Practical and Theoretical Aspects of Volatility Modelling and Trading

Artur Sepp
artur.sepp@juliusbaer.com

Julius Baer

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Content

1. Option replication and trading: the theory vs the real world

2. Derivatives industry and applications of valuation models

3. Volatility modeling and steady-state analysis of stochastic volatility models

4. Volatility trading in practice: the convexity vs the concavity and the volatility risk-premium

It is not true that quantitative/mathematical methods are recent developments in trading applications

Some quotes from a nice book "Reminiscences of a Stock Operator" (1923) about the biography of a legendary trader Jesse Livermore:

- "Wall Street makes its money on a mathematical basis, I mean, it makes its money by dealing with facts and figures"
- "He (the trader) must bet always on probabilities - that is, try to anticipate them"
- "The game of speculation isn't all mathematics or set rules, however rigid the main laws may be"

Livermore was trading spot/futures markets making big bets on trends

As for option trading:

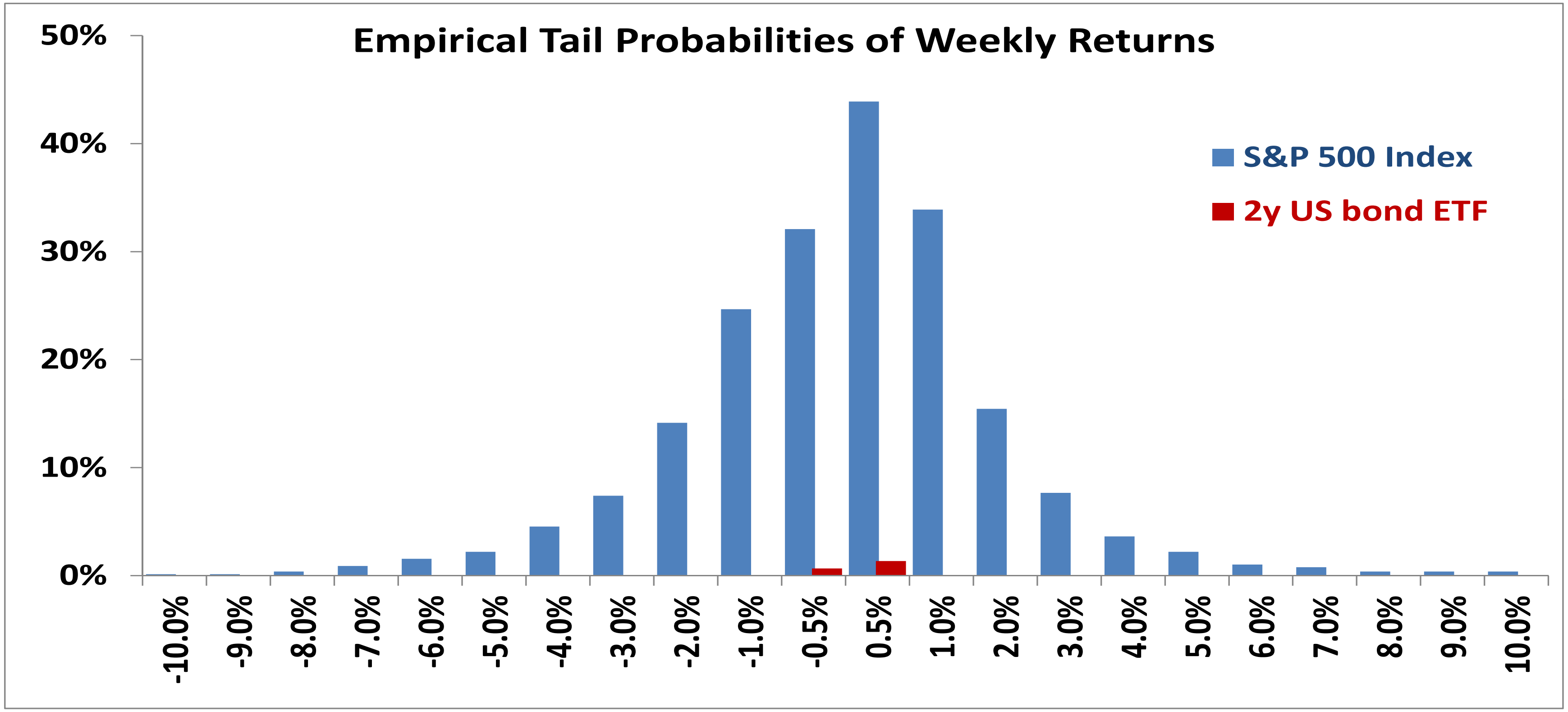
- Options are non-linear securities on underlying prices
- Trading and valuation of derivatives can only be possible using quantitative models and tools

Probability and Volatility

Options valuation includes estimation of probabilities of asset price changes

Volatility is a measure of a likelihood of given price changes

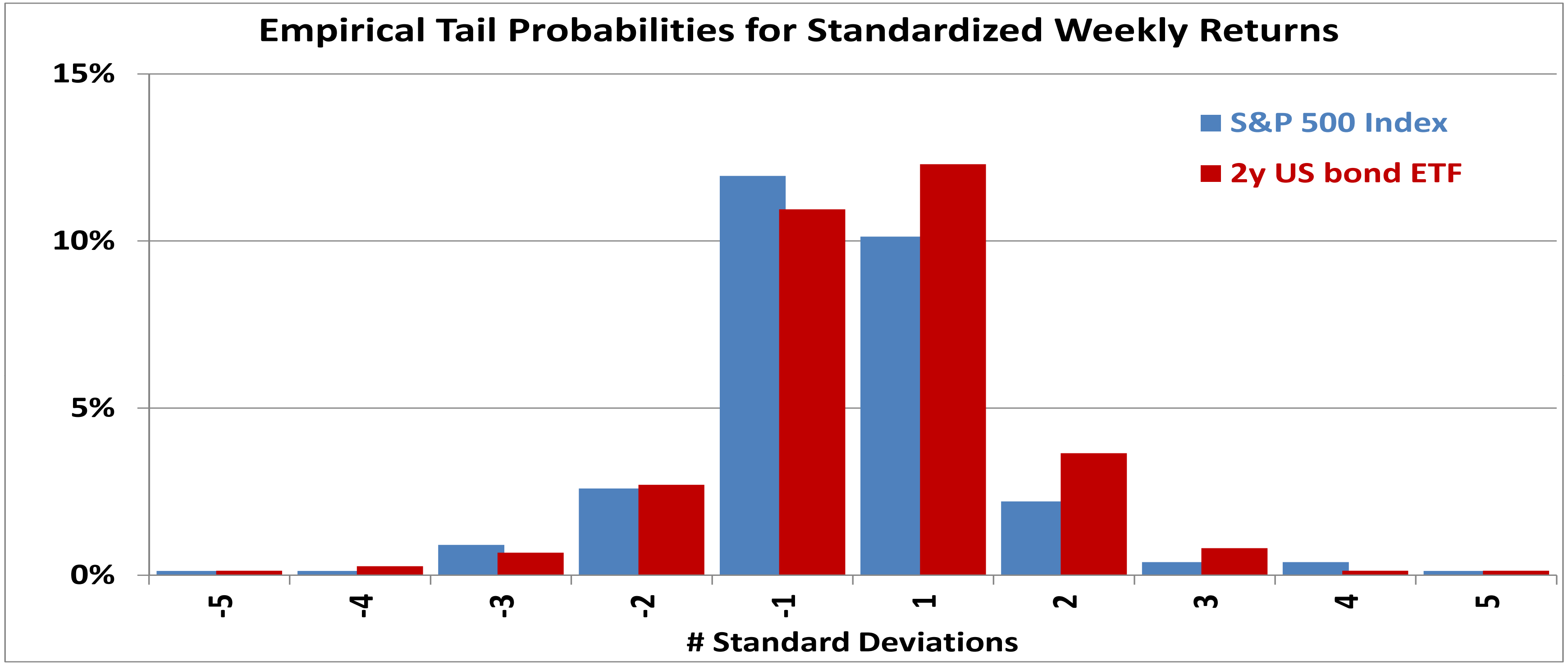
Figure: empirical tail probabilitites of weekly returns on the S&P 500 index (high volatility) and 2year US bond ETFs (low volatility)



Volatility is not the ultimate measure of the risk

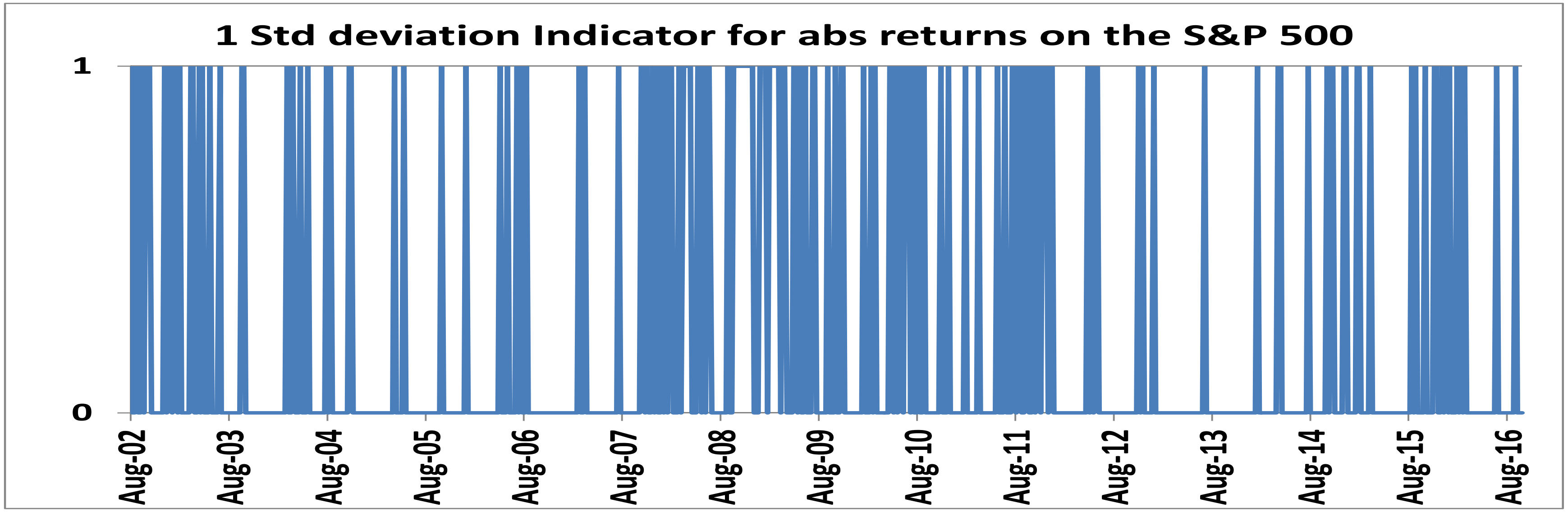
Figure: empirical tail probabilities of weekly **normalized** returns on the S&P 500 index and 2year US bond ETFs

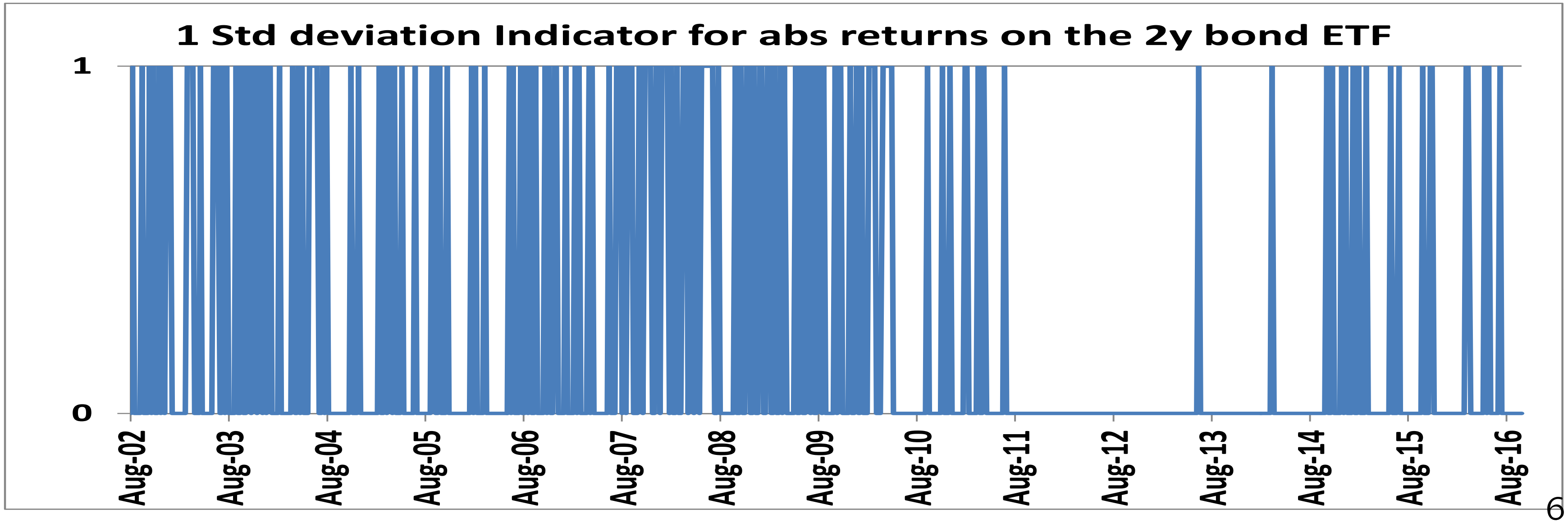
Risk-parity funds: leverage up low volatility assets to a traget volatility



Volatility is clustered

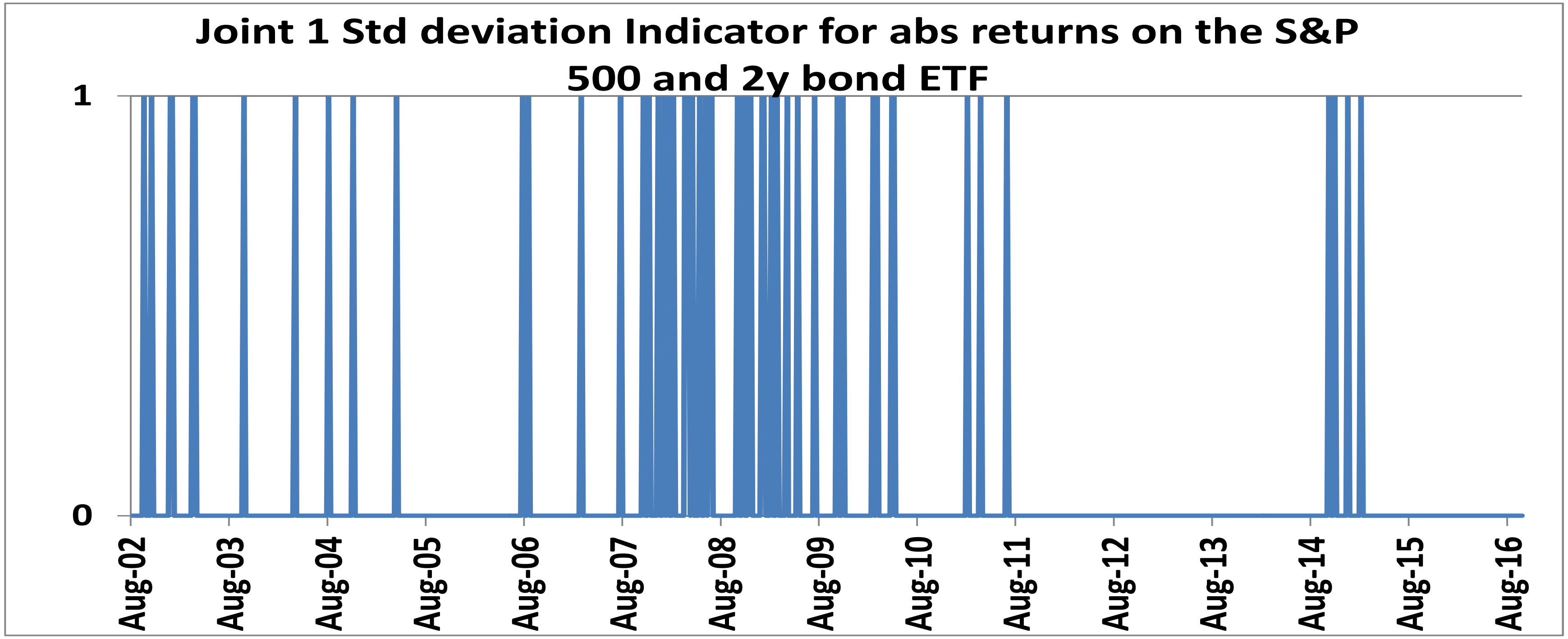
Figure: time series of hitting indicator when absolute returns exceed one standard deviation





Co-dependence is between asset classes ic clustered

Figure: time series of the joint hitting indicator the S&P 500 index and 2year US bond ETFs



Vanilla Put and Call options are primary derivative instruments traded on exchanges

Values and prices of option contracts are derived from the probability of return distributions

Options enable to create strategies related to statistical and market implied probabilities / volatilities

European call option gives the holder the right to buy the asset at maturity time T at strike price K:

$$u(S(T)) = (S(T) - K)^+$$

Put option gives the right to sell:

$$u(S(T)) = (K - S(T))^+$$

Put and call options on major asset classes and stocks represent the bulk of exchanged traded derivative contracts

Any payoff function $u(s)$ on $S(T)$ can be linearly approximated with put and calls

Option pricing in industry (using Oscar Wilde)

A mathematically-oriented quant = "a man who knows the price of everything and the value of nothing" (?)

An empirical quant = "a man who sees an absurd value in everything and doesn't know the market price of any single thing" (?)

