Stability of approximate actions

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Based on joint work with Michael Chapman

Ulam stability

- **General question (Ulam '41):** Is every approximate homomorphism $\Gamma \to G$ close to a homomorphism?

- The answer depends on:
  - The groups Γ and G.
  - What is "approximate"?
  - What is "close"?

Stability of approximate unitary representations

- An amenable group Γ.
- A Hilbert space $\mathcal{H}$.
- A function $f\colon \Gamma \to \mathrm{U}(\mathcal{H})$.
- $\delta < 1/200$.

Theorem (Kazhdan '82)

If

$$\sup_{\gamma_1,\gamma_2\in\Gamma} \|f(\gamma_1\gamma_2) - f(\gamma_1)f(\gamma_2)\|_{op} \le \delta$$

then there is a representation $h\colon \Gamma \to U(\mathcal{H})$ *such that*

$$\sup_{\gamma\in\Gamma} \|h(\gamma) - f(\gamma)\|_{op} \le 2\delta\ .$$

Distance between permutations

Instead of $\mathrm{U}(n)$ and $\| \|_{\mathrm{op}}$, we consider $\mathrm{Sym}(n)$ and:

Definition

The *normalized Hamming metric*:

$$d^H(\sigma,\tau) = \frac{1}{n} \left| \{x \in [n] \mid \sigma(x) \neq \tau(x)\} \right| \quad \forall \sigma,\tau \in \mathrm{Sym}(n) ,$$

where $[n] = \{1,\ldots,n\}$.

Stability of approximate actions

Definition

1. The *uniform local defect* of $f\colon \Gamma \to \mathrm{Sym}(n)$:

$$\mathrm{def}_\infty(f) = \sup_{\gamma_1,\gamma_2\in\Gamma}\left\{d^H(f(\gamma_1\gamma_2), f(\gamma_1)f(\gamma_2))\right\}\ .$$

2. The *uniform distance* between $f,h\colon \Gamma \to \mathrm{Sym}(n)$:

$$d_\infty(f,h) = \sup_{\gamma\in\Gamma}\left\{d^H(f(\gamma), h(\gamma))\right\}\ .$$

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Theorem (Glebsky–Rivera '09)

If Γ is finite and $f\colon \Gamma \to Sym(n)$ then there is a homomorphism $h\colon \Gamma \to Sym(n)$ such that

$$d_\infty(h,f) \le C def_\infty(f) ,$$

where C depends only on Γ (and not on n).

Theorem (Glebsky–Rivera '09)

If Γ is finite and $f : \Gamma \to Sym(n)$ then there is a homomorphism $h \colon \Gamma \to Sym(n)$ such that

$$d_\infty(h, f) \le C\, def_\infty(f)\ ,$$

where C depends only on Γ (and not on n).

- **Question (Lubotzky '18):** Is it true for $\Gamma = \mathbb{Z}$?
- **Question:** Can we replace C by a universal constant?

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- **Question (Lubotzky '18):** Is it true for $\Gamma = \mathbb{Z}$?
- **Question:** Can we replace C by a universal constant?
- **Answers:** No and No.

An instability result

Theorem (B–Chapman '20)

If Γ *acts transitively on* $[n] = \{1, \ldots, n\}$ *then there is* $f : \Gamma \to Sym(n-1)$ *such that*

$$def_\infty(f) \leq \frac{2}{n-1}\,, \tag{1}$$

but

$$d_\infty(h, f) \geq \frac{1}{2} - \frac{1}{n-1} \tag{2}$$

for every homomorphism $h \colon \Gamma \to Sym(n-1)$.

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Proof.

Let

$$f : \Gamma \xrightarrow{\text{transitive}} \mathrm{Sym}(n) \xrightarrow{\mathrm{res}_n} \mathrm{Sym}(n-1)\ ,$$

where $\mathrm{res}_n(\sigma)x = \begin{cases} \sigma(x) & \sigma(x) \neq n \\ \sigma(\sigma(x)) & \sigma(x) = n \end{cases}$ for $x \in [n-1]$. □

Flexibility in the number of points

A relaxed question: Is every approximate homomorphism $f\colon \Gamma \to \mathrm{Sym}(n)$ close to a homomorphism $h\colon \Gamma \to \mathrm{Sym}(N)$, where N is only slightly larger than n?

