Combinatorial Width Parameters for 3-Dimensional Manifolds

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ETH Geometry Zoom Seminar, May 6, 2020

Kristóf Huszár, Jonathan Spreer and Uli Wagner
On the Treewidth of Triangulated 3-Manifolds
Journal of Computational Geometry, **10**(2):70–98, 2019
doi:10.20382/jogc.v10i2a5

Extended abstract: SoCG 2018, Budapest, Hungary

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3-Manifold Triangulations with Small Treewidth
Proceedings of the 35th International Symposium on Computational Geometry (SoCG 2019), Portland, OR, USA, volume 129 of LIPIcs, pages 44:1–44:20. Schloss Dagstuhl, Leibniz-Zentrum für Informatik, 2019
doi:10.4230/LIPIcs.SoCG.2019.44

CG Week 2020 – ETH Zürich, June 23–26, 2020

`https://socg20.inf.ethz.ch`

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1. 3-Manifolds

Triangulations

Graphs

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2. Context & Motivation

Algorithms and computation

Preview

Combinatorial **Width Parameters** for
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1. 3-Manifolds	Triangulations · Graphs
2. Context & Motivation	Algorithms and computation
3. Width Parameters	Treewidth (combinat.) · Heegaard genus (topological)

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4. Overview of Results

Triangulations | Graphs

Algorithms and computation

Treewidth (combinat.) | Heegaard genus (topological)

Quantitative relationship

Combinatorial Width Parameters for 3-Dimensional Manifolds

1. 3-Manifolds — Triangulations | Graphs

2. Context & Motivation — Algorithms and computation

3. Width Parameters — Treewidth (combinat.) | Heegaard genus (topological) — **Quantitative relationship**

4. Overview of Results

5. Methods and Tools — Generalized Heegaard splittings

3-Manifolds: Triangulations and Their Dual Graphs

In this talk only **compact** and **orientable** **3-manifolds** are considered.

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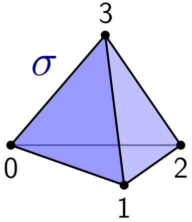

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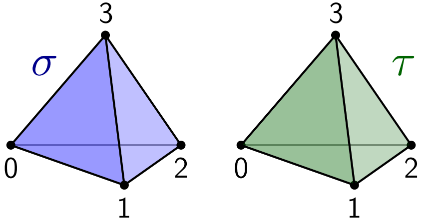

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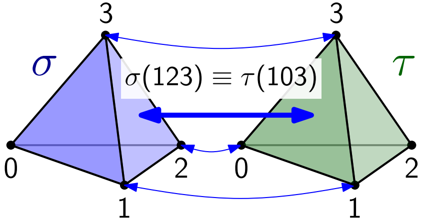

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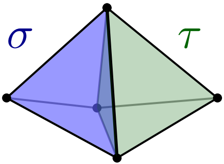

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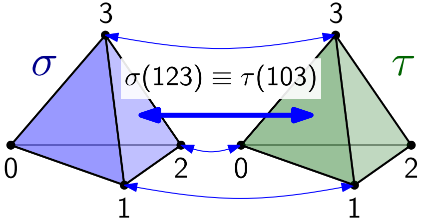

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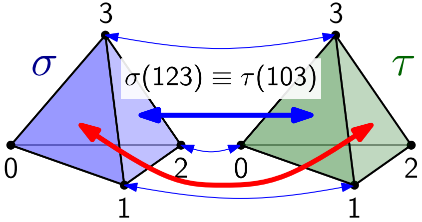

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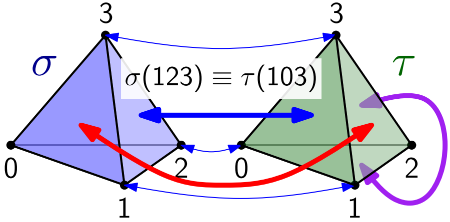

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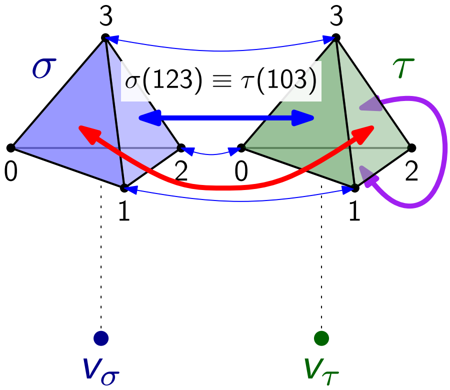
Dual (face pairing) graph

$\Gamma(\mathcal{T})$	**vertices**: tetrahedra of $\mathcal{T}$ **edges**: face gluings

(multigraph, vertex degrees ≤ 4)



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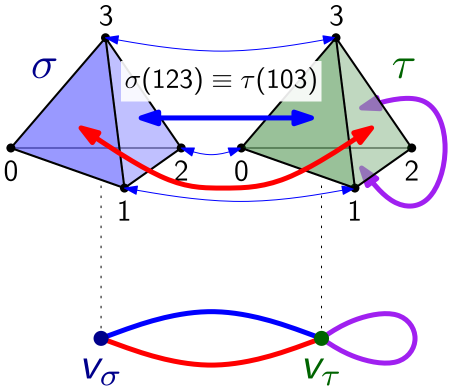