For understanding the practicalities of option trading, we need to understand:

1) The theory of option replication

2) Practicalities of options valuation and trading

3) Empirical features of option trading strategies

4) In particular, the interception of risk-neutral valuation measure and the statistical measure

Fundamental option trading formula is originated by Black-Scholes-Merton (1973) and extended by Harrison-Pliska (1981)

We can assume a general dynamics for the underlying asset under the statistical measure:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$$

where $\mu(t)$ is the drift
$\sigma(t)$ is the volatility of asset returns
$W(t)$ a is standard Brownian motion

The key result of Black-Scholes-Merton replication framework and risk-neutral valuation is:
There exists a trading strategy in the underlying asset with the dynamic weight $\Delta(s)$ such that the terminal payoff $u(S(T))$ of the option can be replicated by trading in the underlying for any realization of price path (!):

$$u(S(T)) = g(S(t)) + \int_t^T \Delta(S(t'))dS(t')$$

Black-Scholes-Merton framework is an idealization of real market conditions

BSM assumptions vs real trading conditions:

- Continuous trading in diffusion-uncertainty market vs discrete trading with gaps (jumps)
- No transaction costs vs transaction costs and market impact costs
- Unlimited borrowing&lending ability at the same risk-free rate vs limited capacity to borrow funds and finance short&long position at different rates
- No exogenous risk factors vs the risk of changes in the volatility, interest rates, dividends, etc
- Instantaneous price discovery vs wide bid-ask spreads and illiquidity
- Flat/zero "end-of-day" risk vs the illusion of daily mark-to-market replication

Black-Scholes-Merton implied volatility

How trading imperfection do affect realized profit&loss?

Black-Scholes-Merton model is based on the log-normal price dynamics under the valuation (risk-neutral, martingale) measure:

$$dS(t) = \sigma_{BSM} S(t) dW(t)$$

where σ_{BSM} is the constant volatility under the valuation measure

Option value $U(t,S)$ solves the BSM PDE (assuming zero borrowing/lending costs):

$$\partial_t U + \frac{1}{2}\sigma^2_{BSM} S^2 \partial_{SS} U = 0 \ , \ U(T,S) = u(S)$$

Given market price of an option we can solve the inverse problem to find the BSM implied volatility σ_{BSM} (equate BSM model value to the market price)

Continuous-time Delta-Hedging P&L is the spread between implied and realized volatilities

Delta-hedging portfolio $\Pi(t)$ for hedging a short position in option $U(t,S)$:

$$\Pi(t) = \Delta(t,S)S(t) - U(t,S)$$

Over the infinitesimal time dt, using the BSM PDE, the delta-hedging P&L is

$$d\Pi(t) = \frac{1}{2}\left\{\sigma^2_{BSM}dt - R(t)\right\} S^2(t)\Gamma(t,S)$$

where $\Gamma(t,S)$ is option gamma $\Gamma(t,S) = \partial_{SS}U(t,S)$
$R(t)$ is the return squared under the statistical measure (!):

$$R(t) = \left(\frac{dS(t)}{S(t)}\right)^2$$

In the limit, $R(t) \to \sigma^2_{STAT}dt$ where σ_{STAT} is returns volatility under the statistical measure

The delta-hedging P&L is zero only if the implied BSM volatility equals to the statistical volatility:

$$\sigma_{BSM} = \sigma_{STAT}$$

The Fundamental Equation relating Implied volatility vs Realized volatility

Real-world imperfections result in the spread between the statistical volatility of returns, σ_{STAT}, and the BSM volatility implied by market prices of options, σ_{BSM}

Fundamental equation for the final P&L of delta-hedging strategy (El Karoui-Jeanblanc-Shreve (1998)):

$$\Pi(T) = \frac{1}{2}\int_0^T \left\{\sigma_{BSM}^2 - \sigma_{STAT}^2\right\} S^2(t')\Gamma(t, S')dt'$$

Even in the ideal conditions with continuous trading in diffusive uncertainty and no trading costs, this result is fundamental because:

1. If implied BSM and statistical volatilities are different, option trading strategies can be designed to take advantage of this spread

2. These strategies still have little dependence on the real-world drift of the underlying asset

This result holds for price dynamics with stochastic volatility and jumps

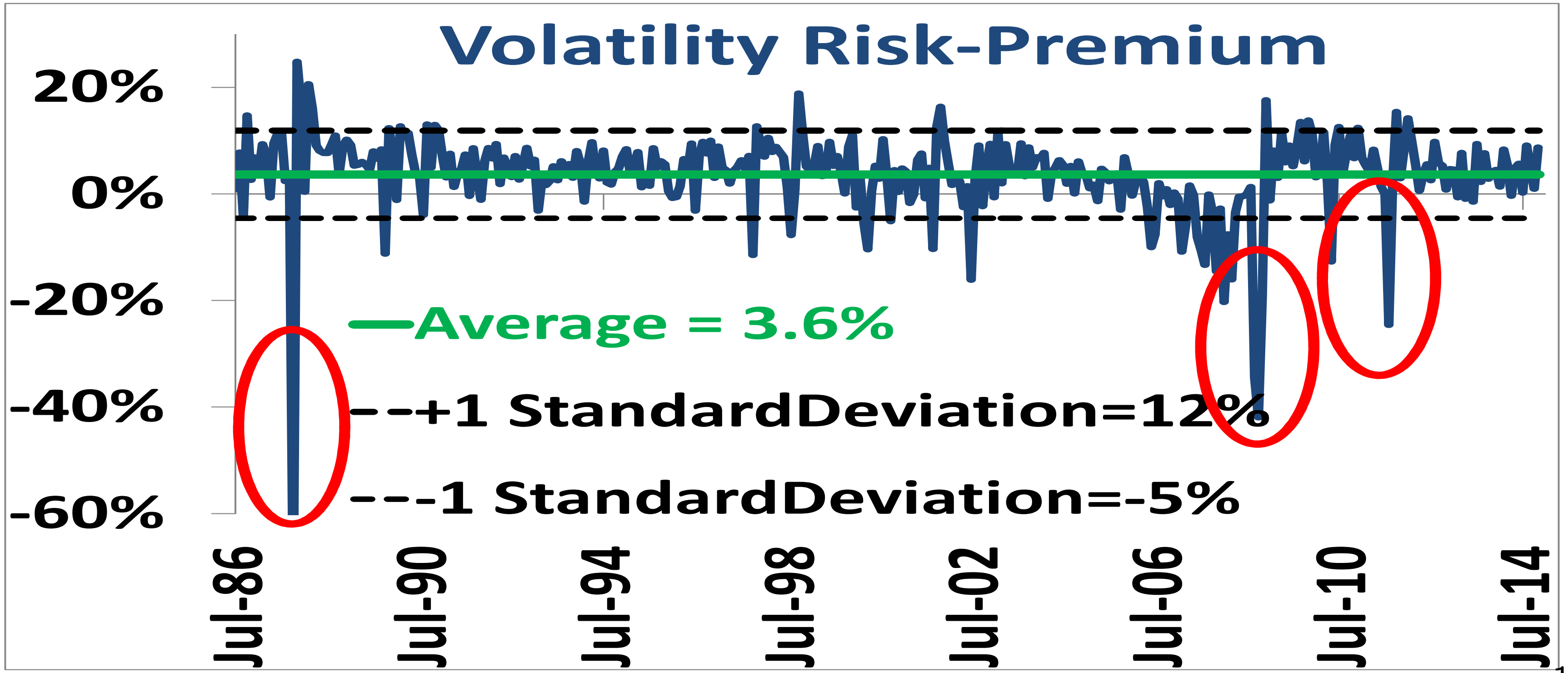
The spread between the statistical realized volatility and the implied volatility is significant and persistent

$$\text{Volatility Risk-premium} = \text{Implied volatility} - \text{Realized volatility}$$

Figure:

$$\text{Proxy Volatility Risk-premium} = \text{VIX at month start} \\ - \text{Realized volatility of S\&P500 in this month}$$

t-statistic is 8.20



Theory vs The Real World

In theory: BSM framework assumes that a derivative security is redundunt because it can be replicated and, as a result, it adds no utility to investors' portfolios

In practice:

1. Retail/institutional investors are not able to delta-hedge and replicate derivatives (no infastracture, little capital for margin, expensive trading costs)
2. A derivative security adds utility to investors' portfolios:
   - Upside speculation (out-of-the money calls)
   - Downside protection (out-of-the money puts)
   - Carry strategies (selling options without hedging to generate income)
3. Hedge funds typically use derivatives for tactical discretionary views
4. Dealers (investment banks) and options market makers stand on the other side of transactions with the goal to generate profits on their capital at risk

Derivatives Industry

The impossibility of replication and the spread between implied and realized volatilities (return distributions) give rise to trading and business opportunities which utilize quantitative models and methods with various levels of complexity

1. Structured derivatives business at invetsment banks

2. Prime brockerage and exchanges (for clearing and margining)

3. Options market makers

4. proprietary trading at hedge funds

Structured Derivatives Business employs the classic applications of derivatives pricing models and tools

1. Broker-dealer sells to a client a stuctured product

2. Risk of this products are computed using a market consistent model

3. The first oder risk, delta and vega, are hedged by trading in exchange traded derivatives

4. Flow driven business

The dealer has the advantage:

1. The client sells volatility to the dealer cheaply so the dealer buys cheap volatility and hedged himself by selling volatility more expensively in the market

2. The client buys volatility from the dealer (by buying principal protected note) at expensive levels, the dealer hedges by byuing cheaper protection in the market

Structured Derivatives Business - modeling tools

1. A model to compute and iterpolate implied BSM volatility from traded option market prices

2. A model for implied volatility surfaces

3. Local and stochastic volatility models, calibrated to implied volatilities, to value and risk-manage structured products

4. Consistency with the statistical dynamics are note relevant as dealers seek to eliminate the first order risks (delta and vega) being compensated by higher spreads from structured products

Prime Brokers, Exchanges and Risk management

Provide clearing, funding and risk-management for exchange traded and OTC derivatives for institutional investors, hedge funds, propriotory traders

Risk management sets trading budgets for trading desks

Objective is to aggregate risk of different instruments by strikes, maturities, underlyings and to provide a "fair" margin for clients

Require the consistency with historical data (both recent data and stress sase data)

Employ time series analysis (PCA) and simple pricing models

Value-at-risk is computed using EWMA and Garch time models to predict the short-term volatility

Option Market Makers

Provide bid-ask quotes for exchange traded options

Primarily apply the BSM model with a function for the implied volatility

Intraday pattern of volatility

Co-dependence with spot price and volatility

Proprietary/systematic trading

Estimate and predict realized volatility

Generate signals by screening cheap/expensive volatility in the market

Volatility models in details

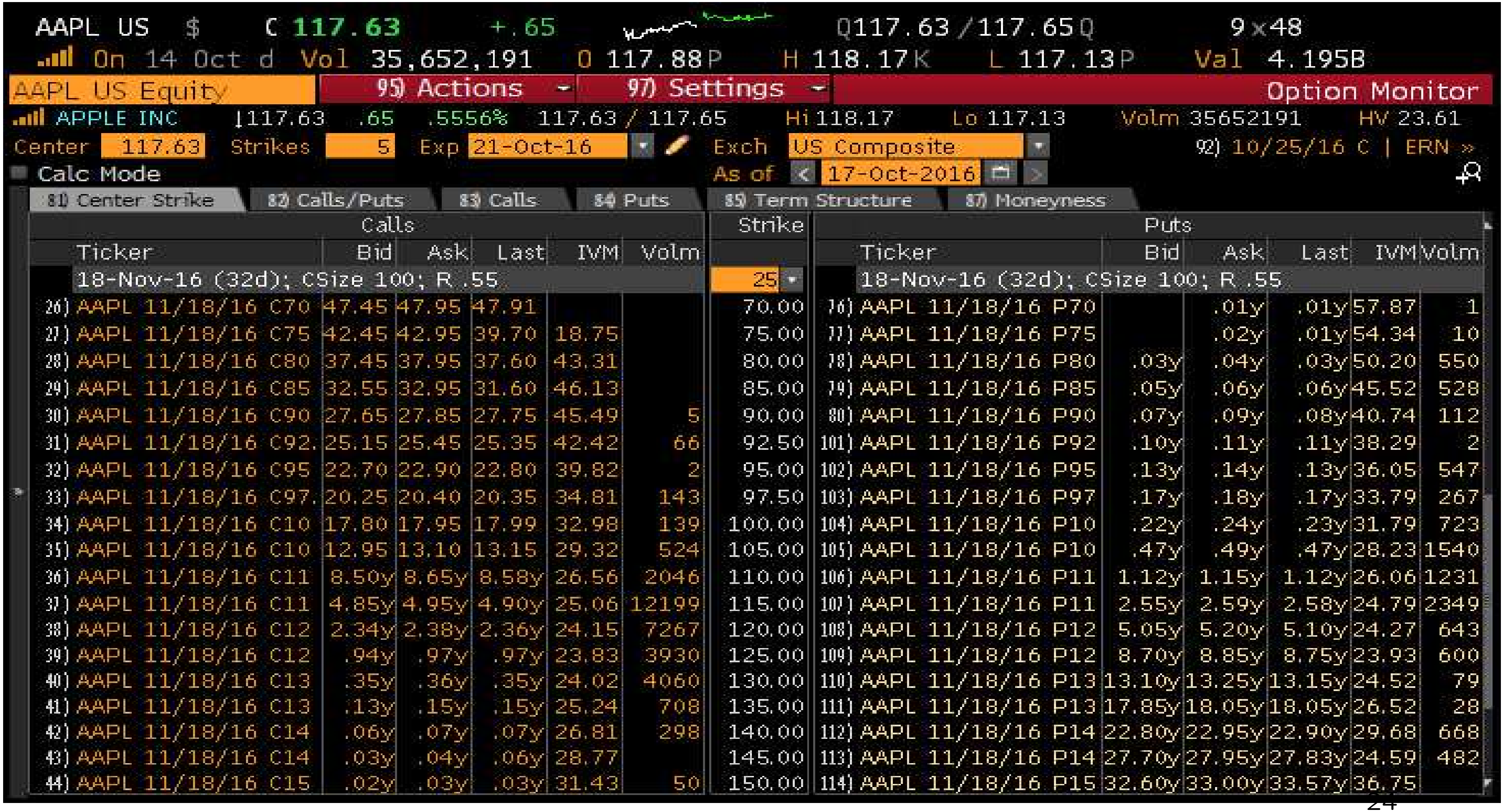
1. Models for Implied Volatility

2. Local Volatility Models

3. Stochastic Volatility Models

Implied Volatility models are applied to interpolate and extrapolate discrete options data

Figure: Snapshot of data for options quotes on Apple stock



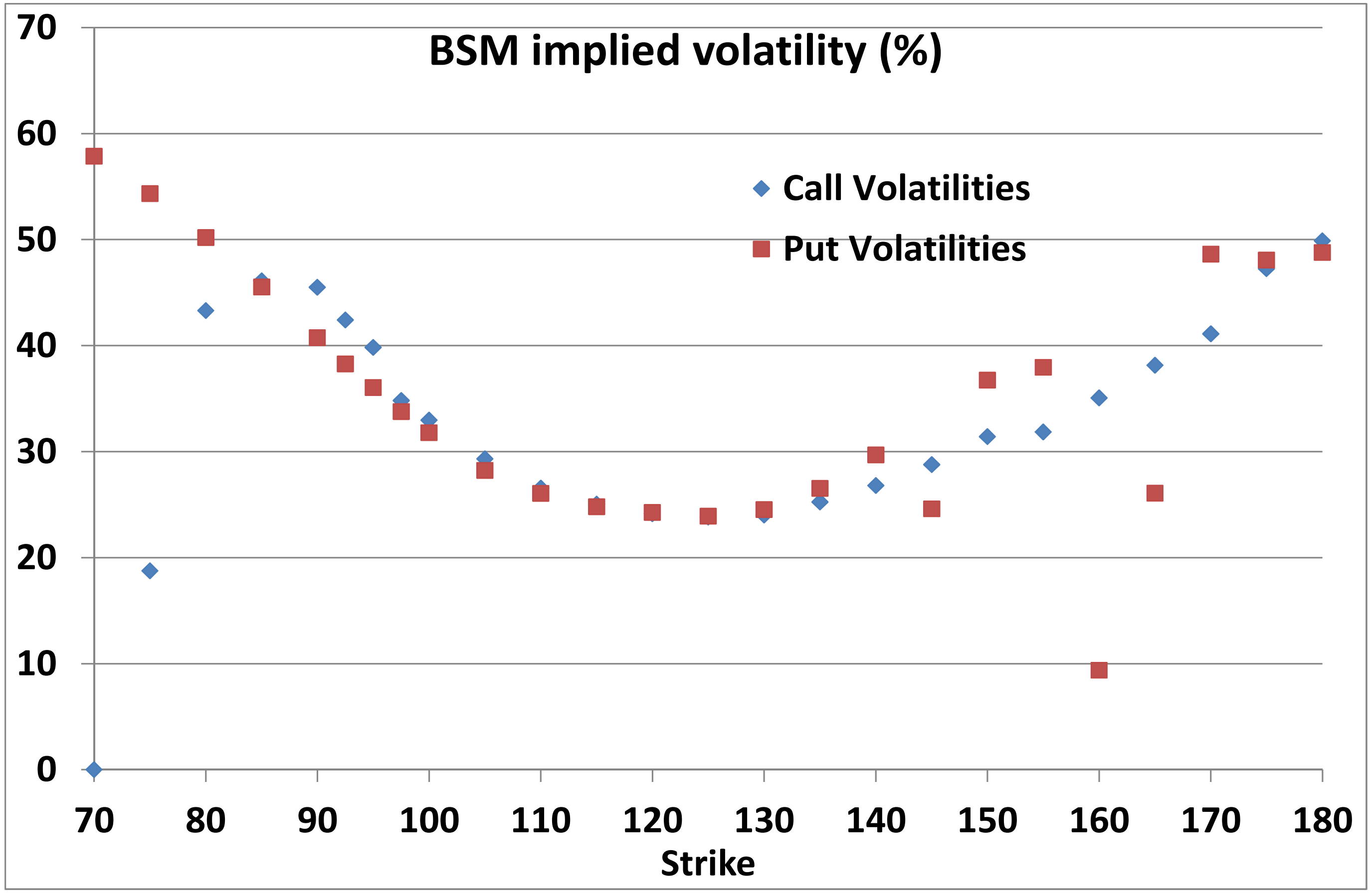
AAPL US $ C 117.63 +.65 Q117.63/117.65Q 9×48
On 14 Oct d Vol 35,652,191 O 117.88P H 118.17K L 117.13P Val 4.195B
AAPL US Equity 95) Actions 97) Settings Option Monitor
APPLE INC ↓117.63 .65 .5556% 117.63 / 117.65 Hi 118.17 Lo 117.13 Volm 35652191 HV 23.61
Center 117.63 Strikes 5 Exp 21-Oct-16 Exch US Composite 92) 10/25/16 C | ERN »
Calc Mode As of 17-Oct-2016
81) Center Strike 82) Calls/Puts 83) Calls 84) Puts 85) Term Structure 87) Moneyness

