ETH - Online Geometry Seminar

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Random Covers of Surfaces and their Asymptotic Statistics

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Introduction and main results

S_N = Symmetric gp

$\Gamma_g = \pi_1($  $) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1,b_1] \cdots [a_g,b_g] = 1 \rangle$

genus g

For simplicity: $g=2$ $\quad \Gamma = \Gamma_2 = \pi_1($  $)$

$= \langle a,b,c,d \mid [a,b][c,d] = 1 \rangle$

We study $\mathrm{Hom}(\Gamma, S_N)$

$$= \{ (\alpha,\beta,\gamma,\delta) \in S_N^4 \mid [\alpha,\beta][\gamma,\delta] = 1 \}$$

Q: $|\mathrm{Hom}(\Gamma, S_N)| = ?$

Frobenius + Hurwitz : G - finite group
1896 1902

$$|\mathrm{Hom}(\Gamma, G)| = |G|^3 \cdot \sum_{\rho \in \mathrm{Irred}(G)} \frac{1}{(\dim \rho)^2}$$

Irreducible representations

pf: $|\mathrm{Hom}(\Gamma, G)| = \sum_{\alpha,\beta,\gamma,\delta \in G} \mathbb{1}_e([\alpha,\beta][\gamma,\delta])$

$$= \frac{1}{|G|} \sum_{\rho} \dim \rho \underbrace{\sum_{\alpha,\beta,\gamma,\delta} \chi_\rho([\alpha,\beta][\gamma,\delta])}_{\text{Frobenius: } |G|^4 \frac{1}{(\dim \rho)^3}}$$

$\mathbb{1}_e(g) = \frac{1}{|G|} \sum_{\rho \in \mathrm{Irred}(G)} \dim \rho \cdot \chi_\rho(g)$

$G = S_N$: Lulov (1996): $\sum_{\rho \in \mathrm{Irred}(S_N)} \frac{1}{(\dim \rho)^2} \xrightarrow[N \to \infty]{} 2$

$$|\mathrm{Hom}(\Gamma, S_N)| \approx 2 \cdot N!^3$$

Our question: Fix $\gamma \in \Gamma$. $\left(\text{e.g. } \underset{[a,b]}{a} \right)$

γ induces a probability distribution on S_N:

Pick $\varphi \in \mathrm{Hom}(\Gamma, S_N)$ at random and

γ induces a probability distribution on S_N:
pick $\phi \in \mathrm{Hom}(\Gamma, S_N)$ at random, and evaluate $\phi(\gamma)$.

<u>Goal</u>: Understand the expected # fixed-points of $\phi(\gamma)$

<u>Motivation</u>: random surfaces

$\rightsquigarrow$ <u>Thm (Magee-Naud-P, 2020)</u>: X is a fixed compact hyp. surface. And Y is a random N-covering of X.
$\forall \varepsilon > 0$ Then the smallest <u>new</u> e-value of Y satisfies $\lambda \geq \frac{3}{16} - \varepsilon$ with prob $\xrightarrow[N\to\infty]{} 1$.

<u>Main Results</u>:

<u>Thm A</u>: There is an infinite sequence of rational numbers $a_{-1}(\gamma), a_0(\gamma), a_1(\gamma), a_2(\gamma), \ldots \in \mathbb{Q}$
s.t. for every $M \in \mathbb{Z}_{\geq 0}$

$$\mathbb{E}\left[\#\mathrm{fix}\,\phi(\gamma)\right] = a_{-1}(\gamma)\cdot N + a_0(\gamma) + \frac{a_1(\gamma)}{N} + \cdots + \frac{a_M(\gamma)}{N^M} + O\left(\frac{1}{N^{M+1}}\right)$$

<u>Example</u>: <u>$\gamma = a$</u>
$1 + \frac{1}{N^2} + \frac{2}{N^3} + \frac{4}{N^4} + \cdots$

<u>$\gamma = id$</u>
N

<u>$\mathbb{E}(\#\mathrm{fix\text{-}points}) - 1$</u>
powers: $\simeq 1$
non-power $\simeq \frac{1}{N}$

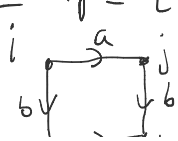
<u>Thm B</u>: If $\gamma \neq id$ write $\gamma = \gamma_0^q$ q maximal.
Then $a_{-1}(\gamma) = 0$
$a_0(\gamma) = \underline{\#\text{ divisors}}$ of q.