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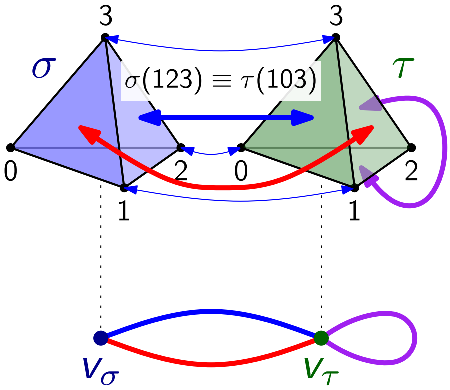

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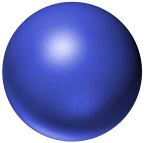
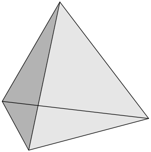
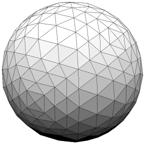
We consider two 3-manifolds the same if they are **homeomorphic**.

Context: Algorithmic Study of 3-Manifolds

Any 3-manifold has infinitely many combinatorially distinct triangulations







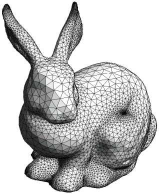


Image Credits: Wikimedia Commons (tetrahedron), Eeo Jun (triangulated sphere) and Daniel Rypl (Stanford bunny)

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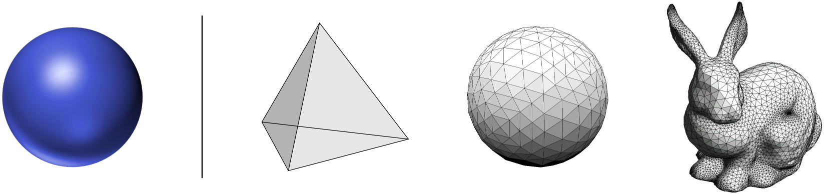

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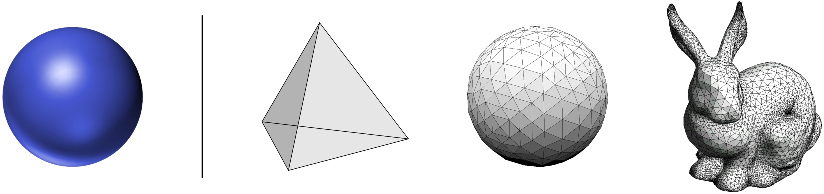

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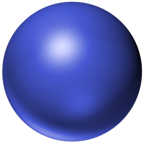

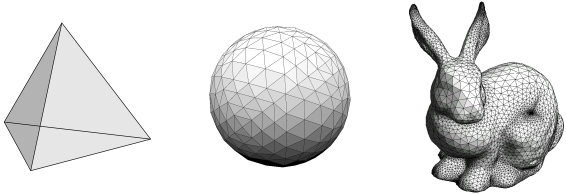

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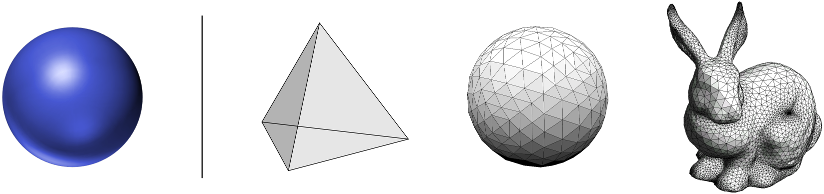

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Thus, in practice, the **HP** is approached via **computable invariants**.

Motivation: Fixed-Parameter Tractable Algorithms

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$\mathrm{tw}(\Gamma(\mathcal{T})) = t$ fixed $\Downarrow$ $O(n)$

Question: Given a 3-manifold $\mathcal{M}$, how **small** $\mathrm{tw}(\Gamma(\mathcal{T}))$ can be?

(This has been asked by several people, incl. at an Oberwolfach meeting in 2015.)

The Treewidth of a Graph

The **treewidth** $\mathrm{tw}(G)$ quantifies the similarity of G to any tree.

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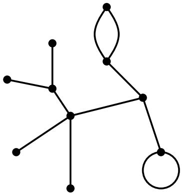
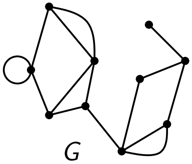
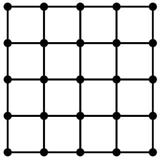


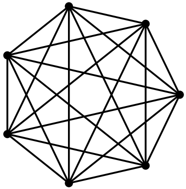
tw(tree) $= 1$



tw(G) $= 2$



tw($k \times k$-grid) $= k$



tw(K_n) $= n - 1$

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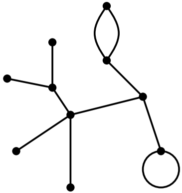

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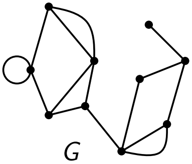

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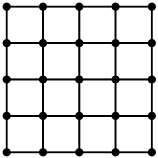

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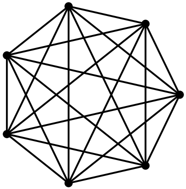