Calls						Strike	Puts					
Ticker	Bid	Ask	Last	IVM	Volm		Ticker	Bid	Ask	Last	IVM	Volm
18-Nov-16 (32d); CSize 100; R .55						25	18-Nov-16 (32d); CSize 100; R .55					
26) AAPL 11/18/16 C70	47.45	47.95	47.91			70.00	76) AAPL 11/18/16 P70		.01y	.01y	57.87	1
27) AAPL 11/18/16 C75	42.45	42.95	39.70	18.75		75.00	77) AAPL 11/18/16 P75		.02y	.01y	54.34	10
28) AAPL 11/18/16 C80	37.45	37.95	37.60	43.31		80.00	78) AAPL 11/18/16 P80	.03y	.04y	.03y	50.20	550
29) AAPL 11/18/16 C85	32.55	32.95	31.60	46.13		85.00	79) AAPL 11/18/16 P85	.05y	.06y	.06y	45.52	528
30) AAPL 11/18/16 C90	27.65	27.85	27.75	45.49	5	90.00	80) AAPL 11/18/16 P90	.07y	.09y	.08y	40.74	112
31) AAPL 11/18/16 C92.	25.15	25.45	25.35	42.42	66	92.50	101) AAPL 11/18/16 P92	.10y	.11y	.11y	38.29	2
32) AAPL 11/18/16 C95	22.70	22.90	22.80	39.82	2	95.00	102) AAPL 11/18/16 P95	.13y	.14y	.13y	36.05	547
33) AAPL 11/18/16 C97.	20.25	20.40	20.35	34.81	143	97.50	103) AAPL 11/18/16 P97	.17y	.18y	.17y	33.79	267
34) AAPL 11/18/16 C10	17.80	17.95	17.99	32.98	139	100.00	104) AAPL 11/18/16 P10	.22y	.24y	.23y	31.79	723
35) AAPL 11/18/16 C10	12.95	13.10	13.15	29.32	524	105.00	105) AAPL 11/18/16 P10	.47y	.49y	.47y	28.23	1540
36) AAPL 11/18/16 C11	8.50y	8.65y	8.58y	26.56	2046	110.00	106) AAPL 11/18/16 P11	1.12y	1.15y	1.12y	26.06	1231
37) AAPL 11/18/16 C11	4.85y	4.95y	4.90y	25.06	12199	115.00	107) AAPL 11/18/16 P11	2.55y	2.59y	2.58y	24.79	2349
38) AAPL 11/18/16 C12	2.34y	2.38y	2.36y	24.15	7267	120.00	108) AAPL 11/18/16 P12	5.05y	5.20y	5.10y	24.27	643
39) AAPL 11/18/16 C12	.94y	.97y	.97y	23.83	3930	125.00	109) AAPL 11/18/16 P12	8.70y	8.85y	8.75y	23.93	600
40) AAPL 11/18/16 C13	.35y	.36y	.35y	24.02	4060	130.00	110) AAPL 11/18/16 P13	13.10y	13.25y	13.15y	24.52	79
41) AAPL 11/18/16 C13	.13y	.15y	.15y	25.24	708	135.00	111) AAPL 11/18/16 P13	17.85y	18.05y	18.05y	26.52	28
42) AAPL 11/18/16 C14	.06y	.07y	.07y	26.81	298	140.00	112) AAPL 11/18/16 P14	22.80y	22.95y	22.90y	29.68	668
43) AAPL 11/18/16 C14	.03y	.04y	.06y	28.77		145.00	113) AAPL 11/18/16 P14	27.70y	27.95y	27.83y	24.59	482
44) AAPL 11/18/16 C15	.02y	.03y	.03y	31.43	50	150.00	114) AAPL 11/18/16 P15	32.60y	33.00y	33.57y	36.75	

BSM implied volatility

Figure: Implied BSM volatility as function of strike

Implied volatilities for out-of-the-money puts and calls are expensive



Arbitrage-free implied volatility function is a key input for computing risks and calibration of more advanced models

Key challenges:

- Data is discrete across strikes and maturities
- Bid-Ask spreads are wide for out-of-the-money options

Typical Approaches:

- Parametric form (SVI, SABR)
- Non-parametric (splines)

Non-parametric local volatility model linkes implied volatility into implied distributions

Breeden-Litzenberger (1978) formula relates market prices $C^{(\text{market})}$ (implied volatilities) into implied terminal distribution under the valuation measure:

$$\mathbb{P}\left[S(T)=K\right]=\partial_{KK}C^{(\text{market})}(T,K)$$

where T is the maturity time and K is the strike

Local volatility model specifies a function $\sigma_{(\text{loc,dif})}(t,S)$ so that price dynamics are consistent with the implied terminal distribution above:

$$dS(t)=\sigma_{(\text{loc,dif})}(t,S(t))S(t)dW(t)$$

Local volatility is computed using Dupire formula (1994):

$$\sigma^2_{(\text{loc,dif})}(T,K)=\frac{C_T(T,K)}{\frac{1}{2}K^2C_{KK}(T,K)}$$

A continuum of options market prices is calibrated perfectly

Problem: options market data are discrete

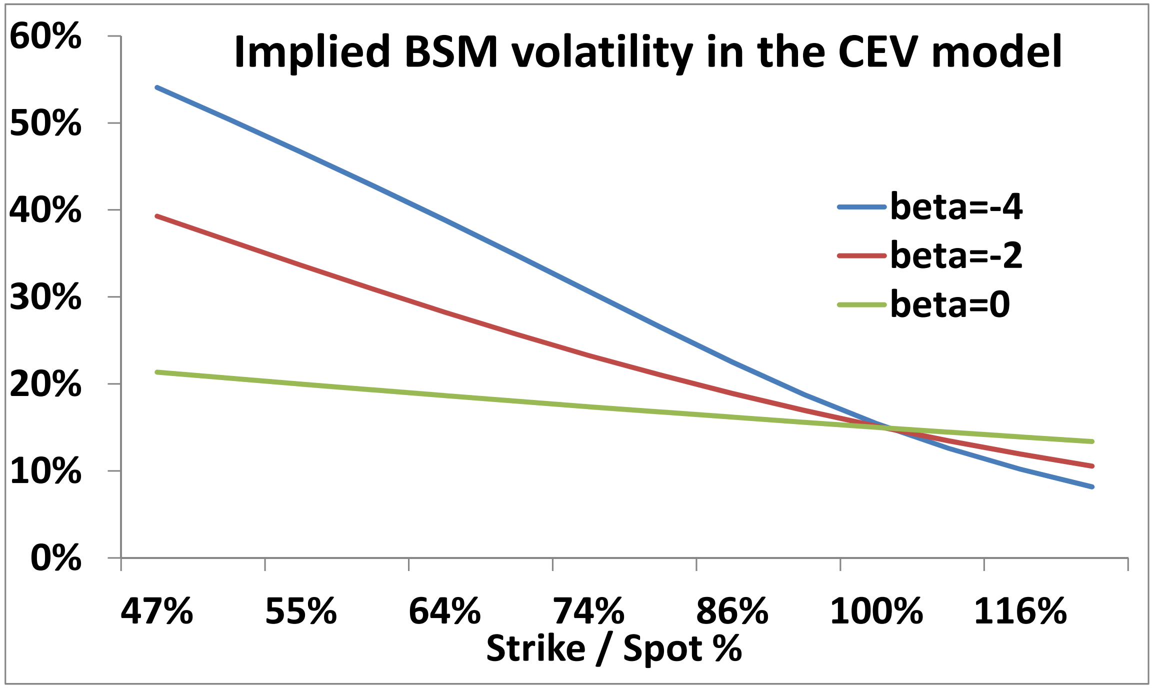
Parametric local volatility models specify parametric functions

This models can be applied to calibrate a small number of market quotes at one maturity (they have a too small number of parameters to fit the whole implied volatility surface)

The classic example is the CEV process (Cox (1975)):

$$dS(t) = \sigma \left(\frac{S(t)}{S(0)} \right)^{\beta} dW(t)$$

Parameter β is the leverage coefficient that allows to calibrate the implied volatility skew



Lipton-Sepp (2011) local volatility model has as many parameters as market quotes

Given a discrete set of market prices $C_{mrkt}\left(T_i, K_j\right)$, $0 \leq i \leq I$, $0 \leq j \leq J_i$

Introduce a tiled local volatility $\sigma_{loc}(T, K)$:

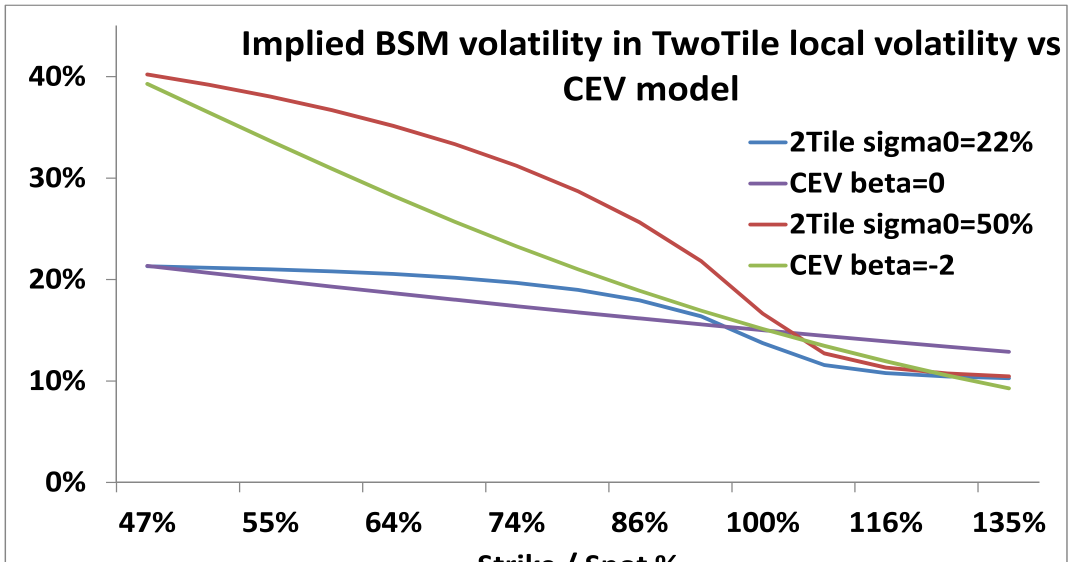
$$\sigma_{loc}(T, S) = \sigma_{ij}, \quad T_{i-1} < T \leq T_i, \quad K_{j-1} < S \leq K_j, \quad 1 \leq i \leq I, \quad 0 \leq j \leq J_i$$

By construction, for every T_i, $\sigma_{loc}(T_i, K)$ depends on as many parameters as there are market quotes

Semi-analytic model using Laplace transform and recursive solution to Sturm-Liouville problem with least-square calibration

Illustration (vs CEV model) using the two-tiled case:

$$\sigma(S) = \begin{cases} \sigma_0, & S \leq S_0, \\ \sigma_1 & S > S_0. \end{cases}$$

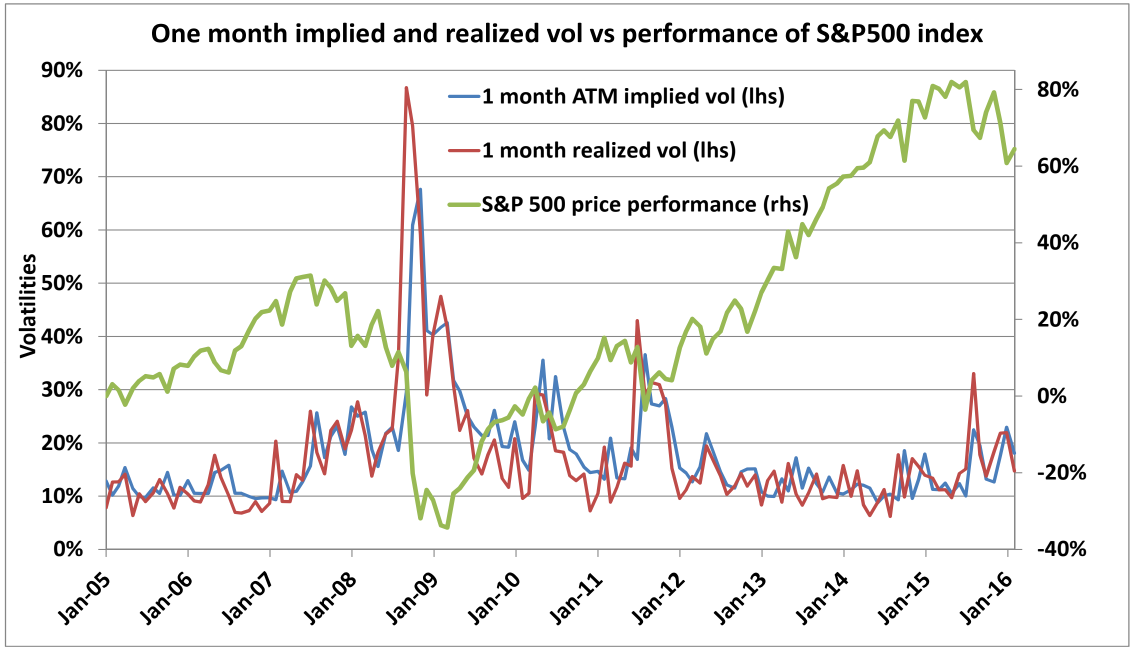


Local Volatility Models

Local volatility models are the most widely used by dealers as interpolation tools from market prices of vanilla products into implied distributions for pricing structured products

Local volatility model serves only as risk-management tool

These models lack dynamical properties, in particular, the mean-reverting features of implied volatilities



Stochastic Volatility Models

Intoruce diffusive uncertainty for the log-price $S(t)$ and variance $V(t)$ dynamics with correlated Brownian motions $W^{(0)}(t)$ and $W^{(1)}(t)$:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sqrt{V(t)}dW^{(0)}(t)$$
$$V(t) = a(V)dt + b(V)dW^{(1)}(t)$$

In practice and literature, the following concepts are studied:

- The instantaneous variance of price returns
$$\mathbb{V}ar[r_t] = V(t)dt\ ,\ r_t = \log(S(t)/S(0))$$

- Quadratic Variance: the integrated instantaneous variance
$$QV(t) = \int_0^t V(t')dt'$$

- Realized Variance: the discrete approximation of the quadratic variance computed over discrete time grid $\{t_k\}$
$$DV(t) = \sum_{t_k \in [0,t]} r_{t_k}^2$$

Stochastic Variance Models

The SDE for the price process $S(t)$ with the stochastic variance $V(t)$:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sqrt{V(t)}dW^{(0)}(t), \;\; S(0) = S$$

Classical analytically tractable models:

- Heston model (1993):

$$dV(t) = \kappa(\theta^2 - V(t))dt + \varepsilon\sqrt{V(t)}dW^{(1)}(t)$$

- 3/2 SV model (Lewis 2002):

$$dV(t) = \kappa V(t)(\theta^2 - V(t))dt + \varepsilon(V(t))^{3/2}dW^{(1)}(t)$$

Stochastic Volatility Models

Price dynamics $S(t)$ with the stochastic volatility process $\sigma(t)$:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW^{(0)}(t)$$

The classic models:

- Stein-Stein model:

  $$d\sigma(t) = \kappa(\theta - \sigma(t))dt + \varepsilon dW^{(1)}(t)$$

  Volatility is normally distributed: not a good feature

- SABR Model (Hagan (2003)):

  $$d\sigma(t) = \varepsilon\sigma(t)dW^{(1)}(t)$$

  The volatility is not mean reverting and explosive: practitioners only use the function for approximation of the model implied volatility

- Log-normal model:

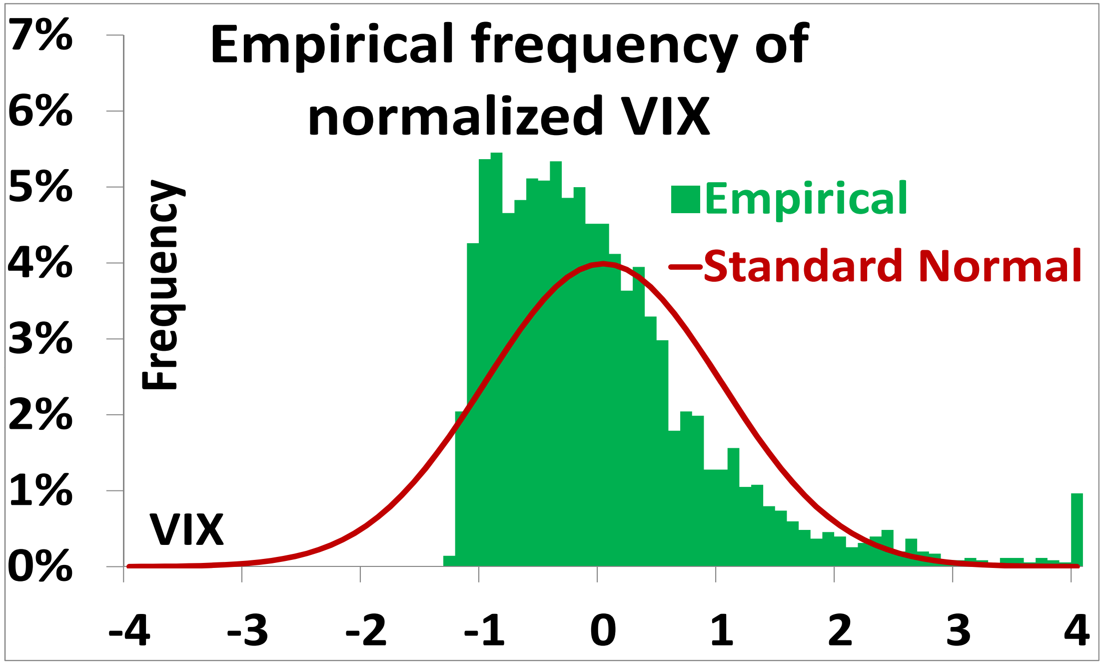
  $$d\sigma(t) = \kappa(\theta - \sigma(t))dt + \varepsilon\sigma(t)dW^{(1)}(t)$$