$\gamma = a^6$ 4

⊛ This is true also if Γ is replaced with a free group [Nica, 1994]

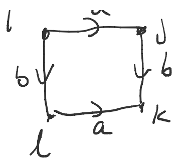
(2) Ideas in Thm A: rational approximation

Consider $\gamma = [a,b] = aba^{-1}b^{-1}$ $\phi(\gamma) \in S_N$



$i, j, k, l \in \{1, \ldots, N\}$

[illegible]

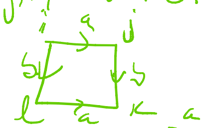


$i, j, k, l \in \{1, \dots, N\}$

compose permutations from left to right.

In free groups: a, b are independent random permutations.

Case I: i, j, k, l all different



$$\frac{N(N-1)(N-2)(N-3)}{N(N-1)\quad N(N-1)} \approx \frac{N^V}{N^e} = N^{V-e} = N^{\chi(\text{graph})}$$

Case II: $i = j$, $\neq$, $k = l$

$$\frac{N(N-1)}{N(N-1)\cdot N} \approx N^{-1} = \frac{1}{N}$$

$\vdots$

7 cases

$$1 + \frac{1}{N-1}$$

So $E[\#\text{fixed-points}]$ of $\phi(w)$ is a rational expression in N.
free-group case

Surface groups:

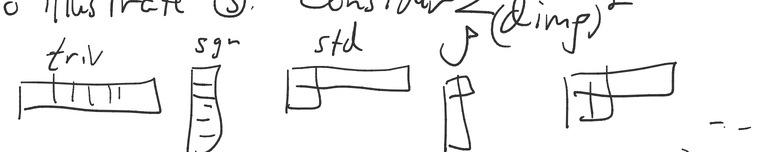
① For each of the 7 graphs give an expression based on irred. repr. of S_N.
(Vershik-Okounkov approach)

For every graph,

② $\forall$ Every "tuple" of representations contributes an analytic function to the #fix-points

③ Show that for every $m \in \mathbb{Z}_{\geq 0}$, there are only finitely many contributions with more than $O\left(\frac{1}{N^m}\right)$.

To illustrate ③: consider $\sum \frac{1}{(\dim \rho)^2}$

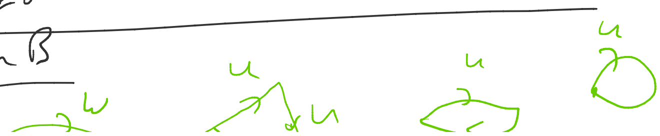


$1 \qquad 1 \qquad \frac{1}{(N-1)^2} \qquad \frac{1}{(N-1)^2} \qquad \left(\frac{2}{N(N-3)}\right)^2$

Liebeck-Shalev (2004), Gamburd (2006): Almost irreps of S_N contribute together $O\left(\frac{1}{N^m}\right)$ $\forall m \geq 0$.

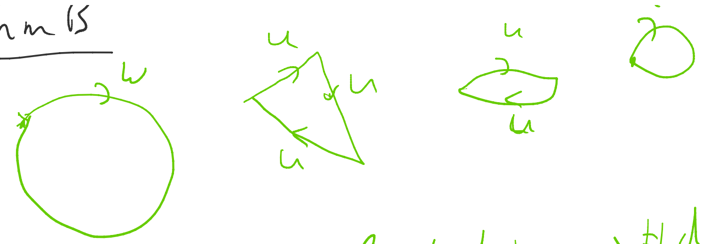
③ Ideas in Thm B

$w = w^6$



(3) Ideas in Thms

$w = u^6$



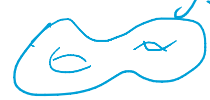
$\Rightarrow$ in the free group: $\mathbb{E}(\#\text{fixed pts}) \longrightarrow \#$ divisors of q

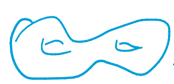
$w = w_0^{\,q}$ $\quad$ q maximal

Surface groups

- Some contributions may have order N^3
- even if you sum up all contributions of a given graph, you don't get the "right" contribution

Solution: Instead of graphs, consider surfaces

$$\mathrm{Hom}(\Gamma, S_N) \longleftrightarrow N\text{-coverings of}$$ 

We look for 2-dimensional pieces in random coverings of .

Thm ① If such a piece Y has nice properties (=short boundary) then the contribution of Y is $N^{\chi(Y)} + O(N^{\chi(Y)-1})$.

② $\chi(Y) = 0$ if and only if Y corresponds to some root of γ.