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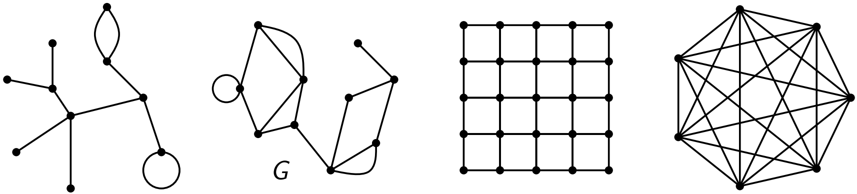

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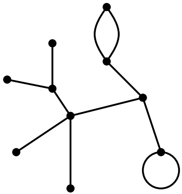

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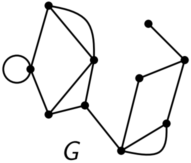

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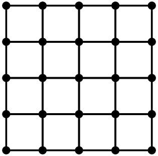

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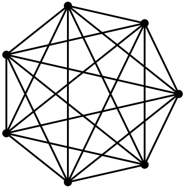

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- Cornerstone of **parametrized complexity theory** (since the 1970s).
- A **zoo of width parameters for graphs**: cutwidth, pathwidth, etc.

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This way, any non-negative graph parameter yields a *topological invariant* for 3-manifolds. We call these **combinatorial width parameters**.

Examples. *pathwidth* $\operatorname{pw}(\mathcal{M})$, *cutwidth* $\operatorname{cw}(\mathcal{M})$, *congestion* $\operatorname{cng}(\mathcal{M}), \ldots$

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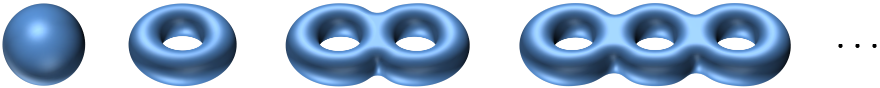
Motif. Understand the **quantitative relation** between treewidth & co. and classical topological invariants of 3-manifolds, e.g., *Heegaard genus*, *hyperbolic volume*, *Scharlemann–Thompson width*, etc.

The Heegaard Genus of a 3-Manifold

A **handlebody of genus g** is a solid body with g holes.



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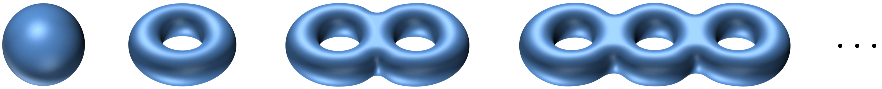

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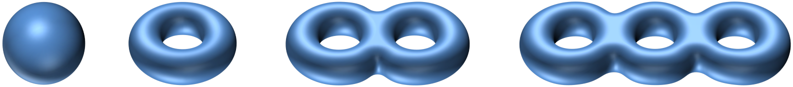

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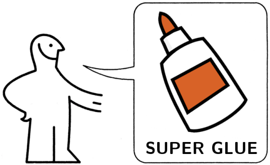

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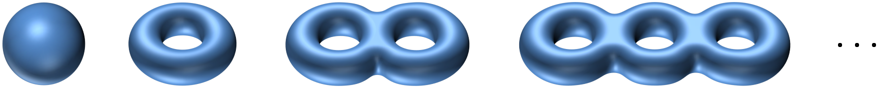

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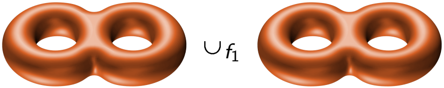
$\mathcal{M} =$ 

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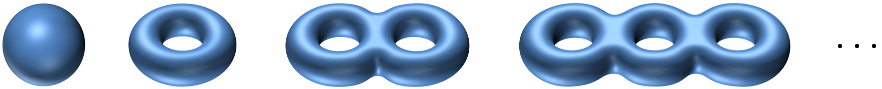

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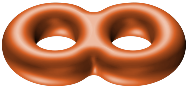
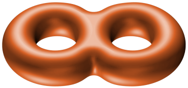
$\mathcal{M} =$  $\cup_{f_1}$ 

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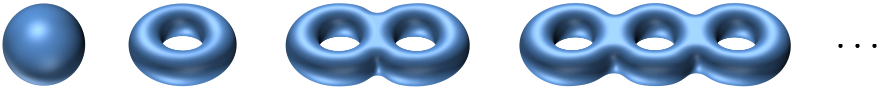

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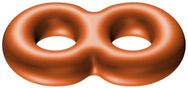
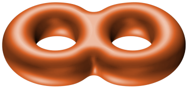
$\mathcal{M} =$  $\cup_{f_1}$ 

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The **Heegaard genus** $\mathfrak{g}(\mathcal{M})$ is the minimum genus of any Heegaard splitting of $\mathcal{M}$.

Results, I. 3-Manifolds with Large Treewidth

Theorem 1 (H–Spreer–Wagner, 2019**).** Let $\mathcal{M}$ be closed, orientable, irreducible, non-Haken. Then its Heegaard genus and treewidth satisfy

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Here $\mathrm{tw}\,(K)$ denotes the minimum treewidth of any *diagram* D of K.

