{S_x}$ there exists a vector field $Z' \in I_x\mathcal{F}_{S_x}$ and a neighborhood S' of x in S_x such that $exp(Z)\mid_{Z'}$ is the time-1 flow of $\{X_t\}_{t\in[0,1]}$.

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Guess: This condition is satisfied whenever $\mathcal{F}$ is closed as a Frechet space...

(This rules out the *extremely* singular cases...)

The holonomy map

Let $(M, \mathcal{F})$ a singular foliation, L a leaf, $x, y \in L$ and S_x, S_y slices of L at x, y respectively.

Theorem (A-Zambon)

There is a well defined map

$$\Phi_x^y : H_x^y \to \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F}) \mid_{S_x}}, h \mapsto \langle \tau \rangle$$

where τ is defined as

- pick any bi-submersion (U, t, s) and $u \in U$ with $[u] = h$
- pick any section $b : S_x \to U$ of s through u such that $(t \circ b)S_x \subseteq S_y$

and define $\tau = t \circ b : S_x \to S_y$.

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It defines a morphism of groupoids

$$\Phi : H \to \cup_{x,y} \frac{GermAut_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F})\mid_{S_x}}$$

Holonomy map and the Bott connection

Conjecture: Φ is injective.

(Proven at points x where $\mathcal{F}$ vanishes and for regular foliations.)

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Let L be a leaf with discrete essential isotropy.

1. The derivative of τ gives

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   Lie groupoid representation of H_L on NL;

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All this justifies the terminology "holonomy groupoid"!

Linearization

Vector field on M tangent to $L \rightsquigarrow$
Vector field Y_{lin} on NL, defined as follows:

$$Y_{lin} \text{ acts on the fibrewise constant functions as } Y\,|_L$$
$$Y_{lin} \text{ acts on } C^{\infty}_{lin}(NL) \equiv I_L/I_L^2 \text{ as } Y_{lin}[f] = [Y(f)].$$

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The linearization of $\mathcal{F}$ at L is the foliation $\mathcal{F}_{lin}$ on NL generated by $\{Y_{lin} : Y \in \mathcal{F}\}$.

Lemma

Let L be an embedded leaf such that $\ker \varepsilon$ is discrete. Then the linearized foliation $\mathcal{F}_{lin}$ is the foliation induced by the Lie groupoid action Ψ_L of H_L on NL.

We say $\mathcal{F}$ is linearizable at L if there is a diffeomorphism mapping $\mathcal{F}$ to $\mathcal{F}_{lin}$.

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Remark: When $\mathcal{F}\langle X\rangle$ with X vanishing at $L = \{x\}$, linearizability of $\mathcal{F}$ means:

There is a diffeomorphism taking X to fX_{lin} for a non-vanishing function f.

This is a weaker condition than the linearizability of the vector field X!

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We don't know yet, but:

Proposition (A-Zambon)

Let L_x embedded leaf with discrete essential isotropy. Assume H_x^x compact. The following are equivalent:

1. $\mathcal{F}$ is linearizable about L
2. there exists a tubular neighborhood U of L and a (Hausdorff) Lie groupoid $G \to U$, proper at x, inducing the foliation $\mathcal{F}|_U$.

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In that case:

- G can be chosen to be the transformation groupoid of the action Ψ_L of H_L on NL.

- $(U, \mathcal{F}|_U)$ admits the structure of a singular Riemannian foliation.

Papers

[1] I. A. and G. Skandalis The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.* **626** (2009), 1–37.

[2] I. A. and M. Zambon Smoothness of holonomy covers for singular foliations and essential isotropy. arXiv:1111.1327

[3] I. A. and M. Zambon Holonomy transformations for singular foliations. arXiv:1205.6008

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Thank you!