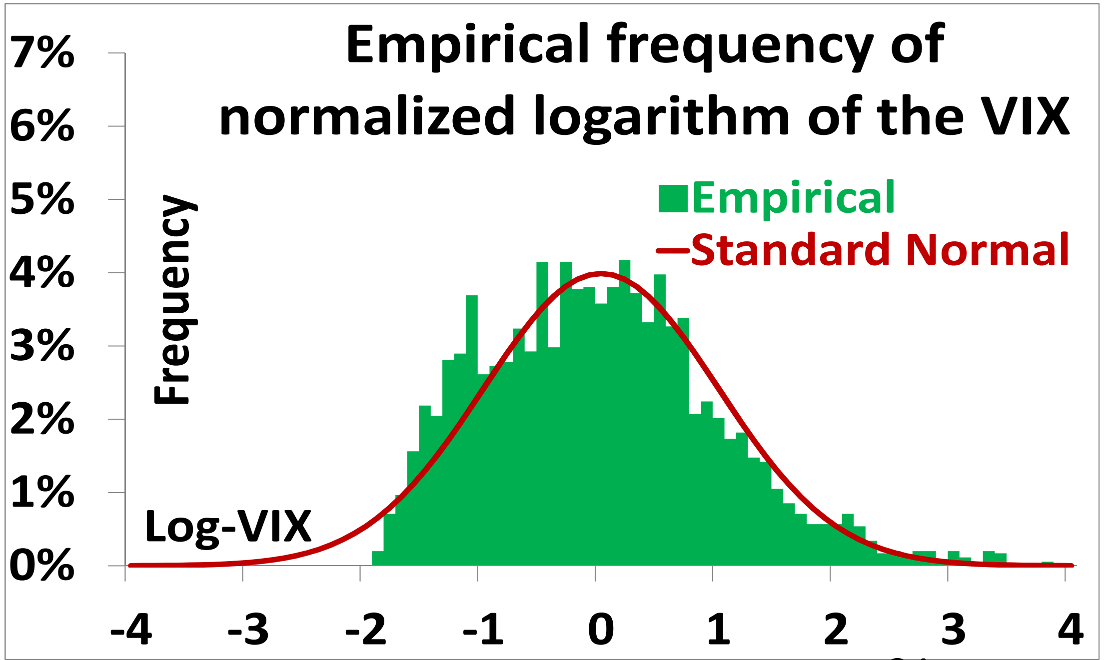
  The model has a very strong empirical evidence (Christoffersen-Jacobs-Mimouni (2010)) but it is not analytic

The distribution of realized volatility is close to log-normal

Left figure: empirical frequency of the VIX for last 20 years: it is definitely not normal

Right figure: frequency of the logarithm of the VIX: it is close to the normal density (especially the right tail)

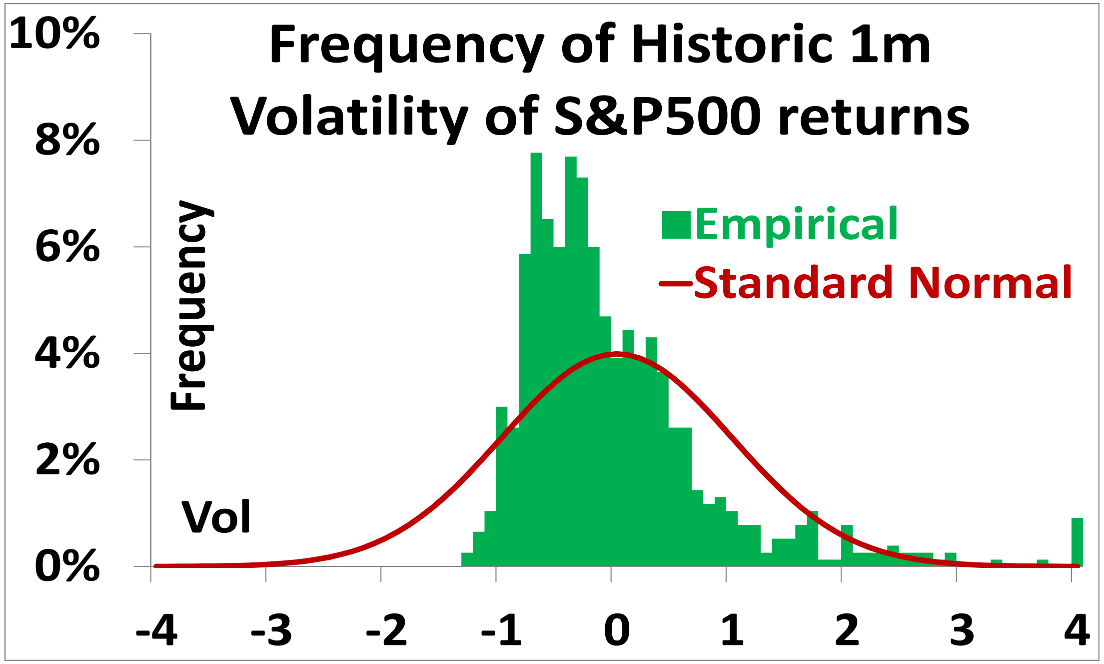


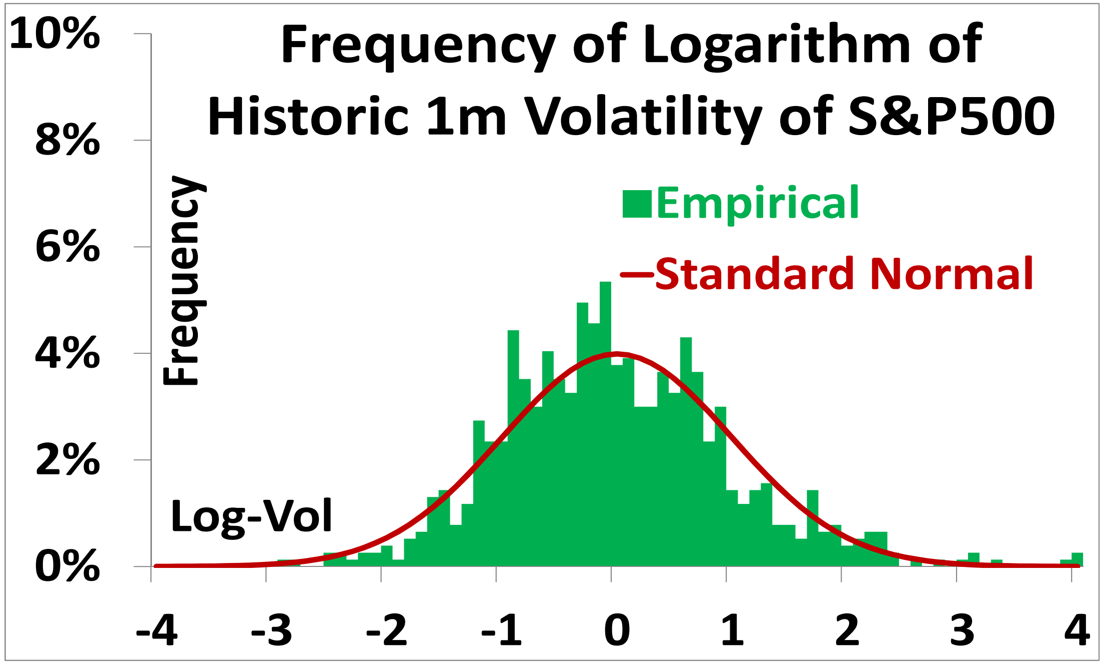


The distribution of implied volatility is also close to log-normal

Left figure: frequency of realized vol - it is definitely not normal

Right figure: frequency of the logarithm of realized vol - again it does look like the normal density (especially for the right tail)





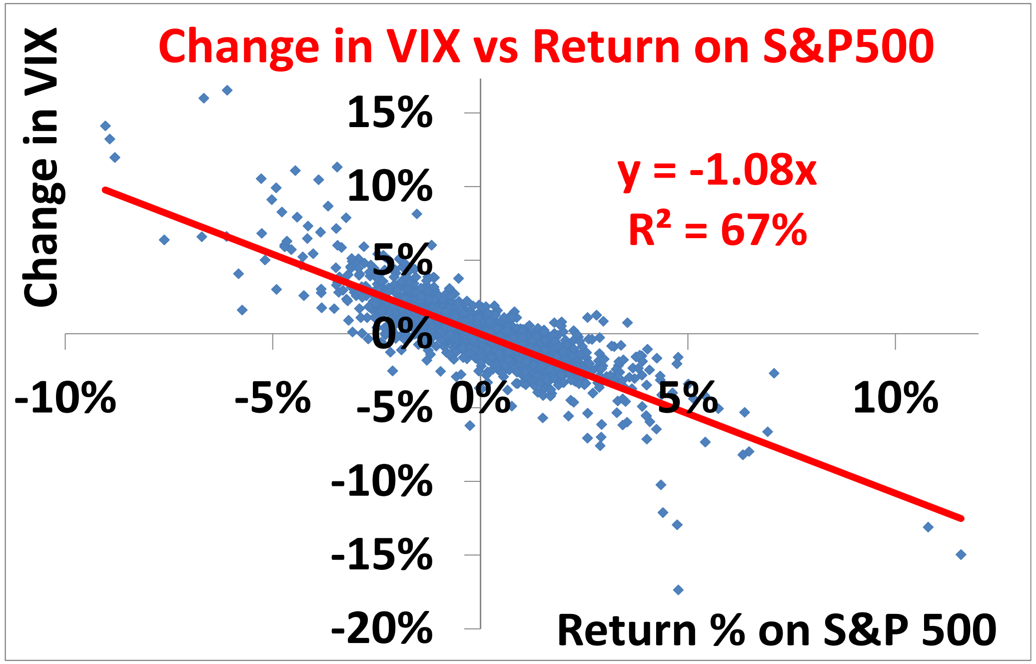
Factor model for changes in volatility (realized or implied) $\sigma(t_n)$ predicted by returns in price $S(t_n)$ is simplier to interprete and estimate:

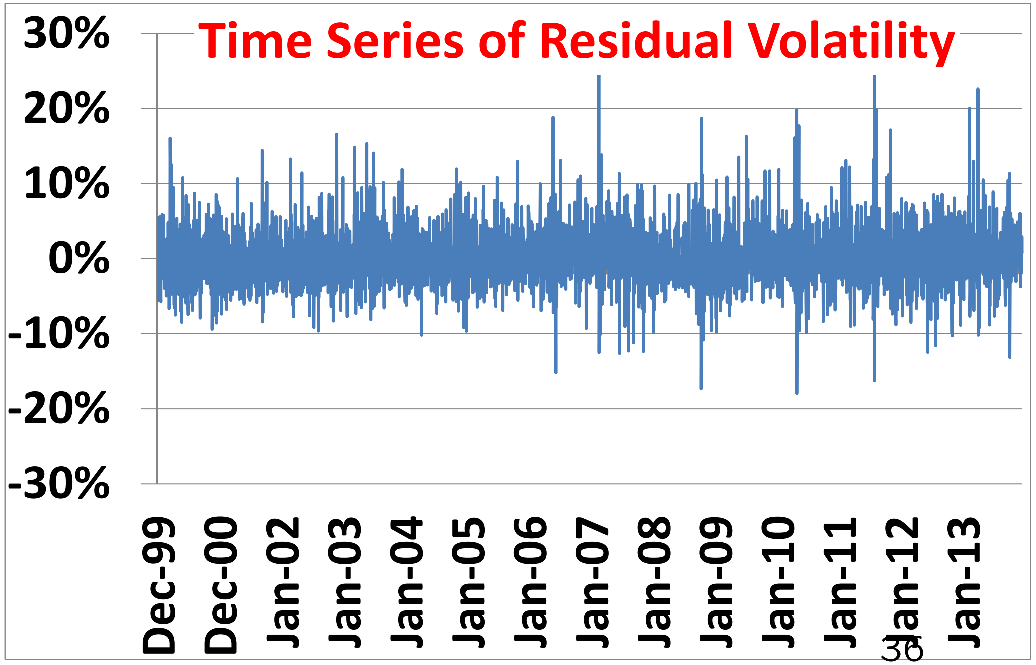
$$\sigma(t_n) - \sigma(t_{n-1}) = \beta \left[\frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})} \right] + \sigma(t_{n-1})\epsilon_n$$

iid normal residuals ϵ_n are scaled by $\sigma(t_{n-1})$ due to log-normality

Left figure: scatter plot of daily changes in the VIX vs returns on S&P 500 for past 14 years: **Volatility beta** $\beta \approx -1.0$ with $R^2 = 80\%$

Right: time series of residuals ϵ_n does not exhibit any systemic patterns



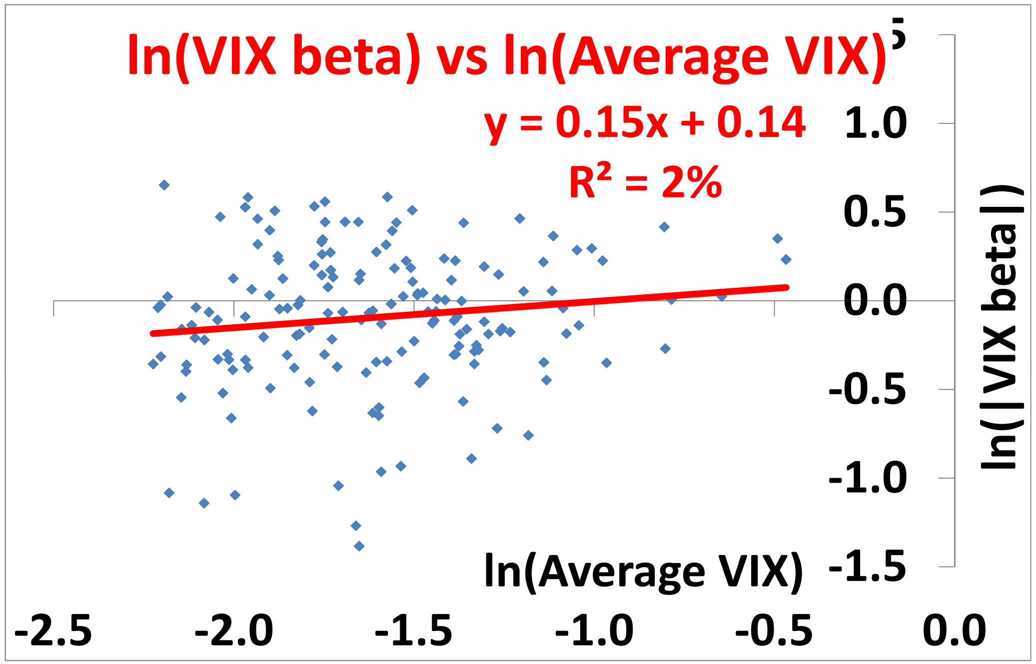


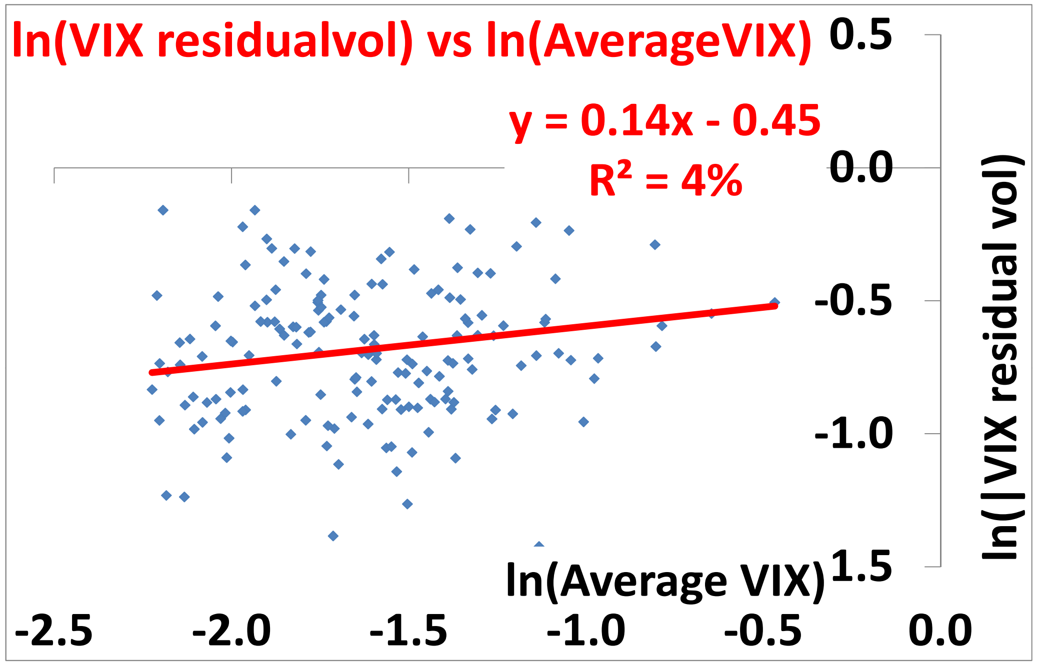
More evidence on log-normal dynamics of volatility using high frequency data: independence of regression parameters on level of ATM volatility

Left figure: test $\widehat{\beta}(V) = \beta V^{\alpha}$ by regression model: $\ln \left|\widehat{\beta}(V)\right| = \alpha \ln V + c$

Right: test $\widehat{\varepsilon}(V) = \varepsilon V^{1+\alpha}$ by regression model: $\ln |\widehat{\varepsilon}(V)| = (1+\alpha) \ln V + c$

The estimated value of elasticity α is small and statistically insignificant





Beta stochastic volatility model (Karasinski-Sepp 2012):

$$
\begin{aligned}
dS(t) &= \sigma(t)S(t)dW^{(0)}(t) \\
d\sigma(t) &= \kappa(\theta - \sigma(t))dt + \beta\frac{dS(t)}{S(t)} + \varepsilon\sigma(t)dW^{(1)}(t) \\
&= \kappa(\theta - \sigma(t))dt + \beta\sigma(t)dW^{(0)}(t) + \varepsilon\sigma(t)dW^{(1)}(t)
\end{aligned}
$$

$\sigma(t)$ is either returns volatility or short-term ATM implied volatility

$W^{(0)}(t)$ and $W^{(1)}(t)$ are independent Brownian motions

β is volatility beta - sensitivity of volatility to changes in price

ε is residual volatility-of-volatility - standard deviation of residual changes in vol

Mean-reversion rate κ and volatility mean rate θ are incorporated for the mean-reverting feature and the stationarity of volatility