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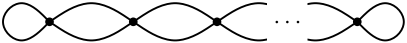

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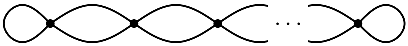

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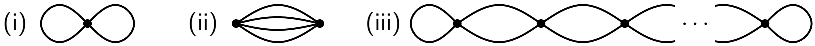

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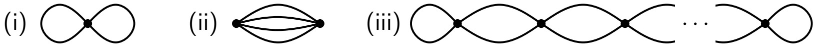

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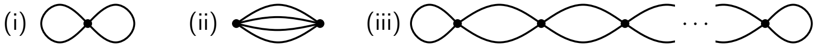

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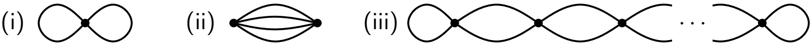

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Corollary Minimal triangulations are *not* always of minimum treewidth.

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Scharlemann–Thompson, 1994; Scharlemann–Schultens–Saito, 2016

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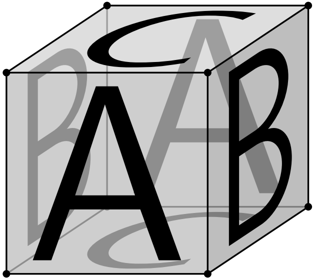

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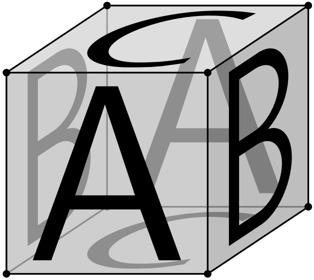

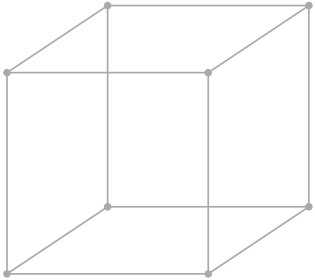

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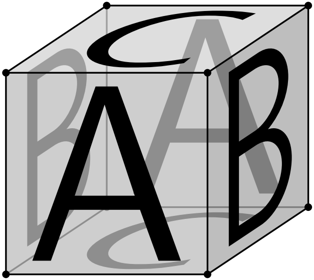
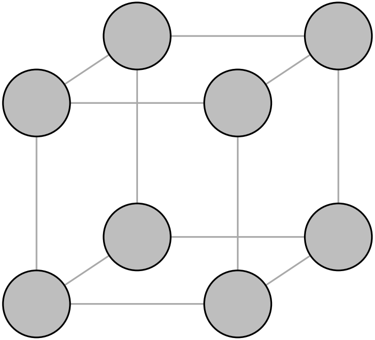


$g(\partial) = 0$



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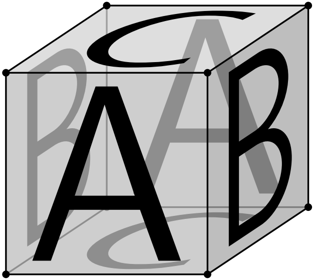

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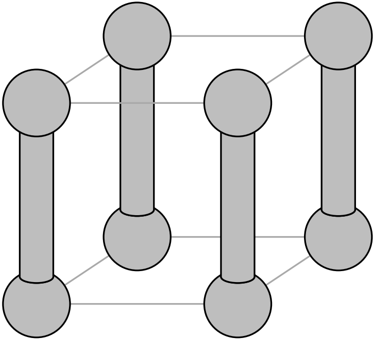

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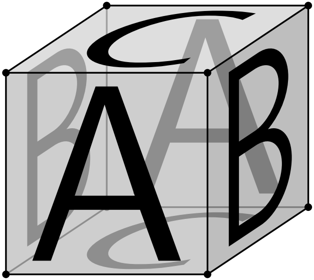

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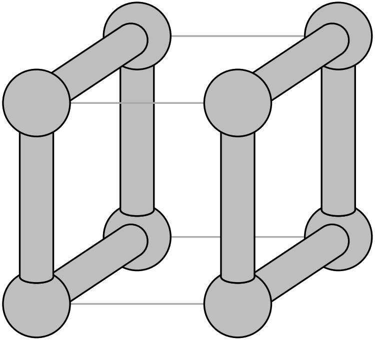

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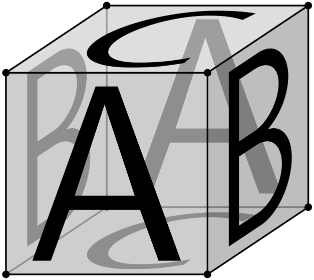

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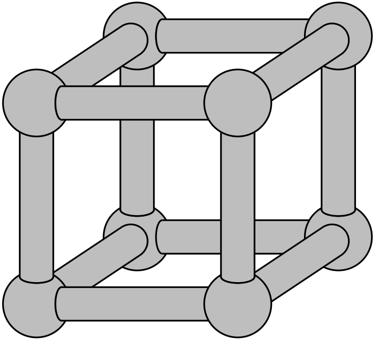

$g(\partial)$:   0   1   2   3

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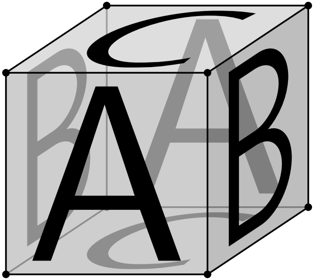

