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Practitioner Seminar in Financial and Insurance Mathematics

ETH Zürich

Modeling Credit Risk of Loan Portfolios in the Presence of Autocorrelation (Part 1)

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Section 1

Background

Banking Background: *Credit Risk*

- One of the fundamental economic purposes of credit institutions is maturity transformation
- They do this by borrowing in the short term and issuing long-term credit
- This practice exposes banks to credit default risk (amongst others)
- "**Credit risk is most simply defined as the potential that a bank borrower or counterparty will fail to meet its obligations in accordance with agreed terms**." [1]

[1] Principles for the Management of Credit Risk, Basel Committee on Banking Supervision (2000)

Regulatory background: *Credit Portfolio Risk*

- "Traditionally, banks have focused on oversight of contractual performance of individual credits in managing their overall credit risk. While this focus is important, **banks also need to have in place a system for monitoring the overall composition and quality of the various credit portfolios**. This system should be consistent with the nature, size and complexity of the bank's portfolios." [2]

[2] ibid. p. 16

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Section 2

Introduction to the Modeling of Credit Portfolio Risk

The Merton structural *credit risk model*

"Equity is a European call option on a firm's assets"[3]

- Consider a toy firm that is financed through equity $(E_t)_{t\in[0,T]}$ and a zero coupon bond $(D_t)_{t\in[0,T]}$ with maturity T and principal K
- The firm's value $(V_t)_{t\in[0,T]}$ is the sum of the values of its securities:

$$V_t = E_t + D_t$$

- The firm's value V_t is assumed to follow a GBM
- Default occurs if the value of the firm is insufficient to repay the debt principal

$$l = \mathbf{1}_{\{V_T<K\}}$$

- In this toy setting, equity holders receive nothing if the firm defaults, but profit from all the upside if the firm is solvent; the payoff to equity holders is therefore

$$E_T = \max\left(V_T - K, 0\right)$$

- The firm's equity can thus be calculated using the Black-Scholes formula

$$E_t = V_t\ \Phi(d_+) - K\ e^{-r\ (T-t)} * \Phi(d_-)$$

$$d_\pm = \frac{\log(V_t/K) + (r \pm 0.5\ \sigma^2)\ (T-t)}{\sigma\ \sqrt{T-t}}$$

[3] R. C. Merton (1974)

The Credit Portfolio Model *Cookbook*

1. Calculate PD (rating), LGD, EAD for each obligor
2. Group obligors $i = 1, ..., I_t$ into segments $j = 1, ..., J$, $J << I_t$
3. Calibrate asset returns to the factor model

$$R_{i,t} = \sum_{p=1}^{P} \beta_{j,p} \; Y_{p,t} + \sigma_j \; Z_{i,t}, \qquad i = 1, \cdots, I_t, \qquad j = 1, \cdots, J,$$

4. Run a Monte Carlo simulation to generate asset return realizations
5. If the asset return realization is **less** than the PD-implied default threshold, the obligor defaults
   – In this scenario, a loss of the amount of $LGD_i * EAD_i$ is registered for obligor i
6. In each scenario, add up the obligor's losses to get the portfolio loss for this scenario
7. Calculate the VaR, cVaR etc. of the loss distribution from Step 6.

Section 3

Estimation of Latent Asset Return Correlations when Returns are Serially Independent

Conditional *Probabilities of Default*

- Consider a one-factor model for the standardized asset log-return of an obligor i

$$R_{i,t} = \log\left(\frac{V_{i,t}}{V_{i,0}}\right) = \sqrt{\rho}\;Y_t + \sqrt{1-\rho}\;Z_{i,t}$$

- We assume that the common factor $Y_t \sim N(0,1)$ describing the overall state of the world is independent from the firm-specific residual return $Z_{i,t} \sim N(0,1)$
- Default occurs if the obligor's assets are insufficient to cover her liabilities
- This threshold is determined using the obligor's PD p
- Denote by $P(Y_t)$ the conditional PD defined as

$$\begin{aligned} P(Y_t) &:= \mathbb{P}\left[R_{i,t} < \Phi^{(-1)}(p) \mid Y_t\right] \\ &= \mathbb{P}\left[Z_{i,t} < \frac{\Phi^{(-1)}(p) - \sqrt{\rho}\;Y_t}{\sqrt{1-\rho}} \mid Y_t\right] = \Phi\left(\frac{\Phi^{(-1)}(p) - \sqrt{\rho}\;Y_t}{\sqrt{1-\rho}}\right) \end{aligned}$$