Semi-analytic solution of log-normal SV model (Sepp 2015)

Introduce the mean-adjusted volatility:

$$Y(t) = \sigma(t) - \theta \ , \ Y(0) = Y = \sigma(0) - \theta$$

The dynamics for log-price $X(t) = \ln(S(t))$ and quadratic variance $I(t)$:

$$\begin{aligned} dX(t) &= -\frac{1}{2}(Y(t)+\theta)^2\, dt + (Y(t)+\theta)\, dW^{(0)}(t) \\ dY(t) &= -\kappa Y(t)dt + \beta\,(Y(t)+\theta)\, dW^{(0)}(t) + \varepsilon\,(Y(t)+\theta)\, dW^{(1)}(t) \\ dI(t) &= (Y(t)+\theta)^2\, dt \end{aligned}$$

The valuation PDE is given on the domain $[0,T] \times \mathbb{R} \times \mathbb{R}_+ \times (-\theta, \infty)$:

$$\begin{aligned} -U_\tau + \left(\mathcal{L}^{(Y)} + \mathcal{L}^{(XI)}\right) U = 0 \\ U(0, X, I, Y) = u(X, I) \end{aligned}$$

where the diffusive operators $\mathcal{L}^{(Y)}$ and $\mathcal{L}^{(XI)}$ are defined on the domain $[0,T] \times \mathbb{R} \times \mathbb{R}_+ \times (-\theta, \infty)$:

$$\begin{aligned} \mathcal{L}^{(Y)}U &= \frac{1}{2}\vartheta^2 (Y+\theta)^2 U_{YY} - \kappa Y U_Y + \beta (Y+\theta)^2 U_{XY} \\ \mathcal{L}^{(XI)}U &= (Y+\theta)^2 \left[\frac{1}{2}(U_{XX} - U_X) + \beta U_{XY} + U_I\right] \end{aligned}$$

Affine decomposition for log-normal SV model

The moment generation function (MGF) of the log-price $X(\tau)$ and the QV $I(\tau)$ with transform variables $\Phi, \Psi \in \mathbb{C}$:

$$G(\tau, X, I, Y; \Phi, \Psi) = \mathbb{E}[e^{-\Phi X(\tau) - \Psi I(\tau)}]$$

MGF G solves the PDE:

$$-G_\tau + \left(\mathcal{L}^{(Y)} + \mathcal{L}^{(XI)}\right) G = 0,\ G(0, X, I, Y; \Phi, \Psi) = e^{-\Phi X - \Psi I}.$$

Theorem. The MGF function can be decomposed into the leading term $E^{[2]}$ and the remainder term $R^{[2]}$:

$$G(\tau, X, I, Y; \Phi, \Psi) = E^{[2]}(\tau, X, I, Y; \Phi, \Psi) + R^{[2]}(\tau, X, I, Y; \Phi, \Psi),$$

The leading term $E^{[2]}$ is given by the exponential-affine form:

$$E^{[2]}(\tau, X, I, Y; \Phi, \Psi) = \exp\left\{-\Phi X - \Psi I + \sum_{k=0}^{4} A^{(k)}(\tau; \Phi, \Psi) Y^k\right\},$$

where the functions $A^{(k)}$ solve the system of ODEs as functions of τ

The remainder term $R^{[2]}(\tau, X, I, Y; \Phi, \Psi)$ solves the following problem:

$$-R^{[2]}_\tau + \left(\mathcal{L}^{(Y)} + \mathcal{L}^{(XI)}\right) R^{[2]} = -F^{[2]}(Y, A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}) E^{[2]}(\tau, X, I, Y; \Phi,$$

$$R^{[2]}(0, X, I, Y; \Phi, \Psi) = 0,$$

where the source term $F^{[2]}$ is a polynomial function in Y

Second-order Affine Decomposition

Corollary. The second-order approximation for the MGF G is obtained by the leading affine term $E^{[2]}$:

$$G(\tau, X, I, Y; \Phi, \Psi) = E^{[2]}(\tau, X, I, Y; \Phi, \Psi),$$

with accuracy given by the estimate for the remainder term $R^{[2]}$:

$$\left| R^{[2]}(\tau, X, I, Y; \Phi, \Psi) \right| \leq \sum_{n=5}^{8} C_n(\tau; \Phi, \Psi) \times M_\sigma^{(n)},$$

where $M_\sigma^{(n)}$ is the n-th central moment of the steady-state volatility, and $C_n(\tau; \Phi, \Psi)$, $n = 5, 6, 7, 8$, are real-valued constants depending on τ and the transform variables Φ and Ψ

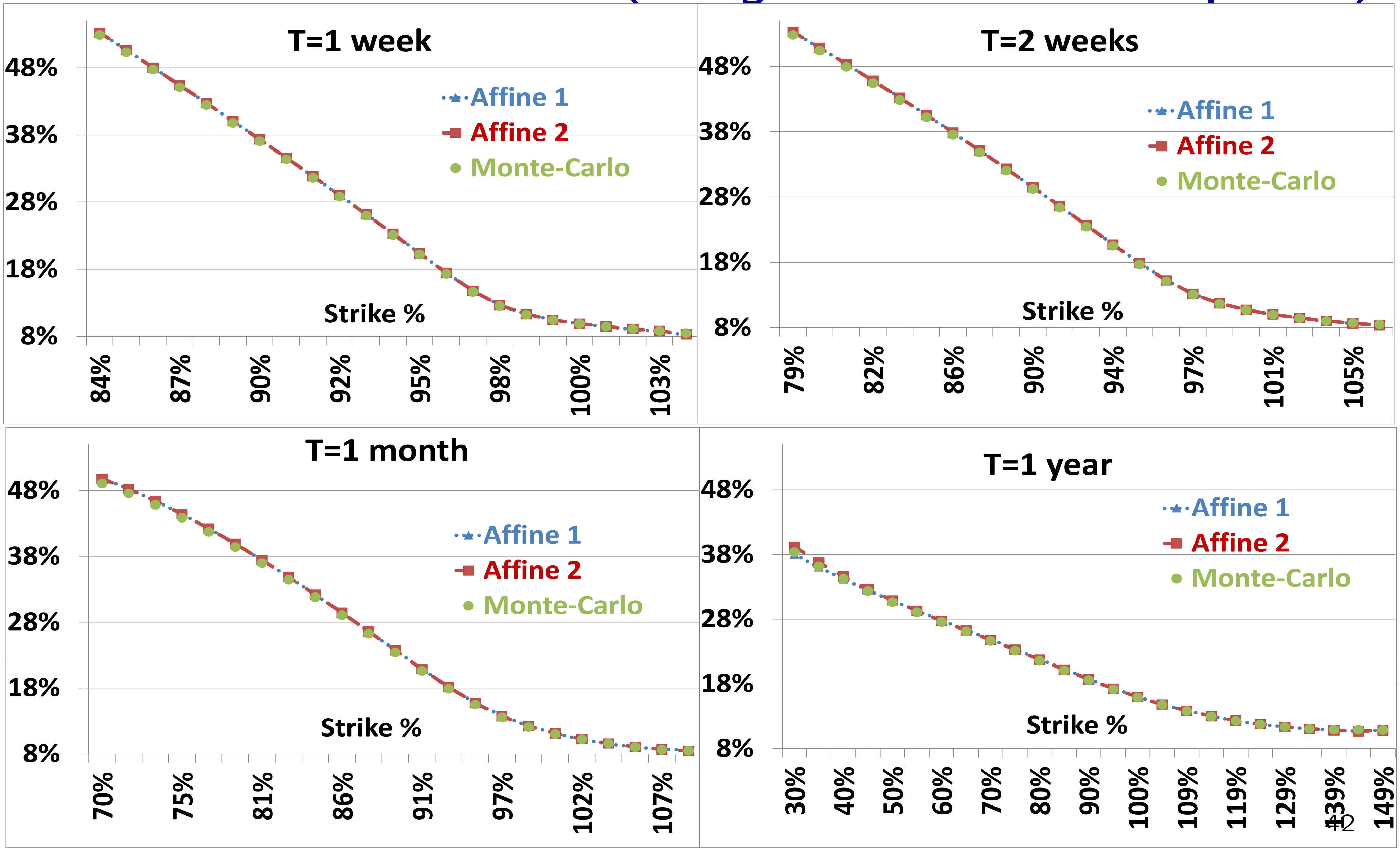
Proposition. There exists a unique and continuous solution for coefficients $A^{(k)}(\cdot, \Phi, \Psi)$, $k = 0, ..., 4$ in the second-order affine decomposition

Proposition. The second-order leading affine term satisfies the martingale condition:

$$E^{[2]}(\tau, X, I, Y; \Phi = 0, \Psi = 0) = 1,$$

Proposition. The second-order leading affine term is consistent with the expected value, variance, and covariance of the log-price and of the QV

Applications for pricing options under the log-normal stochastic volatility model: consistency across different maturities and strikes (using S&P 500 index options)



Equilibrium / Steady-State Analysis of SV models

Different SV models have apparently different dynamics and distributions of returns: how is about their limiting behaviour?

- The steady-state distribution of the volatility σ

- The theoretical distribution of volatility-conditional returns:

$$X \mid \sigma \overset{d}{=} \mathbf{n}\left(0, \sqrt{c\sigma}\right)$$

where c is the scaling factor, $c = 1/252$ for daily returns.

- The empirical distribution of volatility-normalized returns:

$$\widehat{X}_n = \frac{1}{\widehat{\sigma}_{n-1}} \ln\left(S_n/S_{n-1}\right)$$

where $\widehat{\sigma}_{n-1}$ is the empirical stimate of volatility at time t_{n-1}

Empirical distribution of volatility-normalized returns is close to normal distribution

Right: QQ-plot of monthly returns on the S&P 500 index

Left: QQ-plot of monthly returns normalized by the realized historic volatility of daily returns within given month

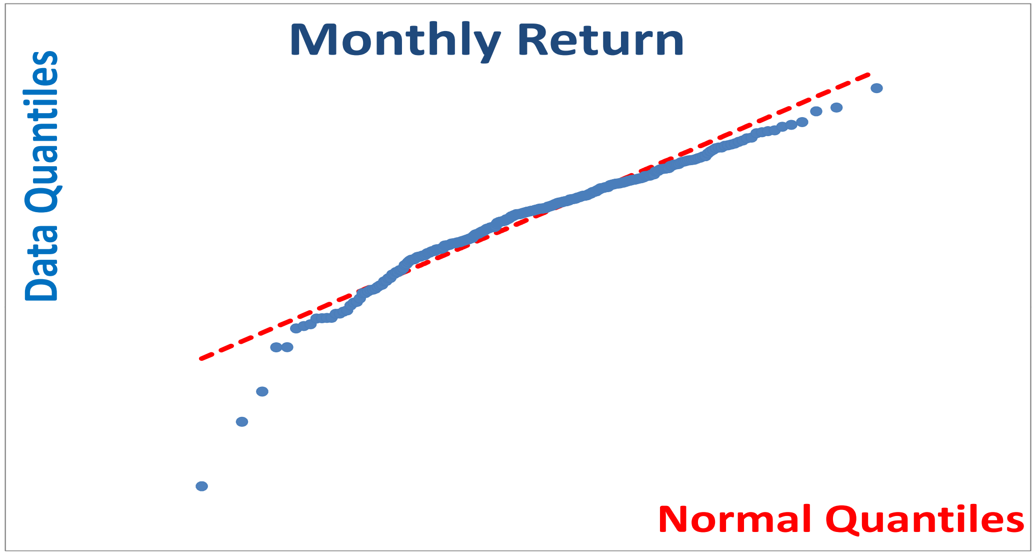
Anderson-Darling test for normality of returns (H0 hypothesis)

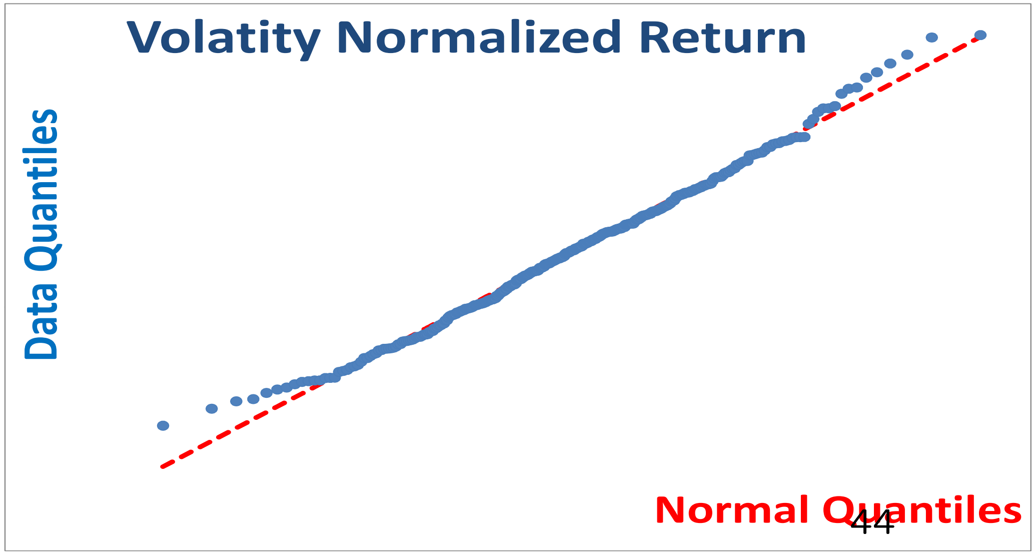
	Returns	Normalized Returns
Reject H0	Yes	No
p-value	0.0005	0.9515

We have strong presumption against normality of returns

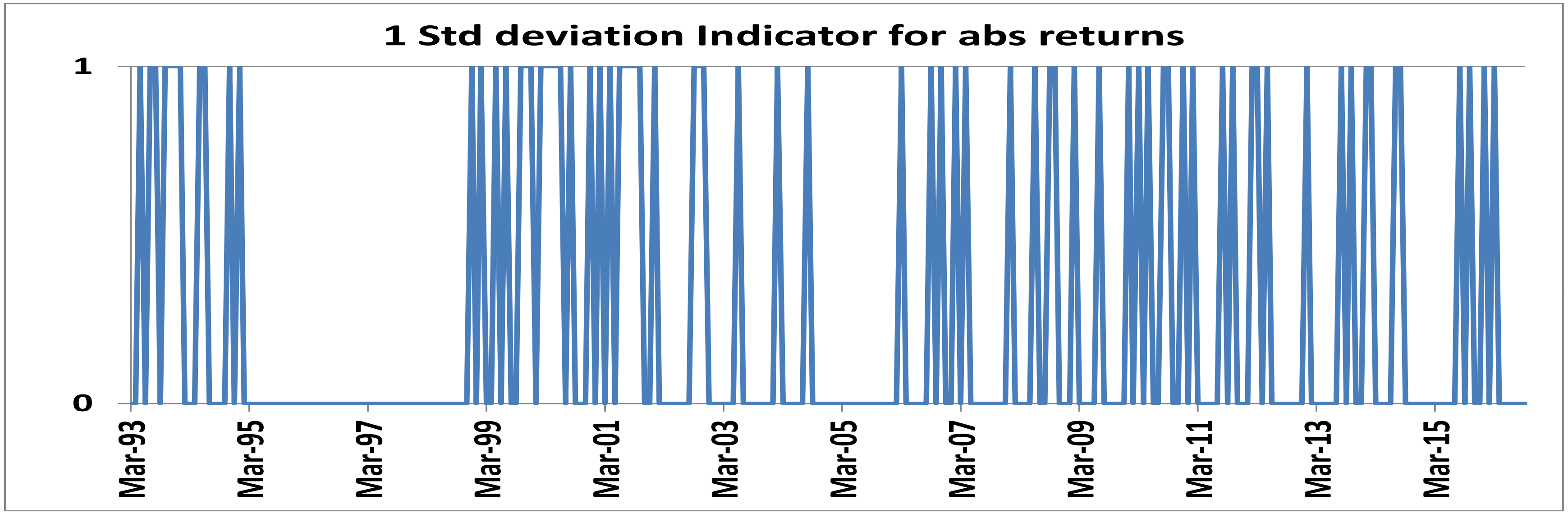
However we cannot reject the hypothesis that volatility-normalized returns

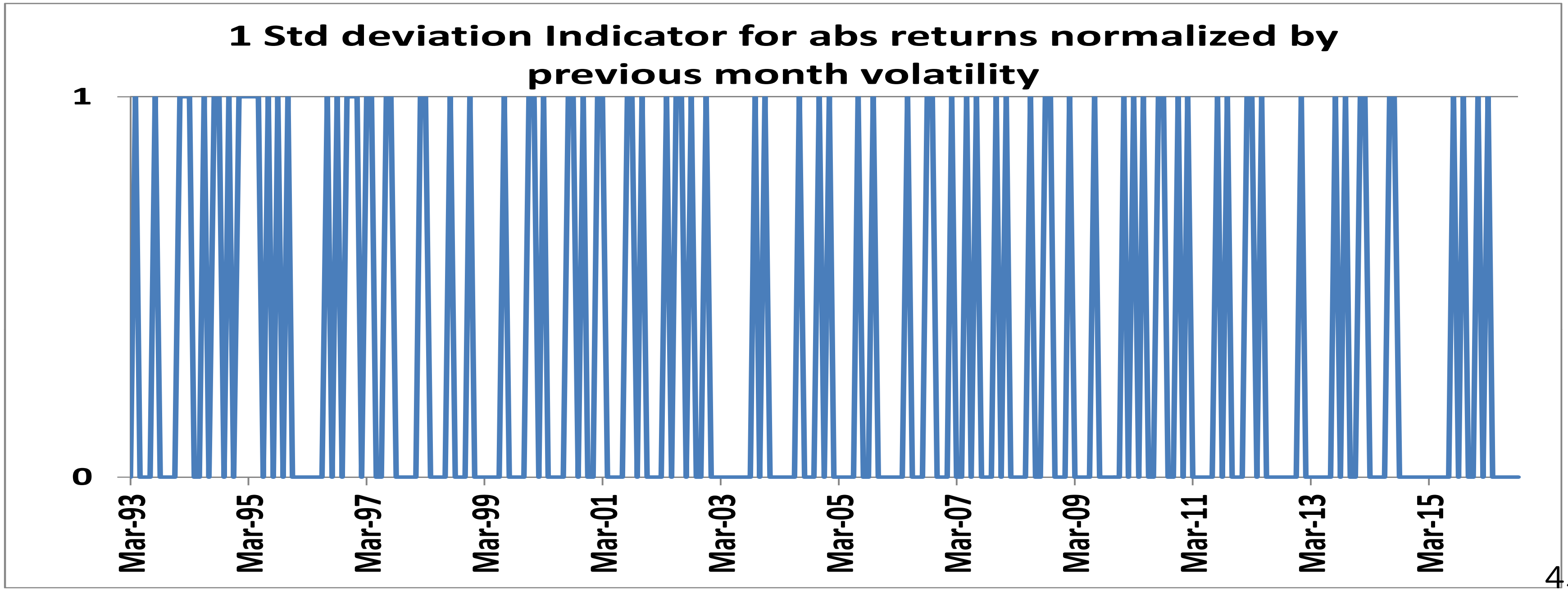
Non-normality of returns can be explained by regimes in the volatility