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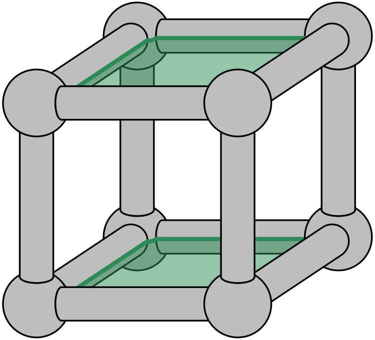

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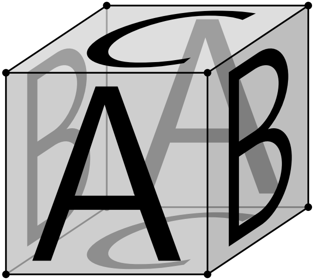

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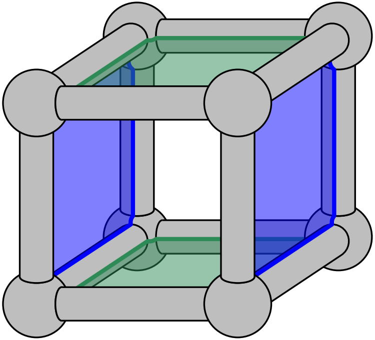

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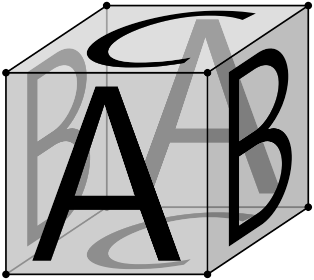

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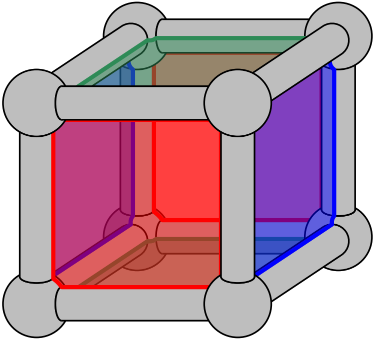

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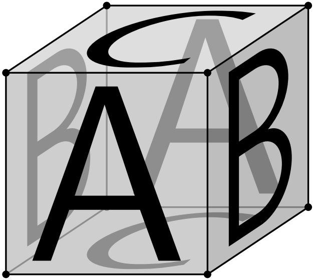
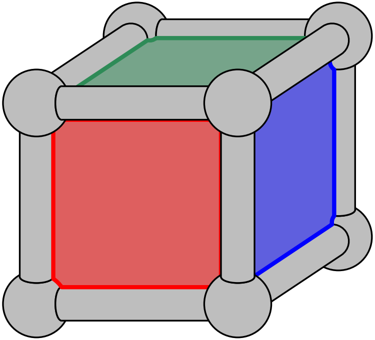
$\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$



$\partial \ = \emptyset$



$g(\partial)$: 0 1 2 3 2 1 0

Generalized Heegaard Splittings: A Case Study

Scharlemann–Thompson, 1994; Scharlemann–Schultens–Saito, 2016

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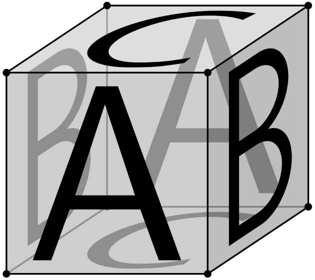

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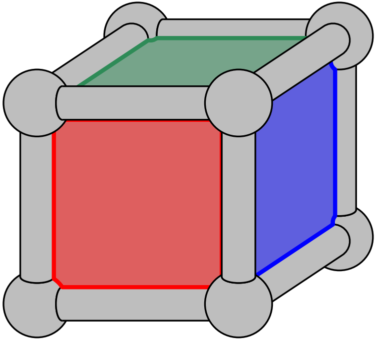

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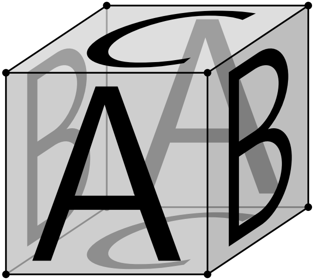

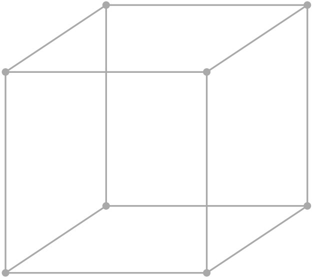

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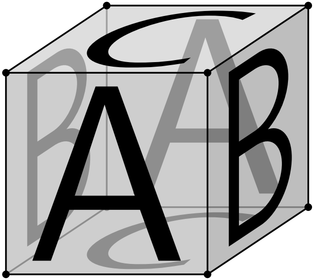
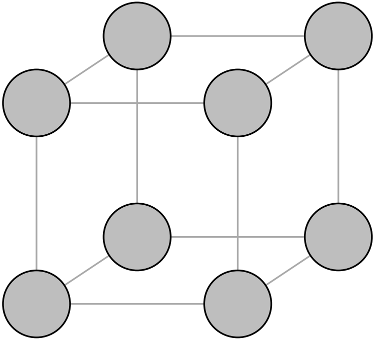


