DENSITY THEOREMS AND THE RAMANUJAN PROPERTY - ONLINE GEOMETRY SEMINAR, ETH

AMITAY KAMBER

1. Intro

Based on joint work with Kosta Golubev.

There are various results that prove "optimal" geometric phenomena which follow from an optimal spectral gap.

- Cutoff, Almost Diameter - Lubetzky-Peres, Sardari (2015)
- Golden Gates - Sarnak-Parzanchevski (2015,2017)
- Diophantine Exponents - Ghosh-Gorodnik-Nevo (2014)

In general, density theorems are meant to replace the optimal spectral gap by a simpler condition, which will deduce similar results.

Following the work of Sarnak and Xue, density theorems can actually be proven using a simple geometric counting argument.

I will start with graphs and then will discuss some more general statements.

2. Graphs

q is fixed. X_t a family of finite $q+1$ regular graph with n vertices.

$$A : L^2(X) \to L^2(X)$$
$$Af(x) = \frac{1}{q+1}\sum_{y\sim x} f(y).$$

$\lambda_0 = 1 > \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n-1}$ eigenvalues.

Definition 2.1 (Expanders and Ramanujan graphs.)**.** The family of graphs in an expander family if $\max\{|\lambda_i|\} \leq \tau < q+1$ for some τ.

The family is Ramanujan if $\max\{|\lambda_i|\} \leq 2\sqrt{q}$.

Ramanujan graphs were constructed by Lubotzky-Philips-Sarnak in the 80's.

Theorem 2.2. *Expander graphs have diameter that is bounded by* $\leq C_\tau \log_q(n)$.

(LPS) Ramaujan graphs have diameter that is bounded by $2\log_q(n)$.

Theorem 2.3 (Sardari, Lubetzky-Peres)**.** *If X is (almost a) Ramanujan graph, then for every $x \in X$, for all but $o(n)$ of $y \in X$, it holds that*

$$d(x,y) \leq (2+\epsilon)\log_q(n).$$

The probelms is that proving that graphs are almost-Ramanujan is very hard.

- The fact that random graphs are almost-Ramanujan is called Alon's conjecture. It was proven by Friedman (see also Bordenave). It is very hard.
- For other "natural" graphs our knowledge is very lacking. But following the "Bourgain-Gamburd Machine", we know that many of them are expanders.

Definition 2.4. +Theorem. TFAE, and then a sequence of graphs satisfies the SX-density property.

- Spectral definition: associate with each exeptional λ_i a p_i according to $\lambda_i = \pm\left(q^{1/p_i} + q^{1-1/p_i}\right)$. For every $\epsilon > 0$, $p > 2$ it holds that
$$\#\left\{i : p_i \geq p\right\} \ll_\epsilon n^{2/p+\epsilon}.$$
- Geometric definition: let $N(t)$ be the number of non-backtracking pathes of length t starting and ending at the same vertex. Then
$$N\left(2\log_q n\right) \ll_\epsilon n^{2+\epsilon}.$$

Theorem 2.5 (Bordenave-Lacoin, Golubev-K.)**.** *If X satisifes the SX-density property, then all but $o\left(n^2\right)$ of $(x,y) \in X \times X$, it holds that*
$$d(x,y) \leq (2+\epsilon)\log_q(n).$$

The Geometric definition is not too hard to prove in various cases. For example, random $q+1$ regular graphs satisfy (a stronger condition) by a classical result of Broder and Shamir. I will remark that the results puder gave last week are analogs for hyperbolic surfaces for the results of Broder and Shamir, and likewise prove a "density theorem" in this context.

Here is something we can prove:
$$P^1(\mathbb{F}_t) = \left\{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{F}_t^2 \backslash \{0,0\}\right\} / \mathbb{F}_t^\times \cong SL_2(\mathbb{F}_t) / \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$$
Here is one of our results.

Theorem 2.6. *S-X for Schreier graphs of SL_2:*

(1) *Schreier* $\left(S \mod t, P^1(\mathbb{F}_t)\right)$ *for* $S = \left\{\begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}\right\}$.

(2) *Schreier* $\left(S \mod t, P^1(\mathbb{F}_t)\right)$ *for random generators.*

3. Some Automorphic Forms

Let $G = SL_n(\mathbb{R})$, $\Gamma = SL_n(\mathbb{Z})$, $\Gamma_n = \ker\left(\Gamma \to_{\mod q} SL_n(\mathbb{F}_q)\right)$. The mod q map is onto by the strong approximation theorem.

The following is a theroem of Sarnak:

Theorem 3.1 (Sarnak,2015, "Optimal Almost-Lifting/ Strong Approximation")**.** *Then for every $\epsilon > 0$, as $q \to \infty$, for a set $Y \subset SL_2(\mathbb{F})$ of size $Y > (1 - o(1))\, SL_2(\mathbb{F}_q)$, for every $x \in Y$ there exists $\gamma \in SL_2(\mathbb{Z})$ of size $\|\gamma\|_\infty \leq q^{3/2+\epsilon}$ such that $\pi_q(\gamma) = x$.*

$\|\|_\infty$- infinity norm on the coordinates. From now on- a set of size $(1 - o(1))$ of the full set - almost every.

Remark: the size of $SL_2(\mathbb{F}_q)$ is $(1+o(1))\,q^3$ and $|\{\gamma \in SL_2(\mathbb{Z}) : \|\gamma\|_\infty \leq T\}| \asymp T^2$ (this is not too hard to show, but also follows from a classical result of Duke, Rudnick, Sarnak). Therefore the exponent $3/2$ is optimal.

Conjecture 3.2. *Let $\pi_q : SL_n(\mathbb{Z}) \to SL_n(\mathbb{F}_q)$ be the mod q map. Then as $q \to \infty$, for almost every $\epsilon > 0$, for almost-every $x \in SL_n(\mathbb{F}_q)$ there exists $\gamma \in SL_2(\mathbb{Z})$ of size $\|\gamma\|_\infty \leq T^{(n^2-1)/(n^2-n)+\epsilon}$ such that $\pi_q(\gamma) = x$.*

We can reduce this question to a counting argument, and ($\tilde{}$) to a spectral question.