Hitting times of volatility normalized returns are not clustered unlike those for absolute returns





Steady-state distribution of volatility

The steady-state distribution of the volatility is obtained by letting the time variable to the infinity

For the log-normal model the steady-state distribution solves the ODE:

$$\frac{1}{2}\vartheta^2\left[\sigma^2 G\right]_{\sigma\sigma} - \left[\kappa(\theta - \sigma)G\right]_{\sigma} = 0$$

Steady-state distribution of the volatility

For log-normal model:

$$G^{(LG)}(\sigma) = \frac{\upsilon^{\nu}}{\Gamma(\nu)} \frac{\exp\left\{-\frac{\upsilon}{\sigma}\right\}}{\sigma^{1+\nu}}, \ \nu = 1 + \frac{2\kappa}{\varepsilon^2}, \ \upsilon = \frac{2\kappa\theta^2}{\varepsilon^2}$$

Inverse Gamma distribution with shape $\alpha = \nu$ and scale $\beta = \upsilon$

For 3/2 model:

$$G^{(3/2)}(\sigma) = \frac{\upsilon^{\nu}}{\Gamma(\nu)} \frac{\exp\left\{-\frac{\upsilon}{\sigma}\right\}}{\sigma^{1+\nu}}, \ \nu = 2 + \frac{2\kappa}{\varepsilon^2}, \ \upsilon = \frac{2\kappa\theta^2}{\varepsilon^2}$$

Inverse Gamma distribution with shape $\alpha = \nu$ and scale $\beta = \upsilon$

Structurally the 3/2 model and the log-normal model are similar

For Heston model:

$$G^{(H)}(\sigma) = \frac{\upsilon^{-\nu}}{\Gamma(\nu)} \frac{\exp\left\{-\frac{\sigma}{\upsilon}\right\}}{\sigma^{1-\nu}}, \nu = \left(\frac{2\kappa\theta^2}{\varepsilon^2}\right)^{-1}, \ \upsilon = \frac{2\kappa}{\varepsilon^2}$$

Gamma distribution with shape $\alpha = \nu$ and scale $1/\upsilon$

Illustration of the steady state density of volatility

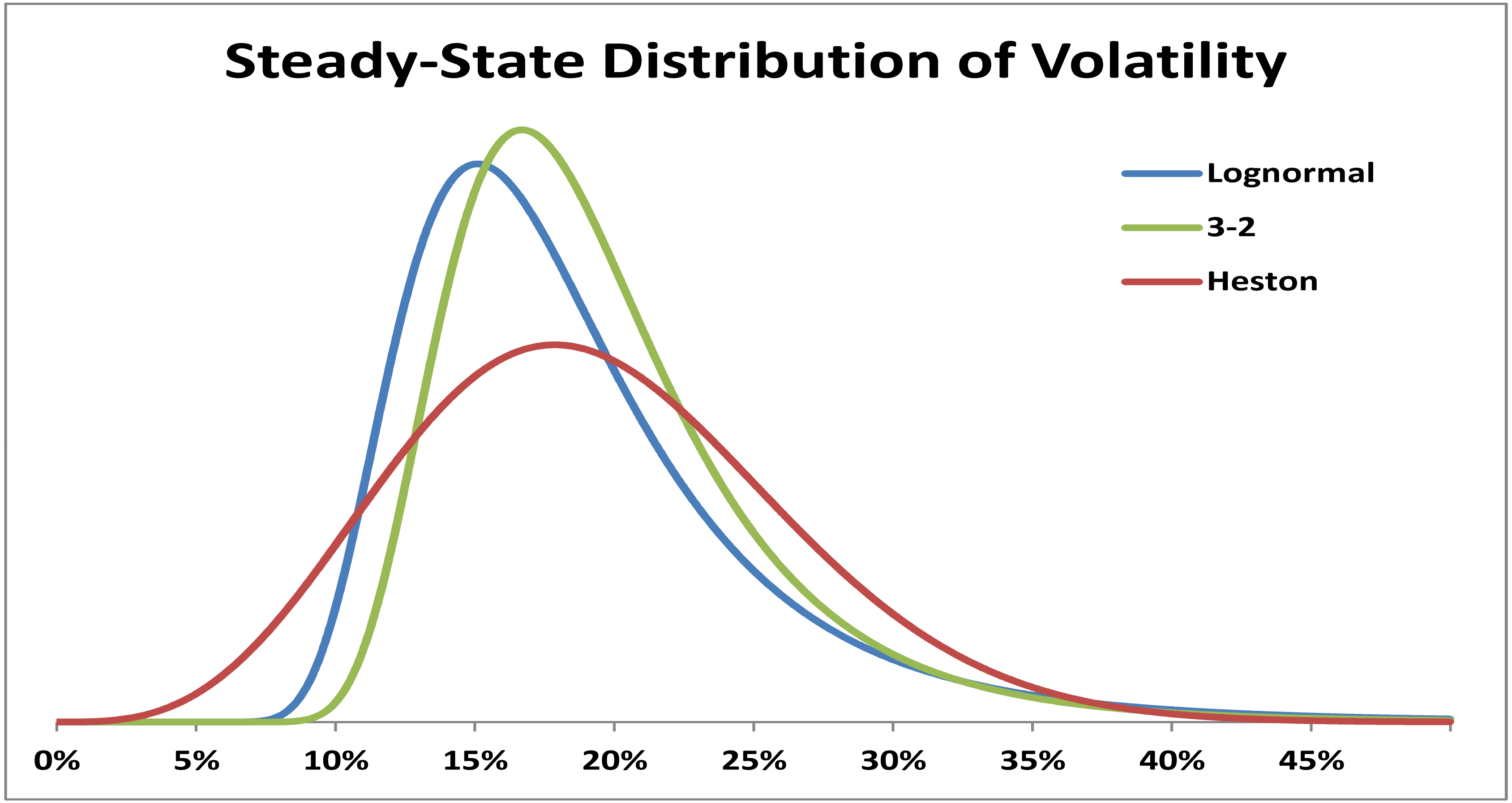
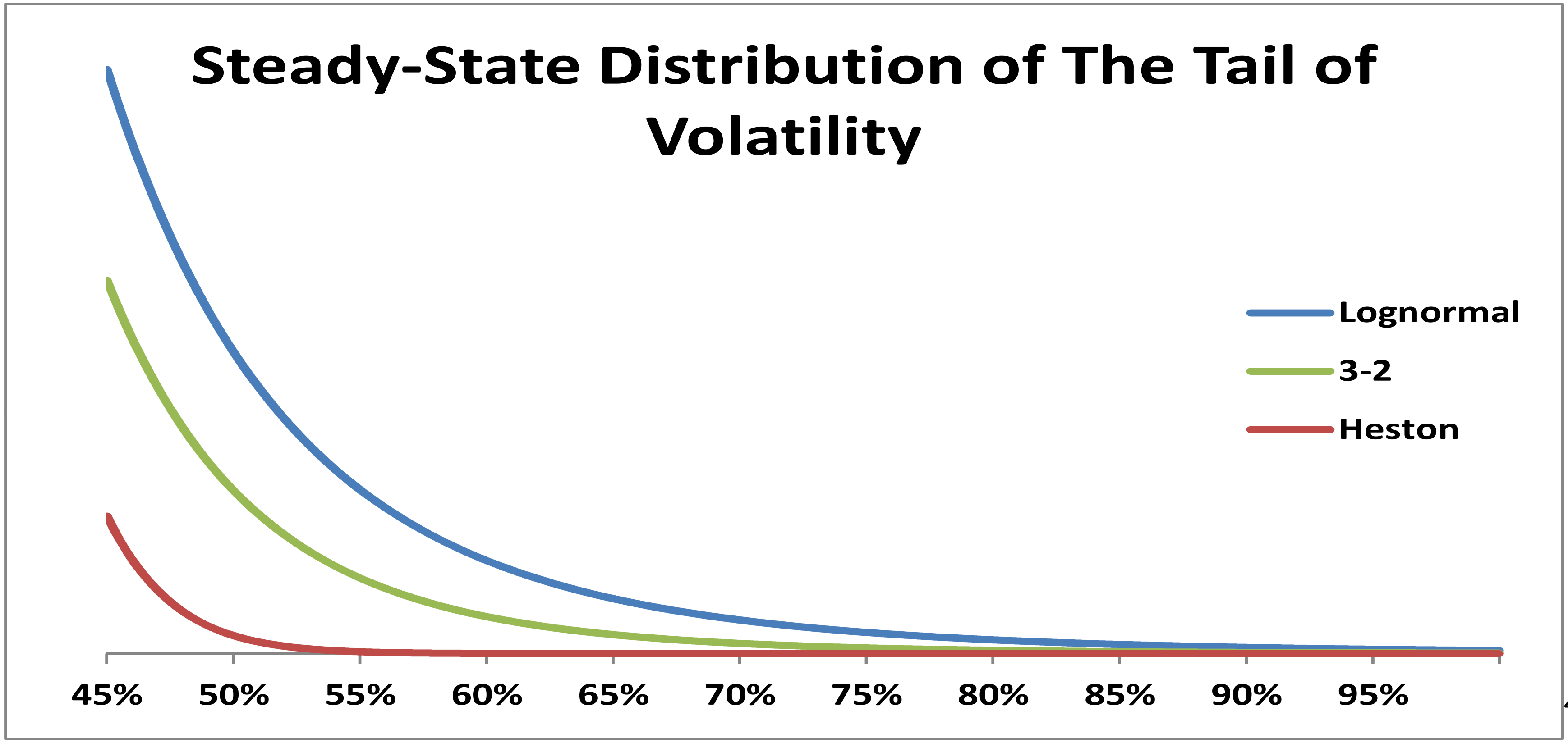


Illustration of the steady state density of volatility: log-normal model implies heavy

For the logarithm of the volatility $L = \log(\sigma)$ under the lognormal model, the distribution is given the extreme-value type PDF:

$$G^{(L)}(L) = \frac{\upsilon^{\nu}}{\Gamma(\nu)} \exp\{-\exp\{-L\}/\upsilon + \nu L\}$$



The Distribution of Conditional Returns

Consider the distribution of returns conditional on the steady-state volatility:

$$X \mid \sigma^\infty \stackrel{d}{=} \mathbf{n}\left(0, \sqrt{c}\sigma^\infty\right)$$

where c is the scaling factor

The unconditional PDF is obtained by the integral:

$$G^{(X)}(X) = \int_0^\infty \frac{1}{\sqrt{2\pi c}\sigma^\infty} \exp\left\{-\frac{1}{2}\frac{X^2}{c(\sigma^\infty)^2}\right\} G^{(\sigma^\infty)}(\sigma^\infty) d\sigma^\infty$$

For the lognormal SV model:

$$G^{(X)}(X) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{2\pi c\upsilon}\,\Gamma\left(\nu\right)} \left(1 + \frac{X^2}{2c\upsilon}\right)^{-\frac{1}{2}(2\nu+1)}$$

This is the Student t-distribution with 2ν degrees of freedom

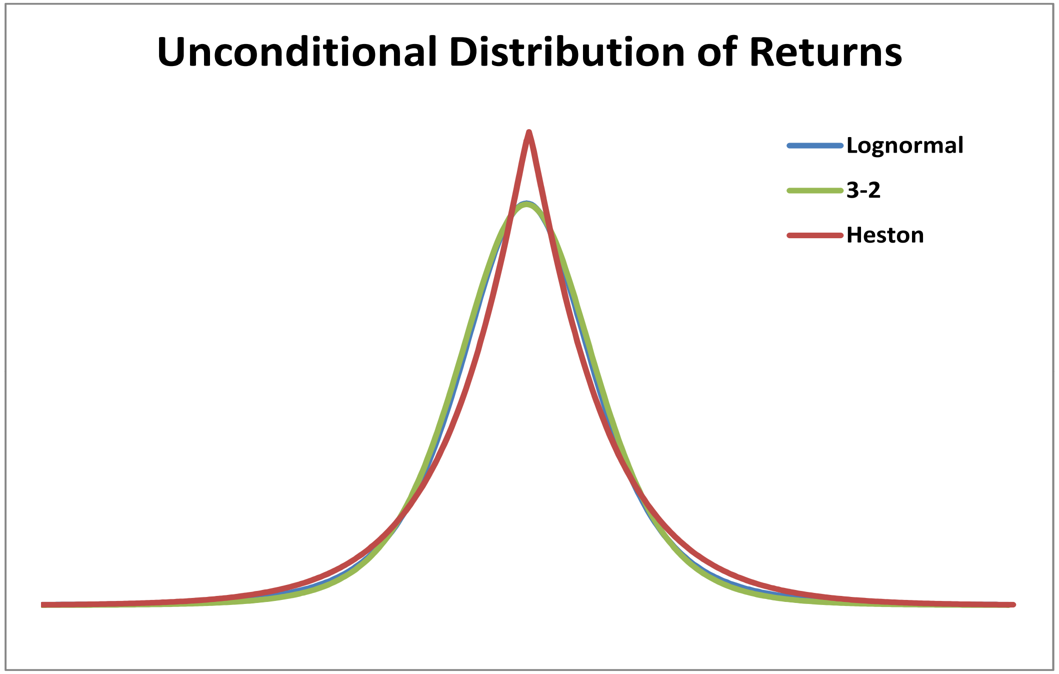
For 3/2 SV model, the Student t-distribution with $2(\nu+1)$ degrees

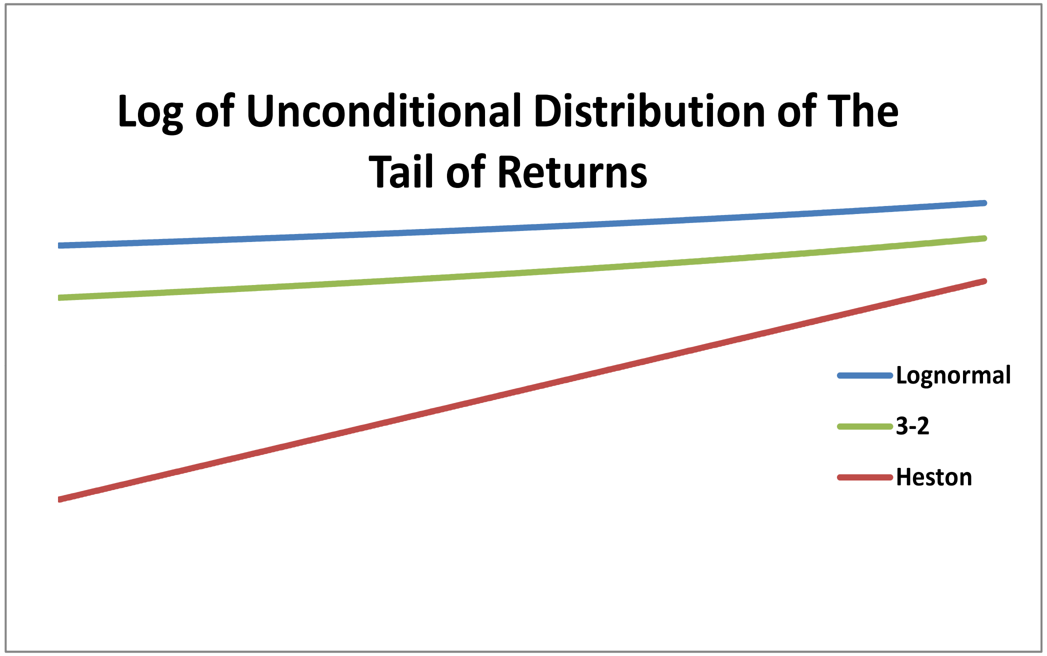
For Heston model:

$$G^{(X)}(X) = \frac{2\left(2c\upsilon\right)^{-\frac{1}{2}\left(\frac{1}{2}+\nu\right)} |X|^{-\frac{1}{2}+\nu}}{\sqrt{\pi}\Gamma(\nu)} \mathbf{K}_{\nu-\frac{1}{2}}\left(2\frac{|X|}{\sqrt{2c\upsilon}}\right)$$

$\mathbf{K}_\nu(x)$ is modified Bessel function of the second kind with index ν

Illustration: the tail of unconditional distribution of returns is heavier under the lognormal SV model