$g(\partial) = 0$



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0

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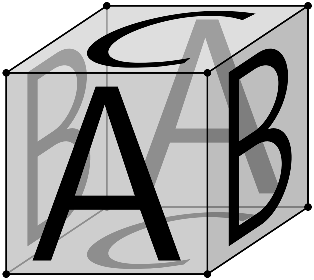
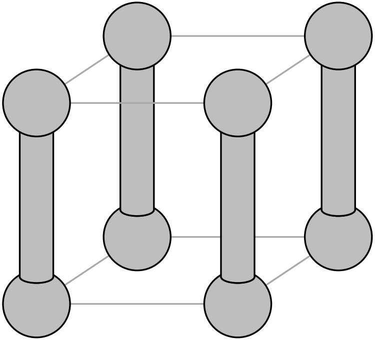


$g(\partial) = 1$



$g(\partial)$: 0 1 2 3 2 1 0 $\rightsquigarrow$ Heegaard splitting of genus 3

0 1

Generalized Heegaard Splittings: A Case Study

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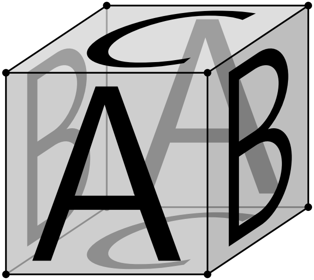
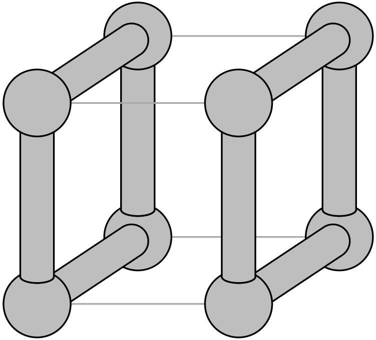


$g(\partial) = 2$



$g(\partial)$: 0 1 2 3 2 1 0 $\rightsquigarrow$ Heegaard splitting of genus 3

0 1 2

Scharlemann–Thompson, 1994; Scharlemann–Schultens–Saito, 2016

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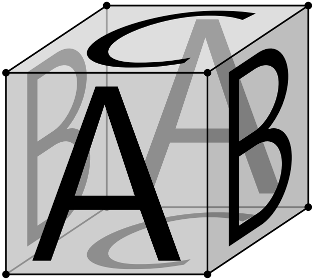

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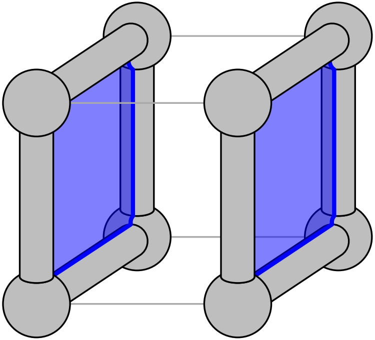

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0 1 2 $\genfrac{}{}{0pt}{}{1}{1}$

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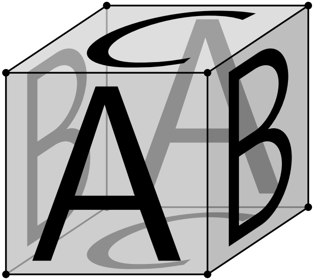

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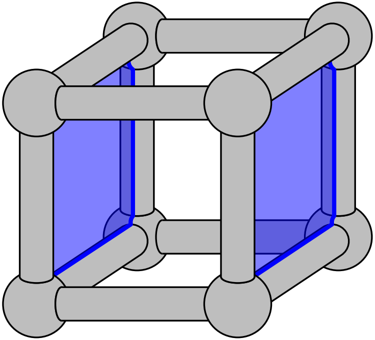

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0 1 2 $\frac{1}{1}$ 2

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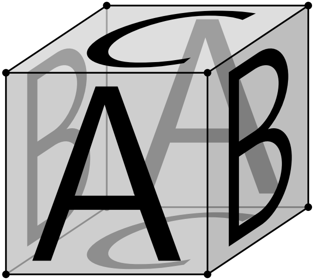

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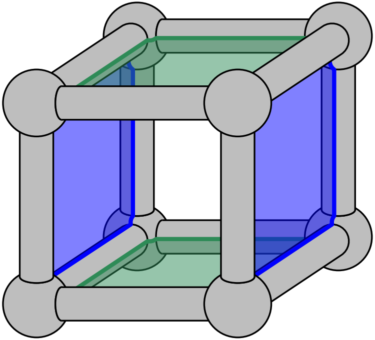

$g(\partial)$: 0 1 2 3 2 1 0 $\rightsquigarrow$ Heegaard splitting of genus 3

0 1 2 $\begin{smallmatrix}1\\1\end{smallmatrix}$ 2 1

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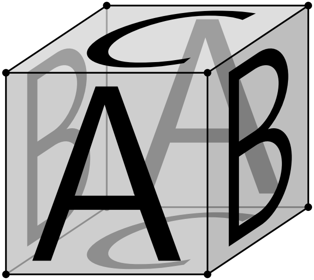
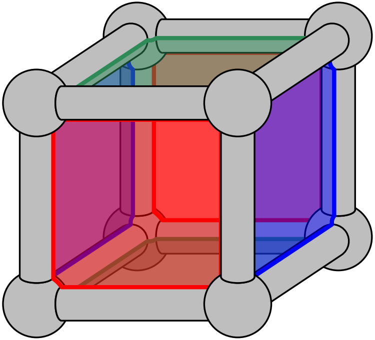