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Theorem (Gowers–Hatami '17, De Chiffre–Ozawa–Thom '19)

If Γ is amenable, $f: \Gamma \to U(n)$, $\delta > 0$ and

$$\|f(\gamma_1\gamma_2) - f(\gamma_1) f(\gamma_2)\|_{hs} \le \delta \quad \forall \gamma_1, \gamma_2 \in \Gamma$$

then there is a representation $h: \Gamma \to U(N)$ and an isometry $T: \mathbb{C}^n \to \mathbb{C}^N$ such that

$$\|h(\gamma) - T^* f(\gamma) T\|_{hs} \le 211\delta \quad \forall \gamma \in \Gamma$$

and

$$n \le N \le \left(1 + 2500\delta^2\right) n .$$

$\|A\|_{\mathrm{hs}} = \left(\frac{1}{n}\mathrm{tr}(A^* A)\right)^{1/2}$ for $A \in \mathrm{U}(n)$

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Definition

For $\sigma \in \mathrm{Sym}(n)$ and $\tau \in \mathrm{Sym}(N)$, $n \le N$,

$$d^H(\sigma,\tau) = d^H(\tau,\sigma) = \frac{1}{N}\left(\left|\{x \in [n] \mid \sigma(x) \neq \tau(x)\}\right| + (N-n)\right).$$

d^H is a metric on $\coprod_{n\in\mathbb{N}} \mathrm{Sym}(n)$.

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Definition

For $f\colon \Gamma \longrightarrow \mathrm{Sym}(n)$ and $h\colon \Gamma \longrightarrow \mathrm{Sym}(N)$,

$$d_\infty(f,h) = \sup_{\gamma\in\Gamma}\left\{d^H(f(\gamma),h(\gamma))\right\} .$$

Question (Kun–Thom '19)

- A finite group Γ.
- A function $f : \Gamma \to \mathrm{Sym}(n)$.

Is there a homomorphism $h : \Gamma \to \mathrm{Sym}(N)$ such that

$$d_\infty(h, f) \le \varepsilon \quad \text{and} \quad n \le N \le (1+\varepsilon)\, n\,,$$

where:

- ε depends only on $\mathrm{def}_\infty(f)$.
- $\varepsilon \to 0$ as $\mathrm{def}_\infty(f) \to 0$?

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Answer (B–Chapman)

Yes, and

- Only assume that Γ is amenable,
- $\varepsilon \le 2039\mathrm{def}_\infty(f)$.

Theorem (B-Chapman)

Let Γ be an amenable group and $f : \Gamma \to Sym(n)$. Then there is a homomorphism $h : \Gamma \to Sym(N)$ such that

$$d_\infty(h, f) \leq 2039 def_\infty(f) \quad and \quad n \leq N \leq (1 + 1218 def_\infty(f))\, n\,.$$

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Rough sketch of proof.

- Set $h(\gamma)x = y$ if $f(t) f\left(t^{-1}\gamma\right) x = y$ for many $t \in \Gamma$.
  "many" is w.r.t. an invariant prob. measure on Γ.
- Each $h(\gamma)$ is a partial injective map $[n] \to [n]$ and

$$h(\gamma_1\gamma_2)x = h(\gamma_1)\, h(\gamma_2)x$$

  for $x \in [n]$ whenever both sides are defined.
- Extend $h(\gamma)$ to a bijection $[N] \to [N]$ such that h is a homomorphism.

□

Theorem (B–Chapman)

Let $f\colon SL_r\mathbb{Z} \to Sym(n)$, $r \geq 3$. Then there is a homomorphism $h\colon SL_r\mathbb{Z} \to Sym(N)$ such that

$$d_\infty(h,f) \leq C def_\infty(f) \quad \text{and} \quad n \leq N \leq (1 + C def_\infty(f))\, n\,,$$

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Sketch of proof (following a method of Burger–Ozawa–Thom).

- Let f_- and f_+ be the restrictions of f to the subgroups of lower and upper triangular unipotent matrices of $\mathrm{SL}_r\mathbb{Z}$.
- Apply the previous theorem to each.
- Both f_+ and f_- are close to homomorphisms, and thus almost vanish on a finite index subgroup $\Delta \triangleleft \mathrm{SL}_r\mathbb{Z}$.
- By bounded generation in $\mathrm{SL}_r\mathbb{Z}$, f almost vanishes on Δ.
- Apply the previous theorem to the finite group $(\mathrm{SL}_r\mathbb{Z})/\Delta$.

