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p-adic Simplicial Volumes

joint work with Steffen Kionke

1. Simplicial Volume & Betti Numbers
2. The p-adic case

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Def. [Gromov] Let M be an occ n-manifold, let R be a normed ring. (oriented closed connected ($\neq\emptyset$, dim>0))

Then the R-simplicial volume of M is

$$\|M\|_R := \inf\Big\{\sum_j |a_j| \;\Big|\; \sum_j a_j\cdot\sigma_j \in C_n(M;R) \text{ fundamental cycle}\Big\} \in \mathbb{R}_{\geq 0}.$$

Classical case: $R=\mathbb{R}$ with standard norm

$\rightsquigarrow$ negative curvature: If M admits a hyperbolic Riemannian metric, then [Gromov, Thurston]
$$\|M\|_{\mathbb{R}} = \frac{\mathrm{vol}(M)}{v_{\dim M}}.$$

$\rightsquigarrow$ amenability: If $\pi_1(M)$ is amenable, then [Gromov, Ivanov]
$$\|M\|_{\mathbb{R}} = 0.$$

Open problem [Gromov]. Let M be an occ aspherical mfd

$$\|M\|_{\mathbb{R}} = 0 \overset{?}{\Longrightarrow} \chi(M) = 0$$

$$\chi(M) = \sum_j (-1)^j \cdot b_j^{(2)}(M) \Longleftarrow \forall_j\; b_j^{(2)}(M) = 0$$

understood if
- $\pi_1(M)$ amenable
- 3-mfds

$\|M\|_{\mathbb{R}} = 0 \overset{?}{\Longrightarrow} \forall_j\; b_j^{(2)}(M)=0$

if $\pi_1(M)$ is residually finite:

$$\|M\|_{\mathbb{R}} = 0 \overset{?}{\Longrightarrow} \|M\|_{\mathbb{Z}}^{\infty} = \lim_{\Gamma\to 1} \frac{\|\overline{M}_\Gamma\|_{\mathbb{Z}}}{[\pi_1(M):\Gamma]} = 0 \underset{\text{(see below)}}{\overset{\text{easy}}{\Longrightarrow}} \forall_j\; b_j^{(2)}(M)=0$$

[Lück] $b_j^{(2)}(M) = \lim_{\pi_1(M)>\Gamma\to 1,\ \text{f.i.}} \frac{b_j(\overline{M}_\Gamma)}{[\pi_1(M):\Gamma]}$

Related problem: What can be said about
$$\frac{\log \# \operatorname{tors} H_j(\overline{M}_\Gamma;\mathbb{Z})}{[\pi_1(M):\Gamma]} \xrightarrow[\Gamma\to 1]{} ?$$

in particular: hyp 3-mfds ?!

Observation. [Gromov] Let M be an occ mfd and let R be a PID. Then

$$\forall j \in \mathbb{N} \quad \mathrm{rk}_R H_j(M;R) \leq \underbrace{\|M\|_{(R)}}_{\text{trivial norm}}.$$

Proof idea: use Poincaré duality: let $\underbrace{c}_{=\sum_k a_k \cdot \sigma_k} \in C_n(M;R)$ be a fundamental cycle. Then

$$H^{n-j}(M;R) \xrightarrow{\cdot \cap [c]} H_j(M;R) \text{ is an iso.}$$

$$[f] \longmapsto \left[\sum_k a_k \cdot f(\sigma_k\rfloor_{n-j}) \cdot {}_j\lfloor\sigma_k \right] \qquad \square$$

Hope: For p-part of log tors $H_j(M;\mathbb{Z})$: look at $\|M\|_{\mathbb{Z}_p}$.

$\boxed{2}$ Def. Let M be an occ mfd and let $p \in \mathbb{N}$ be a prime. Then

$$\|M\|_{(\mathbb{F}_p)} := \|M\|_{\mathbb{F}_p, \text{ trivial norm}}$$

$$\|M\|_{\mathbb{Z}_p} := \|M\|_{\mathbb{Z}_p, |\cdot|_p} \qquad |p^m|_p := p^{-m}$$

$$\|M\|_{\mathbb{Q}_p} := \|M\|_{\mathbb{Q}_p, |\cdot|_p}$$

Warning: There is no obvious duality for $\|\cdot\|_{\mathbb{Q}_p}$ in terms of bounded cohomology because the norm on $C_*(M;\mathbb{Q}_p)$ is a mix of

- archimedean world (ℓ^1-norm)
- non-archimedean world ($|\cdot|_p$)

(no suitable version of Hahn–Banach!)

Properties. [Kionke, L]. Let M be an oc mfd and let $p \in \mathbb{N}$ prime.