$g(\partial) = 0$



$g(\partial)$: 0 1 2 $\color{red}{3}$ 2 1 0 $\rightsquigarrow$ Heegaard splitting of genus 3

0 1 2 $\genfrac{}{}{0pt}{}{1}{1}$ 2 1 0

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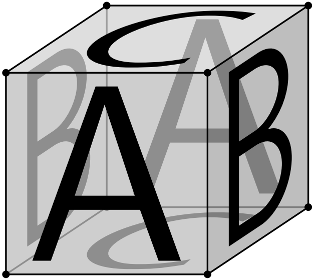
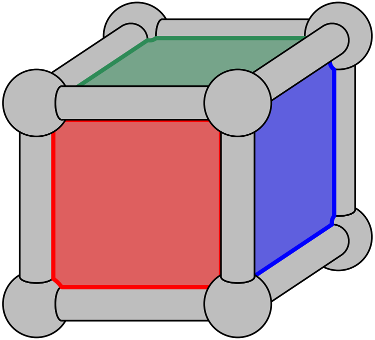


$\partial = \emptyset$



$g(\partial)$: 0 1 2 3 2 1 0 $\rightsquigarrow$ Heegaard splitting of genus 3

0 1 2 $\begin{smallmatrix}1\\1\end{smallmatrix}$ 2 1 0

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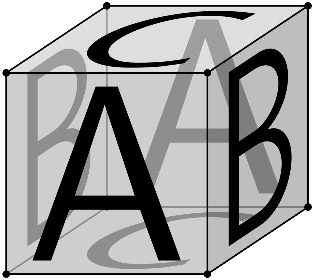

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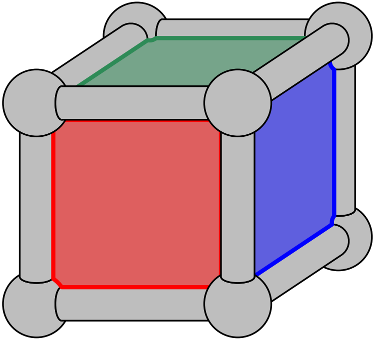

$g(\partial)$: $0 \quad 1 \quad 2 \quad 3 \quad 2 \quad 1 \quad 0 \quad \rightsquigarrow$ Heegaard splitting of genus 3

$0 \quad 1 \quad 2 \quad {}^{1}_{1} \quad 2 \quad 1 \quad 0 \quad \rightsquigarrow$ gen. Heegaard splitting of *width* $(2, 2)$

Generalized Heegaard Splittings: A Case Study

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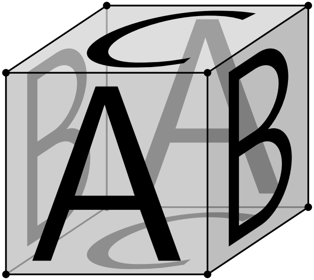
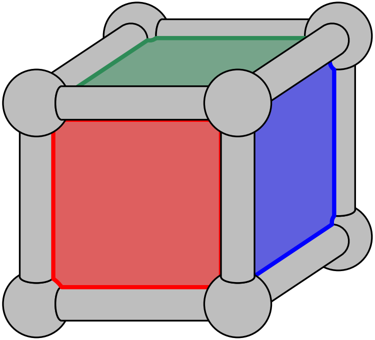


$\partial \;= \emptyset$



$g(\partial)$: 0 1 2 3 2 1 0 $\rightsquigarrow$ Heegaard splitting of genus 3

0 1 2 $\genfrac{}{}{0pt}{}{1}{1}$ 2 1 0 $\rightsquigarrow$ gen. Heegaard splitting of *width* $(2, 2)$

Gen. Heegaard splitting of *lexicographically minimal* width: **thin position**

Proof of Theorem 1

Theorem 1 (H–Spreer–Wagner, 2019**).** Let $\mathcal{M}$ be closed, orientable, irreducible, non-Haken. Then we have $\mathfrak{g}(\mathcal{M}) \leqslant 18\,(\mathrm{tw}\,(\mathcal{M}) + 1)$.

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Strategy Triangulation $\mathcal{T}$ of $\mathcal{M} \rightsquigarrow$ Heegaard splitting with small genus.

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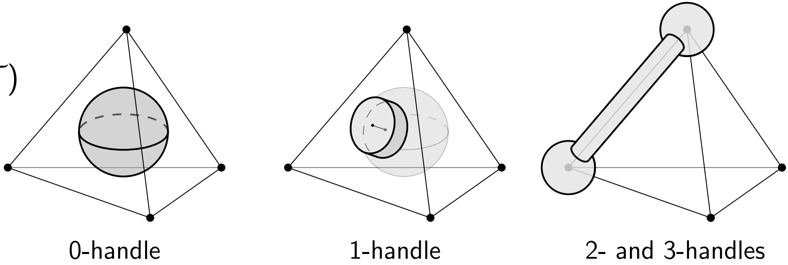
chd($\mathcal{T}$)