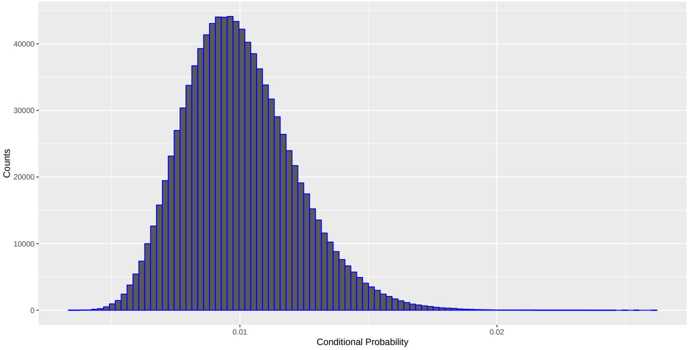
- Given the realization $Y_t = y_t$, the conditional probability of default for this scenario is thus given by

$$P(y_t) = \Phi\left(\frac{\Phi^{(-1)}(p) - \sqrt{\rho}\;y_t}{\sqrt{1-\rho}}\right)$$

Estimation of *Latent Asset Return Correlations*

Recall from the preceding slide that

$$P(Y_t) = \Phi\left(\frac{\Phi^{(-1)}(p) - \sqrt{\rho}\, Y_t}{\sqrt{1-\rho}}\right)$$



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Moment Matching - *Theory*

The problem at hand now consists in finding the latent asset return correlation ρ.
One can calculate that

- $\mathbb{E}[P(Y_t)] = p$
- $\mathbb{V}[P(Y_t)] = \Phi_2\left(\Phi^{(-1)}(p), \Phi^{(-1)}(p); \rho\right) - p^2$

While these theoretical moments are of course unknown, we may, by the strong law of large numbers, assume that they can be approximated by the sample mean and the sample variance of the observed default rate time series (if certain conditions are satisfied).

Moment Matching - *Practice*

In practice, we use the time series of observed default rates $(\Delta_t)_{t=1}^{T}$ for in order to determine the latent asset return correlations.

- The default rate is defined as the ratio of the number of defaulted obligors in a portfolio over the total number of obligors, i.e.

$$\Delta_t = \frac{\#\text{ Obligors that defaulted in } [t-1,t)}{\#\text{ All Obligors in } [t-1,t)}$$

In view of our modelling assumptions,

- $\overline{\Delta}_T := \frac{1}{T}\sum_{t=1}^{T}\Delta_t = \frac{1}{T}\sum_{t=1}^{T} P(y_t) =: \overline{P(y)}_T$
- $\frac{1}{T-1}\sum_{t=1}^{T}\left(\Delta_t - \overline{\Delta}_T\right)^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(P(y_t) - \overline{P(y)}_T\right)^2$

$$\frac{1}{T-1}\sum_{t=1}^{T}\left(P(y_t) - \overline{P(y)}_T\right)^2 \xrightarrow{T\to\infty} \Phi_2\left(\Phi^{(-1)}(p), \Phi^{(-1)}(p); \rho\right) - p^2$$

- One can back out ρ using a standard numerical solver.

Maximum Likelihood Estimation

In many applications, MLE is preferred to MoM. Recall that

$$P(y_t) = \Phi\left(\frac{\Phi^{(-1)}(p) - \sqrt{\rho}\; y_t}{\sqrt{1-\rho}}\right)$$

The MLE for the estimation of latent asset return correlations consists in finding $\rho_0 \in (0,1)$ maximizing the likelihood function

$$L(\rho) = \int_{\mathbb{R}^T} P(y_t)^{D_t}\ (1-P(y_t))^{N_t-D_t} \mathrm{d}F(y_1,\cdots,y_T), \tag{1}$$

where D_t is the number of defaulted obligors in $[t-1,t)$, N_t is the total number of obligors in $[t-1,t)$, and F is the joint distribution of $(Y_1,\cdots,Y_T)$.

- If $(Y_1,\cdots,Y_T)$ are i.i.d. Gaussian, then (1) becomes

$$\boxed{L(\rho) = \prod_{t=1}^{T}\left[\int_{-\infty}^{\infty} P(y_t)^{D_t}\ (1-P(y_t))^{N_t-D_t}\ \mathrm{d}\Phi(y_t)\right]}$$

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Section 4

Estimation of Latent Asset Return Correlations when Returns are Serially Dependent

What to do *in the presence of autocorrelation*?