□

Approximate homomorphisms away from homomorphisms

Theorem (B–Chapman)

Let Γ be a group that surjects onto a nonabelian free group. Then there is a sequence of functions

$$f_k \colon \Gamma \to Sym(n_k), \quad n_k \xrightarrow{k\to\infty} \infty$$

such that

$$def_\infty(f_k) \leq \frac{2}{k}$$

but

$$d_\infty(h_k, f_k) \geq 1 - \frac{5}{k}$$

for every homomorphism $h_k \colon \Gamma \to Sym(N_k)$ for all $N_k \geq n_k$.

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- Proof by explicit construction inspired by the construction of quasimorphisms by Rolli (masters thesis supervised by A. Iozzi, '09).
- **Open Q:** Is the theorem true for nonelementary word-hyperbolic groups?

Property testing

Let Γ and G be finite groups.

Theorem (Blum–Luby–Rubinfeld)

If $f : \Gamma \to G$ disagrees with every homomorphism $\Gamma \to G$ on at least $\varepsilon |\Gamma|$ elements, $\varepsilon \leq 1/3$, then

$$\Pr_{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma}(f(\gamma_1\gamma_2) \neq f(\gamma_1) f(\gamma_2)) \geq \varepsilon/2 \, .$$

For $\alpha > 0$, repeat the test $\frac{2\log(1/\alpha)}{\varepsilon}$ times to increase the rejection probability to $1-\alpha$.

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What if $G = \mathbf{Sym}(n)$ and n is very large?

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What if $G = \mathbf{Sym}(n)$ and n is very large?

Theorem (B–Chapman)

If $f:\Gamma \to Sym(n)$ satisfies $\int d^H(f(\gamma),h(\gamma))\,dm(\gamma) \ge \varepsilon$ for every homomorphism $h\colon \Gamma \to Sym(N)$, $N \ge n$, then

$$\Pr_{(\gamma_1,\gamma_2,x)\in\Gamma\times\Gamma\times[n]}\left(f(\gamma_1\gamma_2)x \ne f(\gamma_1)f(\gamma_2)x\right) \ge \varepsilon/2913\,.$$

Property testing + Quantum Computing

- **Recall**: Gowers–Hatami Theorem about stability of homomorphisms $\Gamma \to \mathsf{U}(n)$, w.r.t. $\|\cdot\|_{\mathsf{hs}}$ and Γ finite.
- This too can be viewed in the property testing setting.
- Used by Vidick and Natarajan, and eventually in the negative answer for the Connes Embedding Problem.

Uniform stability (as studied in the previous slides):
Given $(f_k\colon \Gamma \longrightarrow \mathrm{Sym}(n_k))_{k=1}^{\infty}$ such that

$$\sup_{\gamma_1,\gamma_2\in\Gamma} d^H(f_k(\gamma_1\gamma_2), f_k(\gamma_1) f_k(\gamma_2)) \overset{k\to\infty}{\longrightarrow} 0\,,$$

is there a sequence of homomorphisms $(h_k\colon \Gamma \to \mathrm{Sym}(N_k))_{k=1}^{\infty}$ such that

$$\sup_{\gamma\in\Gamma} d^H(h_k(\gamma), f_k(\gamma)) \overset{k\to\infty}{\longrightarrow} 0\,?$$

Pointwise stability (studied a lot in recent years):
Given $(f_k\colon \Gamma \longrightarrow \mathrm{Sym}(n_k))_{k=1}^{\infty}$ such that

$$d^H(f_k(\gamma_1\gamma_2), f_k(\gamma_1) f_k(\gamma_2)) \overset{k\to\infty}{\longrightarrow} 0 \quad \forall \gamma_1,\gamma_2\in\Gamma,$$

is there a sequence of homomorphisms $(h_k\colon \Gamma \to \mathrm{Sym}(N_k))_{k=1}^{\infty}$ such that

$$d^H(h_k(\gamma), f_k(\gamma)) \overset{k\to\infty}{\longrightarrow} 0 \quad \forall \gamma\in\Gamma?$$

Thank you for your attention!