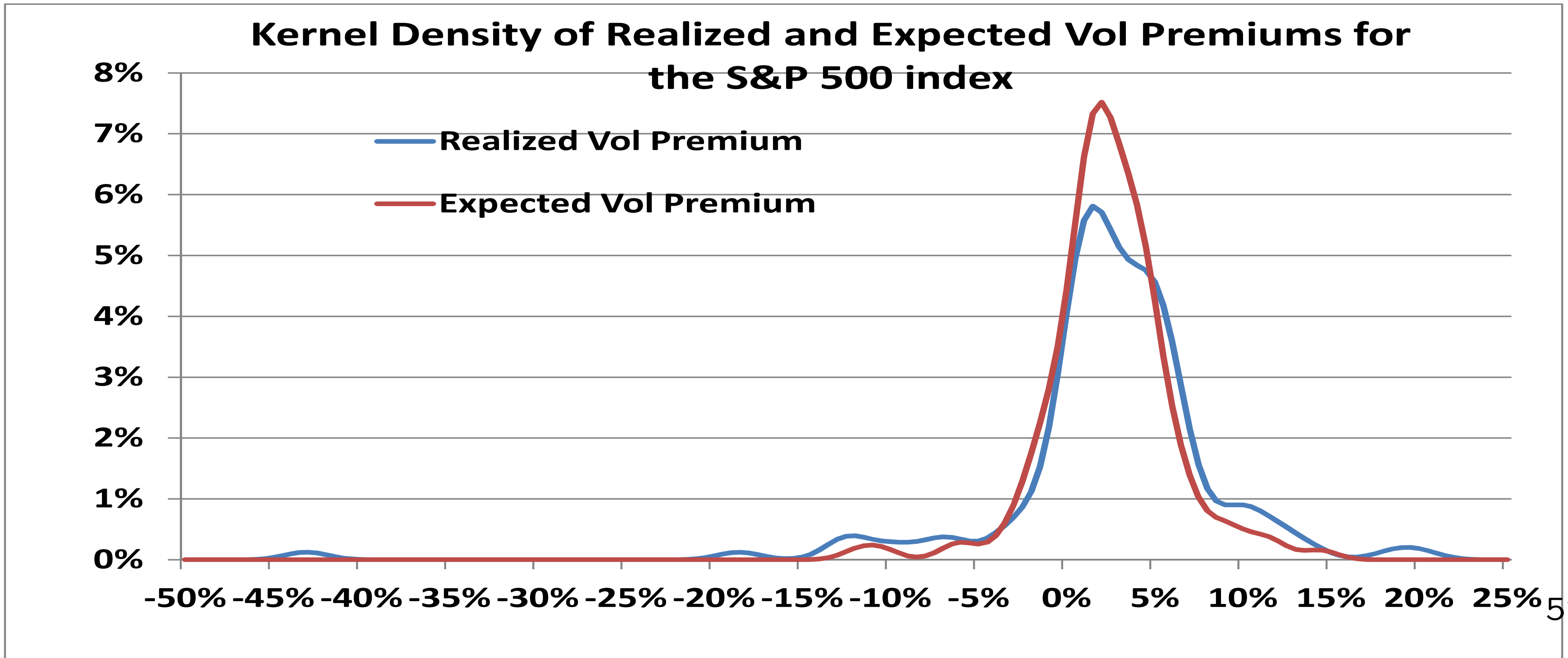


Summary of stochastic volatility models: applications for trading volatility risk-premium

The strength of SV models:

1. Calibration to market options prices for estimation of the implied distributions
2. Forecasting of the expected volatility and its probabilistic range conditional on the current observables
3. Steady-state analysis of systematic volatility trading strategies

Figure: expected vs realized volatility risk-premium computed using the log-normal SV model



Volatility trading in practice: Usage of Options and Structured Products

- Hedging (controversial)

- Investment products with limited downside:

  1. To get a convex pay-off (limited downside with large upside)

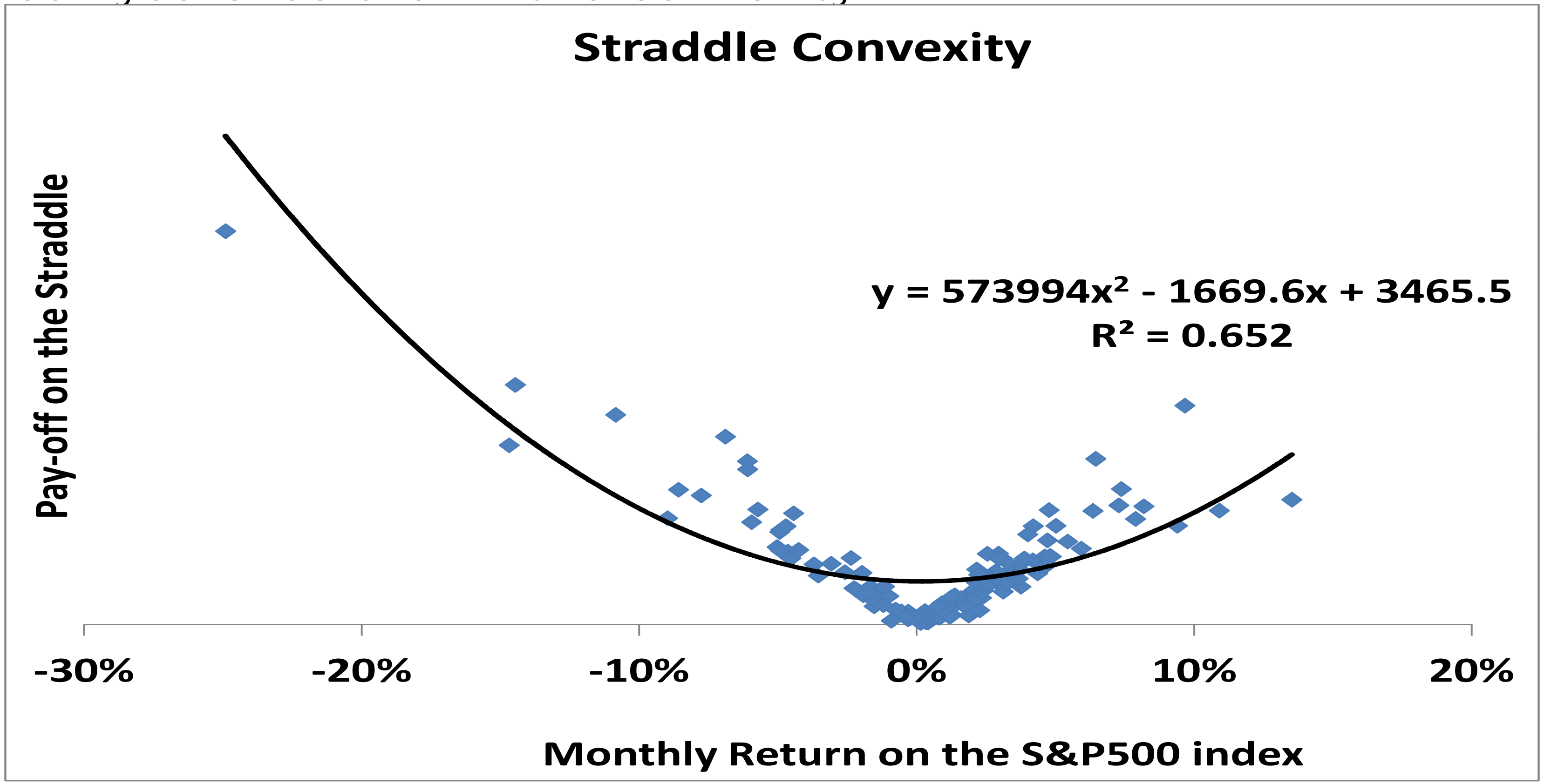
  2. The key is the volatility premium

The convexity profile without costs is appealing

Figure: the realized convexity of the straddle (long at-the-money call and put options) rolled monthly on S&P 500 index from 2005 up to 2016

The convexity of returns is a very attractive profile for any investment strategy

Taleb: you should own the convexity



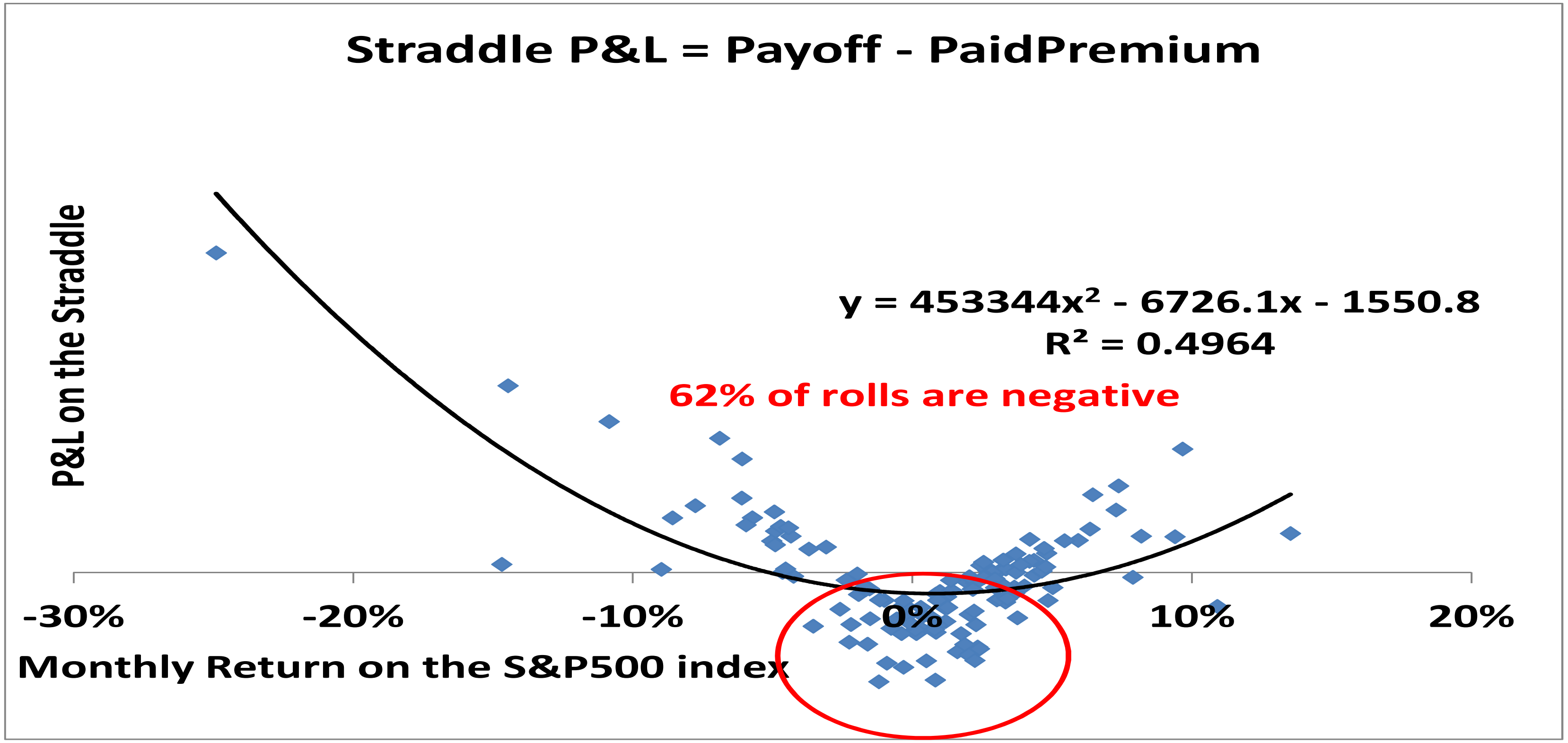
The convexity profile accounting to costs is not appealing

Figure: the convexity of straddle rolled monthly on S&P 500 index from 2005 up to 2016 adjusted to market price of straddle

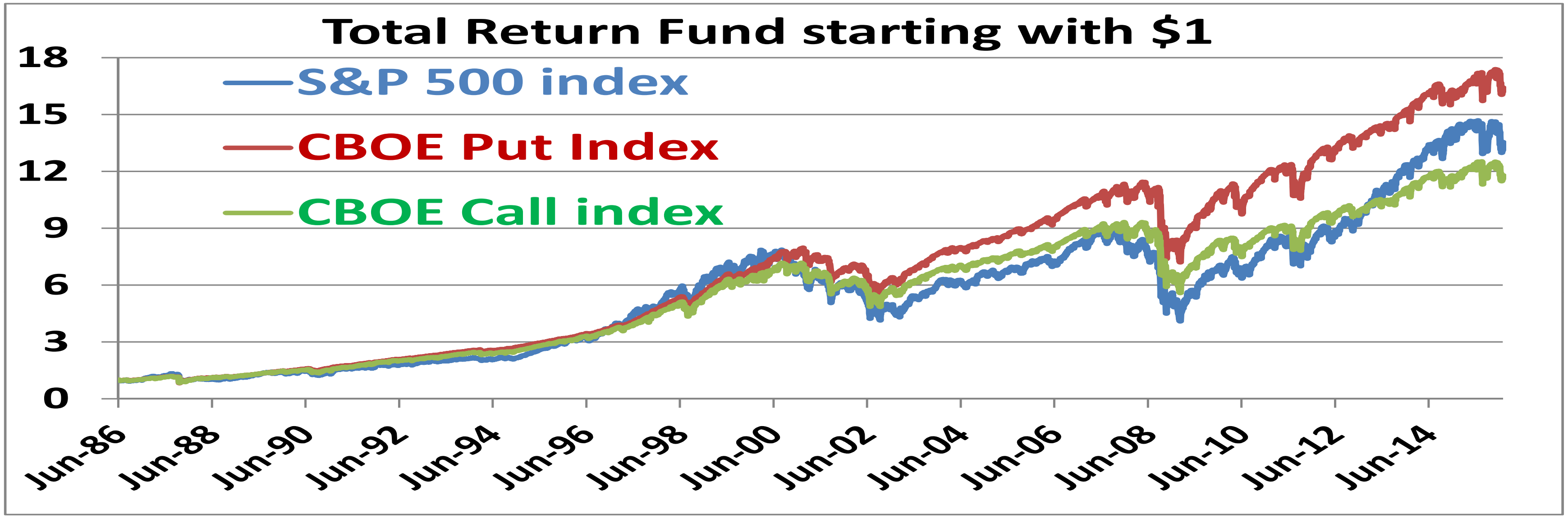
Why Taleb's advice does not work in practice: the convexity is overpriced

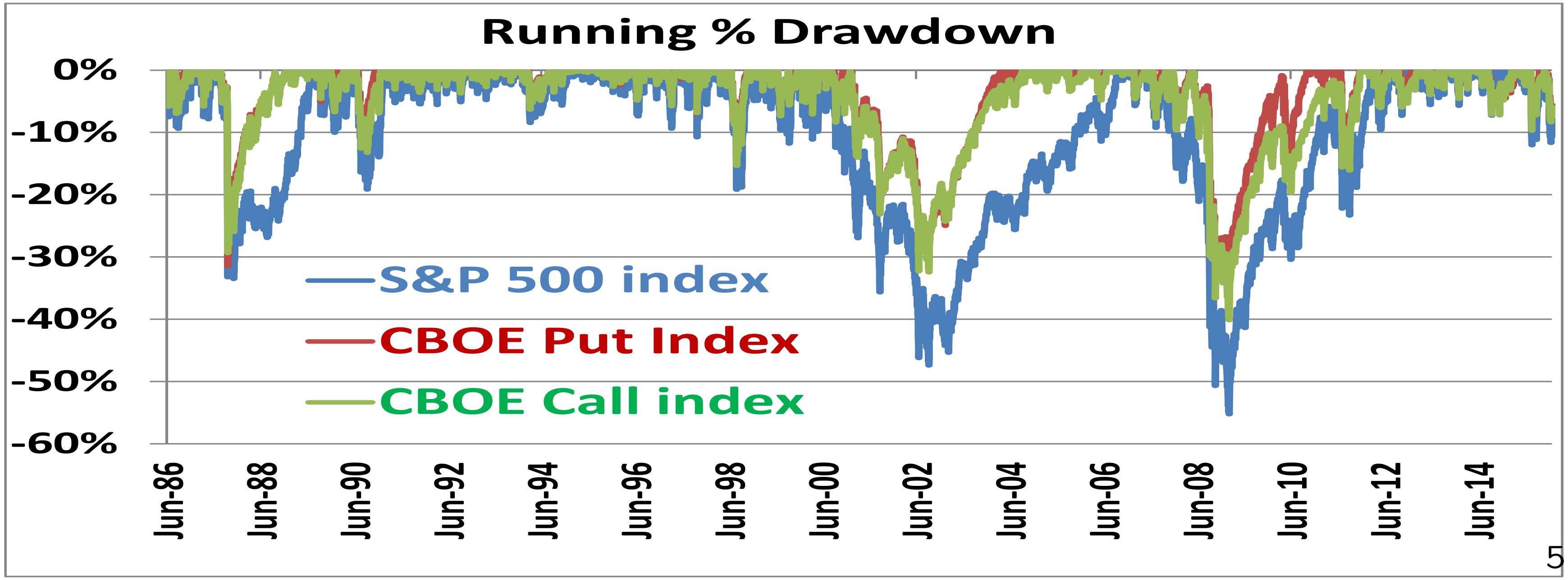
Link to the behavior science (GMO LLC: "What the Beta Puzzle Tells Us about Investing") and preference for "lottery" payoffs

Empirical evidence: the concavity profile (short options) provides higher returns than the convexity profile (long options) in the long term



Empirical evidence: systematic short convexity strategies out-perform the benchmark with smaller risk





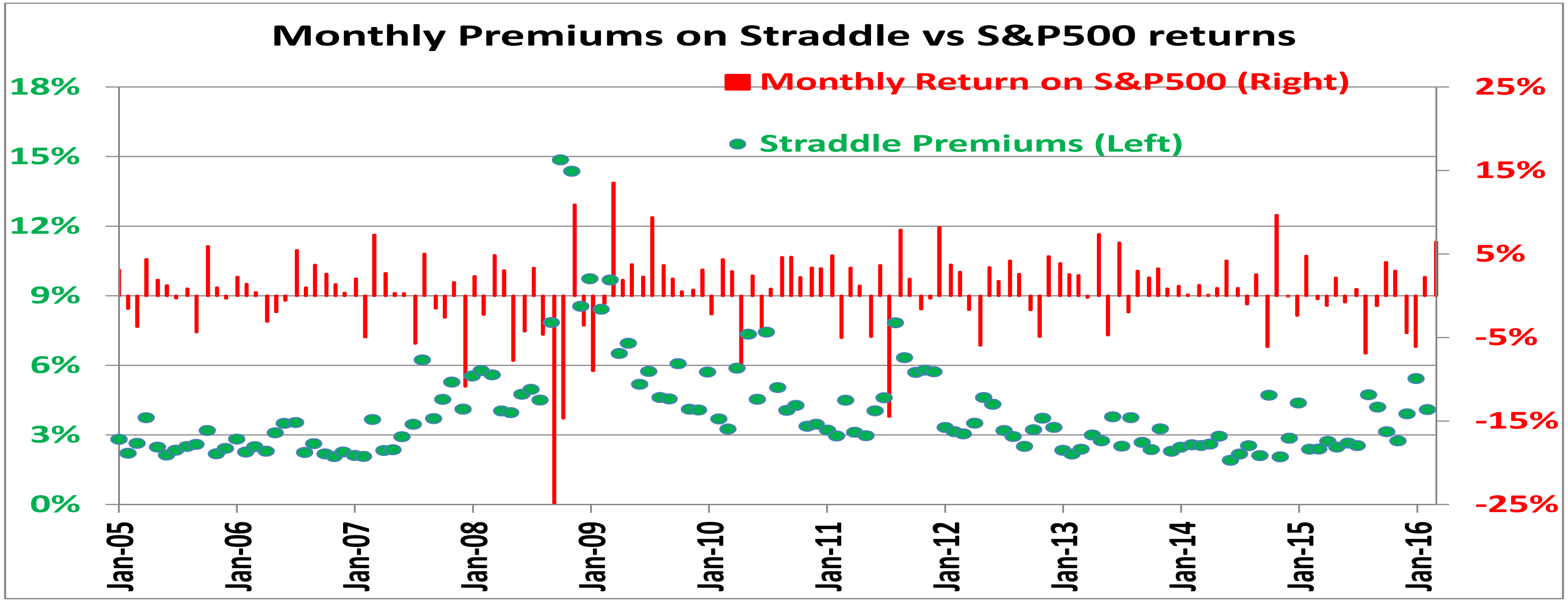
The cyclicality of the volatility risk-premium makes trend-following with options prohibitive