- If $(Y_1, \cdots, Y_T)$ are not i.i.d., the classical SLLN fails to apply
- There is an extension of the SLLN to *stationary* L_2 *processes*[4]
  - A sequence of rv's (Y_t) is called *stationary* if each Y_t has the same distribution and the distribution of $(Y_{t_1+h}, \cdots, Y_{t_k+h})$ does not depend on h, for any k
  - The sequence is called L_2 if each Y_t has finite variance σ^2
- (Theorem) Let the auto-covariance $\gamma_{s,t}$ $(s, t \in \mathbb{R})$ of the stationary process Y, defined by $\gamma_{s,t} = \mathbb{E}[(Y_s - \mathbb{E}[Y])\ (Y_t - \mathbb{E}[Y])]$, be *summable*, i.e., $\sum_{t=-\infty}^{\infty} |\gamma_{s,t}| \leq c < \infty$. Then

$$\boxed{\frac{1}{T}\sum_{t=0}^{T} Y_t \xrightarrow{T\to\infty} \mathbb{E}[Y] \qquad \text{a.s.}}$$

- For the problem at hand, we invoke this theorem for the convergence of the sample variances - in which case we need that the auto-cokurtoses of Y are summable.

[4] C. Frei, M. Wunsch (2017) - Preprint

Example: *A process with summable auto-comoments*

Consider the AR(1) process

$$X_t = \alpha \; X_{t-1} + \sigma Z_t,$$

where $|\alpha| < 1$, $\sigma > 0$, and $Z_t \sim N(0,1)$ for all t, so that

- $X_t \sim N\left(0, \sigma^2/(1-\alpha^2)\right)$
- $\gamma_{s,t} = (\sigma^2 \alpha^{|s-t|})/(1-\alpha^2)$

Hence $\gamma_{s,t}$ satisfies the summability condition of the theorem, so that the generalized SLLN holds for AR(1) sequences.
Notice that for the auto-cokurtosis $\kappa_{s,t}^{(2,2)}$ we have

$$\kappa_{s,t}^{(2,2)} := \frac{\mathbb{E}[(X_s)^2 \; (X_t)^2]}{\gamma_{s,s} \; \gamma_{t,t}} = \frac{3\sigma^4}{1-\alpha^4} \; \frac{\alpha^{\left(2^{|s-t|}\right)}}{\frac{\sigma^4}{(1-\alpha^2)^2}} = \frac{3(1-\alpha^2)}{1+\alpha^2} \; \alpha^{\left(2^{|s-t|}\right)}$$

Analogous summable expressions hold for the other auto-cokurtoses.
This implies that we can invoke the generalized SLLN for the almost sure convergence of the second moment of an AR(1).

MLE *in the presence of autocorrelation*

Let us assume that $(Y_1, \cdots, Y_T)$ is AR(1), i.e.

$$R_{i,t} = \sqrt{\rho}\, Y_t + \sqrt{1-\rho}\, Z_{i,t} \qquad Y_t \perp Z_{i,t}, \qquad k = i, ..., I_t,$$

$$Y_t = \alpha\, Y_{t-1} + \sqrt{1-\alpha^2}\, \Upsilon_t, \qquad Y_{t-1} \perp \Upsilon_t, \qquad t = 2, \cdots, T,$$

In this case, the likelihood function (1) takes the form

$$\boxed{L(\rho) = \int_{\mathbb{R}^T} P(y_t)^{D_t}\, (1 - P(y_t))^{N_t - D_t}\, \mathrm{d}\Phi_T((y_1, \cdots, y_T)'; \mathbf{0}, \Sigma_T)}$$

where the covariance matrix is given by

$$\Sigma_T = \begin{pmatrix} 1 & \alpha & \cdots & \alpha^T \\ \alpha & 1 & \cdots & \alpha^{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^T & \alpha^{T-1} & \cdots & 1 \end{pmatrix}.$$

McNeil and Wendin (2007) solve this MLE using Gibbs sampling.

Latent Asset Correlations: *Problems and Solutions*

We summarize what we have learnt so far in the table below.

	Problem	*Solution*	*Implementation*
MLE	Complicated likelihood function	Gibbs Sampling	Intricate
MoM	Slow convergence rate	$\rightsquigarrow$ Part 2	Straightforward

Thank you very much for your attention!