- Then

$$
\begin{tikzcd}
\mathbb{F}_p & & \\
& \mathbb{Z}_p \arrow[ul] \arrow[dl] & \mathbb{Z} \arrow[l, hook'] \\
\mathbb{Q}_p & &
\end{tikzcd}
$$

$$b_*(M;\mathbb{F}_p) \le \|M\|_{(\mathbb{F}_p)} \le \|M\|_{\mathbb{Z}_p} \le \|M\|_{\mathbb{Z}} \quad \text{from}$$

$$b_*(M;\mathbb{Q}) \le \|M\|_{\mathbb{Q}_p} \le \|M\|_{\mathbb{Z}_p}$$

Proof:

$$b_j(M;\mathbb{Q}) \overset{\text{UCT}}{\le} \underbrace{\dim_{\mathbb{F}_p} p^m \cdot H_j(M;\mathbb{Z}/p^{m+1})}_{\text{related to } p\text{-torsion part of } H_*(M;\mathbb{Z})} \overset{(B)}{\le} \|p^m\cdot[M]\|_{(\mathbb{Z}/p^{m+1})}$$

$$\le p^m \cdot \|p^m\cdot[M]\|_{\mathbb{Z}/p^{m+1},\, |\cdot|_p}$$

$$\le p^m \cdot \|p^m\cdot[M]\|_{\mathbb{Z}_p}$$

$$\rightsquigarrow b_j(M;\mathbb{Q}) \le \lim \cancel{\inf}_{m\to\infty} p^m \cdot \|p^m[M]\|_{\mathbb{Z}_p} \underset{\text{clear denominators}}{=} \|M\|_{\mathbb{Q}_p}.$$

- If $\|M\|_{\mathbb{Q}_p} < p$, then $\|M\|_{\mathbb{Q}_p} = \|M\|_{\mathbb{Z}_p}$.
- For almost all primes p

$$\|M\|_{(\mathbb{Q})} = \|M\|_{\mathbb{Q}_p} = \|M\|_{\mathbb{Z}_p} = \|M\|_{(\mathbb{F}_p)}.$$

Question. - When does $\|M\|_{\mathbb{Q}_p} < \|M\|_{\mathbb{Z}_p}$ happen ?

- Is this related to the p-torsion part ?

Examples.

look at double cover $S^3 \to \mathbb{RP}^3$

- $$\|\mathbb{RP}^3\|_{\mathbb{Z}_p} = \begin{cases} 1 & p \neq 2 \\ 2 & p = 2 \end{cases}$$

requires concrete computations

- $$\|\Sigma_g\|_{\mathbb{Z}_p\,(\mathbb{F}_p)} = \|\Sigma_g\|_{\mathbb{Z}} = 4g-2.$$

occ surface of genus $g \geq 1$

$$\rightsquigarrow \quad \|\Sigma_g\|^{\infty}_{\mathbb{Z}\,(\mathbb{F}_p)} = 4g-4 = \|M\|_{\mathbb{R}}$$

Open problem: For occ aspherical 3-mfds M:

reasonable lower bounds ?? $\longrightarrow$

$$\|M\|^{\infty}_{\mathbb{F}_p} \leq \|M\|^{\infty}_{\mathbb{Z}} = \frac{\mathrm{hypvol}(M)}{v_3} = \|M\|_{\mathbb{R}}$$

$= ?$

Fauser, L., Moraschini, Quintanilla

Soma

a

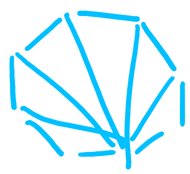
k

$\frac{\alpha}{k}$

Prop. [Kianke, L] If R is a PID and $g \in \mathbb{N}_{\geq 1}$, then

$$\|\Sigma_g\|_{(R)} = 4g-2.$$

Proof. $\leq$: standard construction:



+ signs, orientations

$4g$-gon

$\geq$: Let K be the quotient field of R

($\leadsto$ only need to show $\|\Sigma_g\|_{(K)} \geq 4g-2$)

$\|\Sigma_g\|_{\mathbb{R}} \geq 4g-4$: geodesic straightening + integration

Let $c \in C_2(\Sigma_g; K)$ be a fcl cycle $c = \sum_{j=1}^{k} a_j \cdot \sigma_j$ with minimal k.

$b_1(\Sigma_g; \mathbb{F}_p) = 2g$ ← not good enough

$\leadsto$ semi-simplicial set X assoc. with c

$\leadsto$ chain complex

$$C_0(X;K) \xleftarrow{\partial_1} C_1(X;K) \xleftarrow{\partial_2} C_2(X;K)$$

We know:

- $\dim_K C_2(X;K) = k$
- $\dim_K H_1(X;K) \overset{PD}{\geq} \dim_K H_1(\Sigma_g;K) = 2g$ $X \xrightarrow{\text{"}c\text{"}} \Sigma_g$
- $\dim_K \ker \partial_2 = 1$ (minimality of k + lin. algebra)
- $2 \cdot \dim_K C_1(X;K) \leq 3 \cdot k$ (cycle cond. + combin. of Δ^2)

$$\leadsto 2g \leq \dim_K H_1(X;K) \leq \underbrace{\dim_K C_1(X;K)}_{\leq \frac{3}{2}\cdot k} - \underbrace{\dim_K \operatorname{im} \partial_2}_{k-1} = \frac{1}{2}k + 1$$

$\leadsto 4g-2 \leq k.$ $\square$