Two major investment approaches:

1. Follow the trend: buy high and sell higher
2. Contrarian: bet on reversions or range bounds

Trend following using options is prohibitive as the volatility risk-premium is cyclical and the option value decays the fastest at a high volatility

Figure: prior month return on the S&P 500 index and option premium for monthly straddles at the third Fridays



Convexity is equivalent to volatility

Given monthly returns, consider two estimators of the volatility:

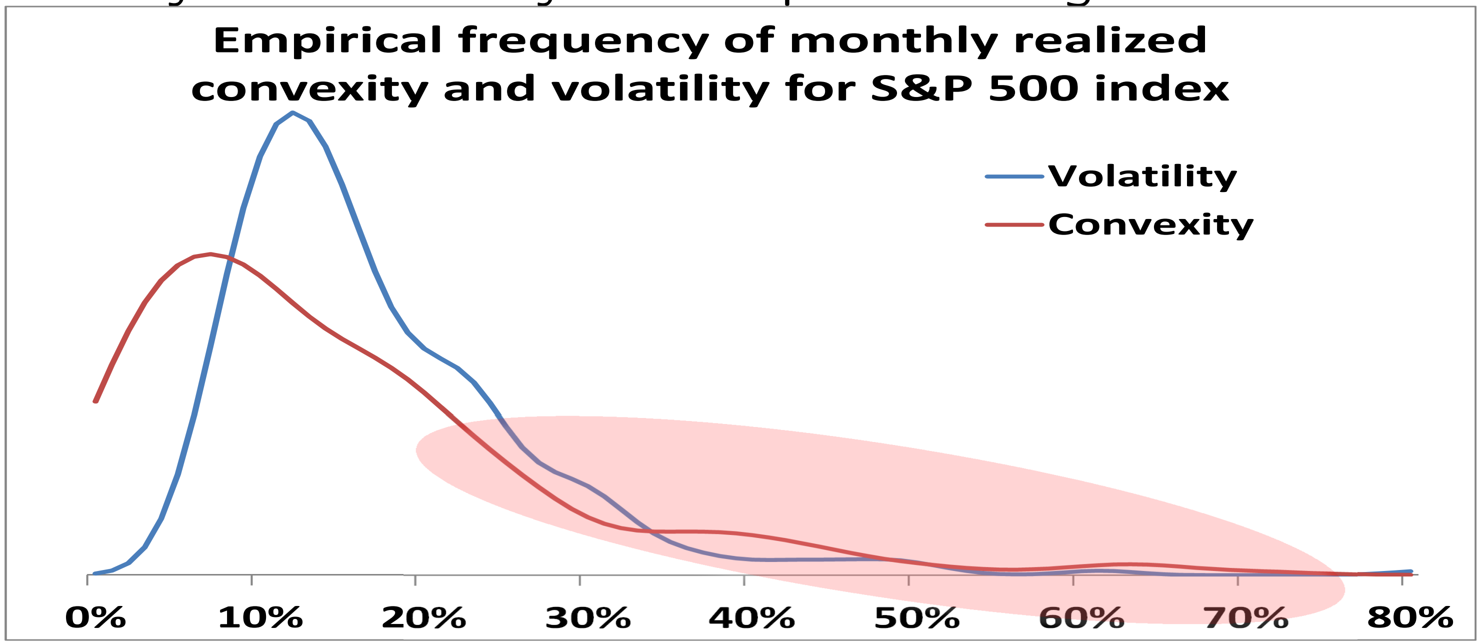
- The convexity estimator using the monthly return (equivalent to P&L on the straddle):

$$\hat{\sigma}_n^{(conv)} = \sqrt{12} \times \sqrt{\frac{\pi}{2}} \left| \frac{S(t_n)}{S(t_{n-1})} - 1 \right|$$

- The volatility estimator using daily returns within the month (equivalent to P&L on the straddle delta-hedged daily):

$$\hat{\sigma}_n^{(vol)} = \sqrt{12} \times \sqrt{\frac{1}{N} \sum_{t_k \in (t_{n-1}, t_n]} \left(\frac{S(t_k)}{S(t_{k-1})} - 1 \right)^2}$$

Figure: Convexity and volatility have equivalent right tails



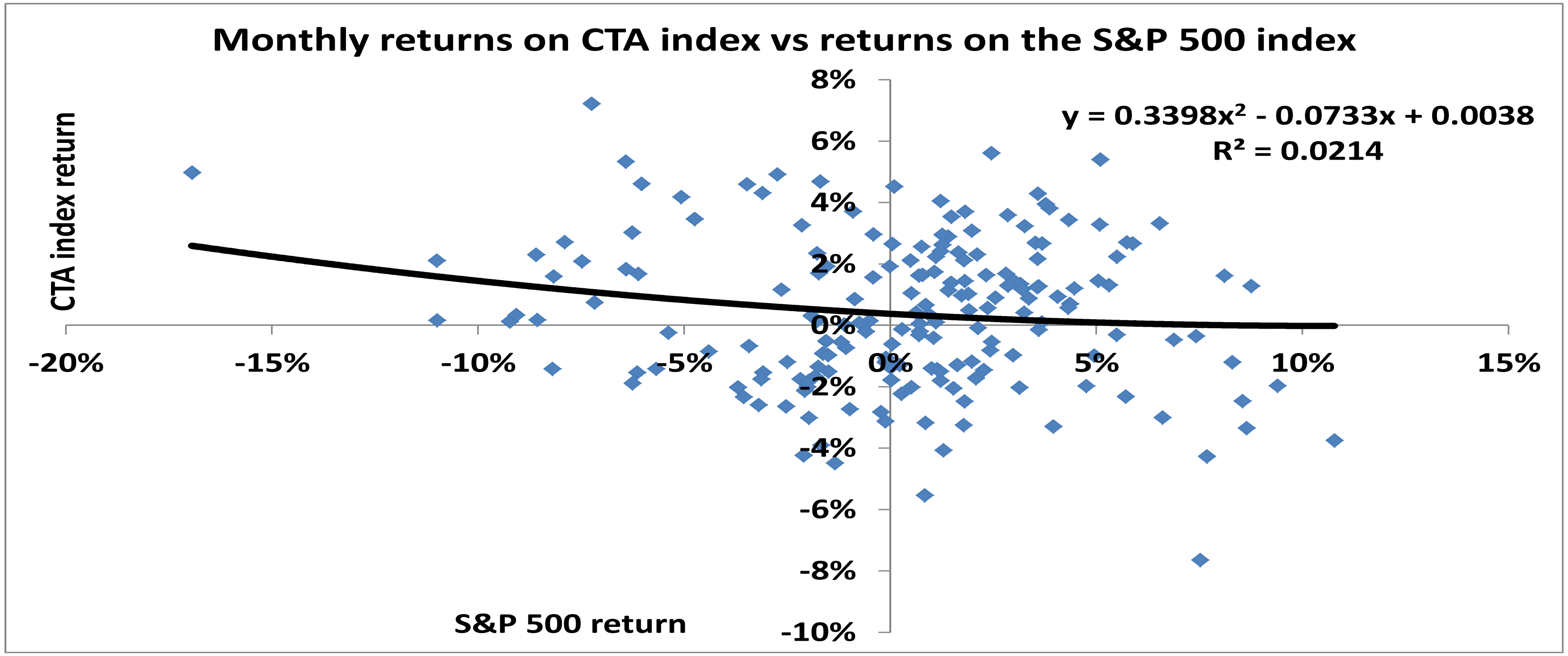
CTAs (commodity trading advisors) are able to create convex return profiles by applying quant strategies for trend-following

Figure: Monthly returns on SC CTA index (tracking 20 largest CTAs) from 2001 to 2016 vs monthly returns on the S&P 500 index

CTAs attemp to replicate option pay-off without actually buying options (long convexity but short volatility similar to trading with a stop-loss)

CTAs seek to rank trends by volatility (stong trends with small volatility)

CTAs attract much more investments than volatility funds



CTAs vs volatility strategies

CTAs derive its convex pay-off from a positive auto-correlation (Bouchaud *et al* (2016))

Replication costs of CTAs are linked to the short-term realized volatility

Long options strategy can create a similar convexity profile but its replication costs are derived from implied volatilitites, which are expensive

Short options strategy works well when auto-correlations are negative (no trend) and the implied volatility is expensive

Combination of CTAs with short volatility strategies can create a more desirable risk profile than each of the alone

Why it is so difficult to make profits being long volatility

Being long volatility requires for a trader to make an intelligent assessment about the trend of the underlying:

- In a strong trending market, the trader should hedge infrequently (let the delta-risk to accumulate)

- In a choppy range-bound market, the trader should hedge very frequently (reduce the delta-risk fast)

It is not only enough to estimate the expected realized volatility

If options are purchased on the buy-to-hold basis without delta-hedging, the trade can make money only for a strong trend in the underlying:

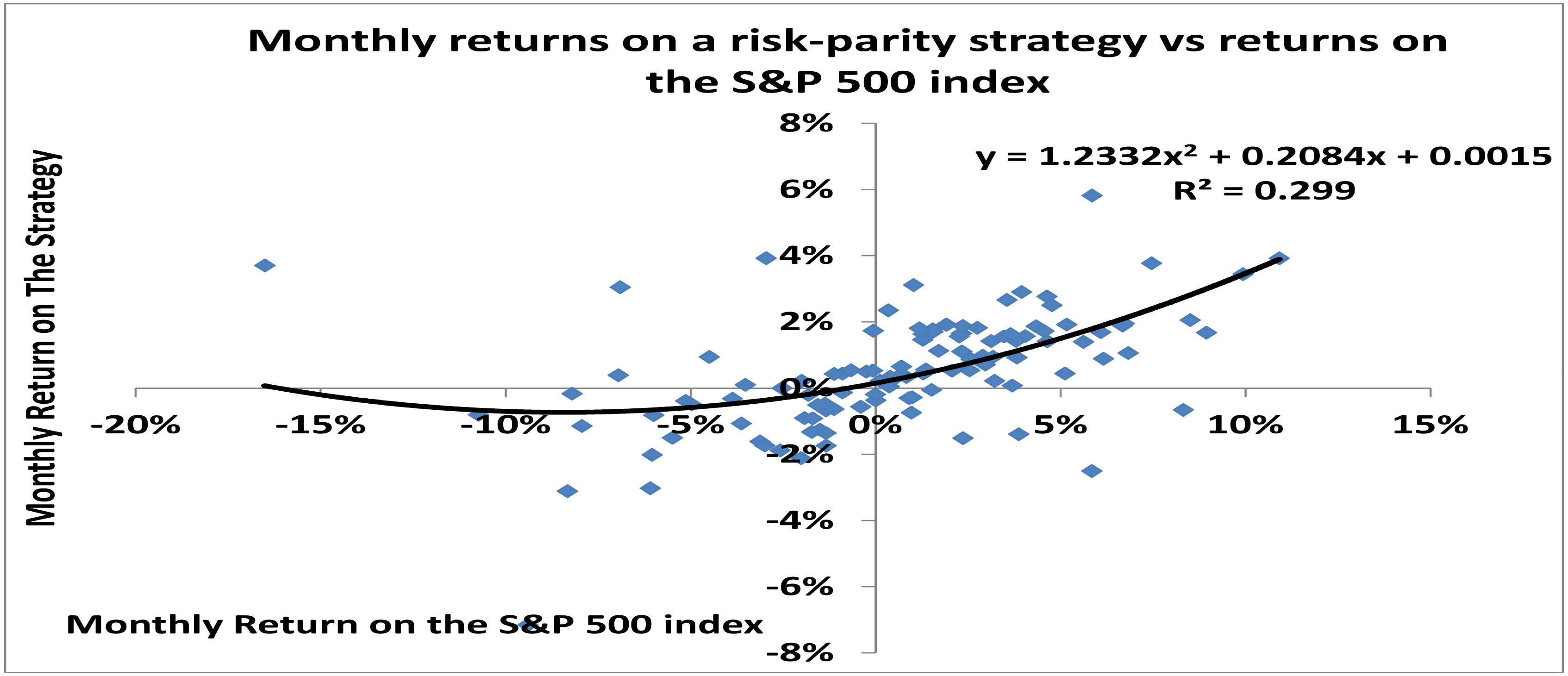
- Recall the cyclicality of the volatility risk-premium: the timing ability is crucial

- Longer-dated options to reduce the timing risk along with pre-defined profit taking

The convexity profile of returns can also be created by using the statistical volatility as a risk control

Figure: the convexity profile of a proprietary risk-parity strategy

1. Statistical volatility is typically negatively correlated to expected return: apply the estimated volatility for asset allocation
2. Key idea behind the risk-parity and minimum volatility funds
3. These strategies tend to outperform over long-term
4. Very strong interest and inflows by the investment community



Conclusions: the cyclicality of markets dynamics

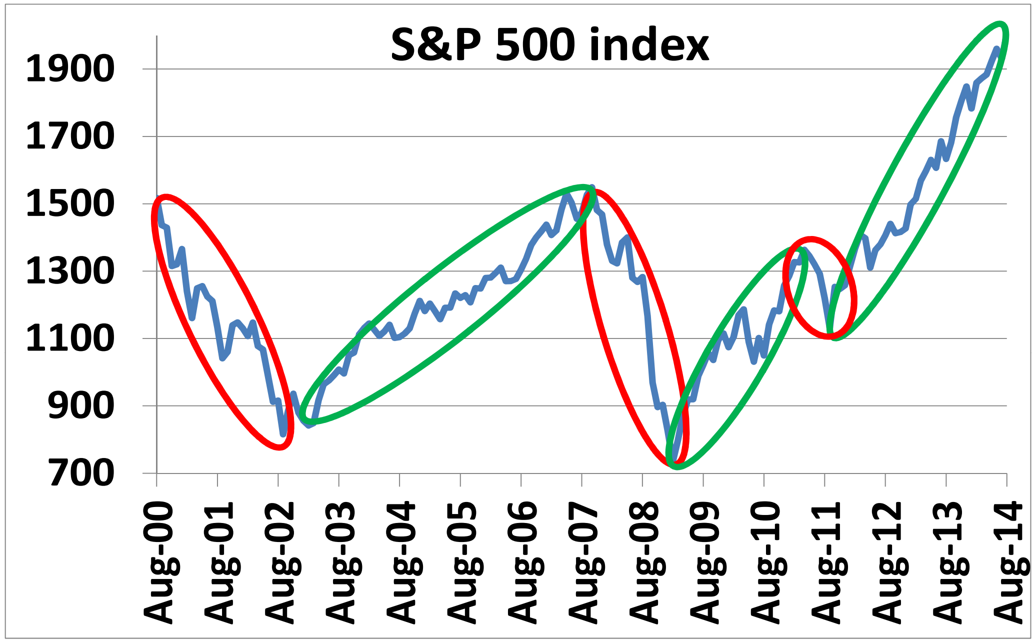
The classic derivative theory deals with option replication assuming ideal conditions and using models that are static in nature

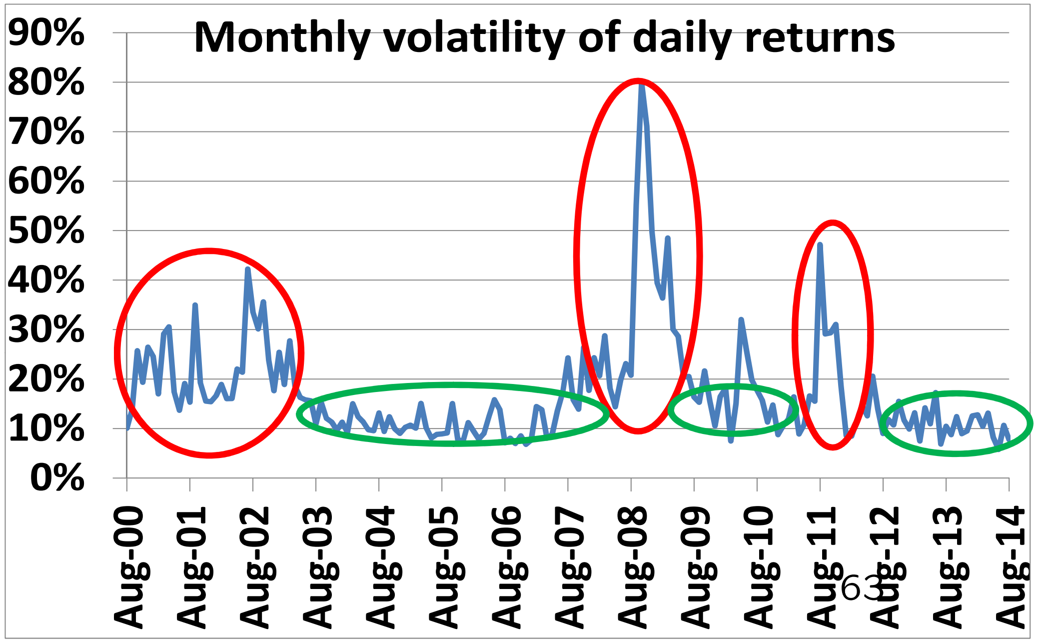
Well-known empirical facts: persistent trends, risk-aversion, over-pricing of tail probabilities

Classic quantitative investment strategies (trend-following and volatility trading) seek for a statictical arbitrage of these anomalies

The volaility risk-premium is part of a factor-based investment approach

Quantitative strategies using statistical volatility as a risk-control can also generate the convexity profile of long option strategies:
Stong investor demand for minimum volatility and risk-parity products





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