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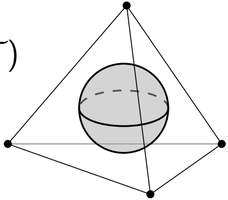
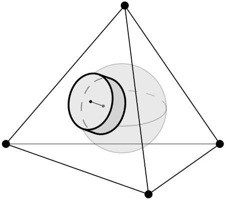
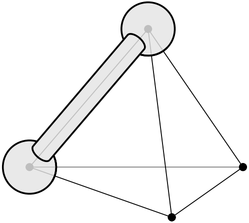


0-handle



1-handle



2- and 3-handles

$\mathcal{H}_1 = \{$0-handles$\} \cup \{$1-handles$\}$
$\mathcal{H}_2 = \{$2-handles$\} \cup \{$3-handles$\}$

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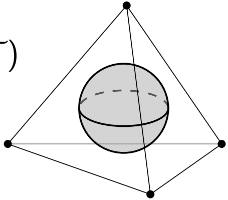

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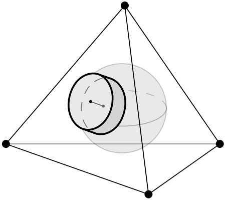

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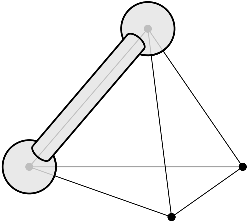

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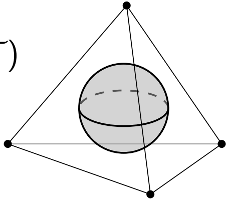

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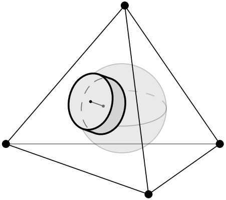

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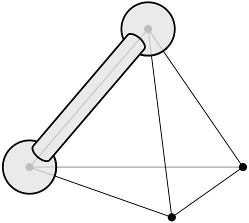

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Problem If $\mathcal{T}$ has n tetrahedra, then $g(\mathcal{S}) = n + 1 \Rightarrow$ Too large!

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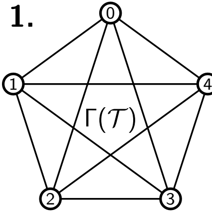
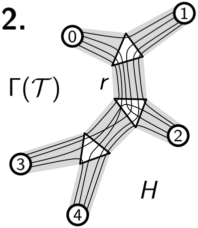
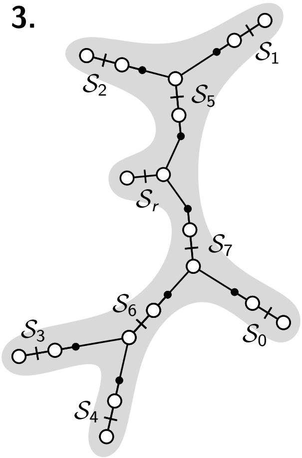


2.



3.



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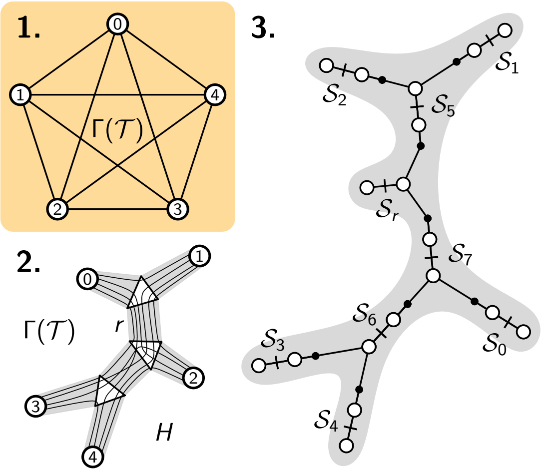

1. $\mathcal{T}$: $\mathrm{tw}\,(\Gamma(\mathcal{T})) = \mathrm{tw}\,(\mathcal{M})$

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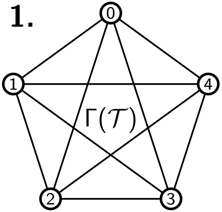

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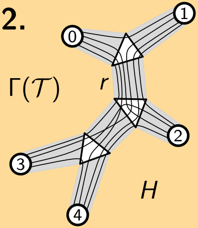

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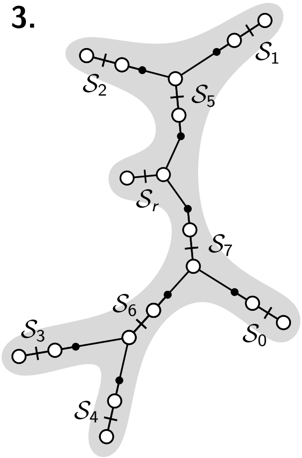

1. $\mathcal{T}$: $\mathrm{tw}\,(\Gamma(\mathcal{T})) = \mathrm{tw}\,(\mathcal{M})$

$\Downarrow$ [Bienstock 1990]

2. Low-congestion layout

Proof of Theorem 1

Theorem 1 (H–Spreer–Wagner, 2019**).** Let $\mathcal{M}$ be closed, orientable, irreducible, non-Haken. Then we have $\mathfrak{g}(\mathcal{M}) \leqslant 18\,(\mathrm{tw}\,(\mathcal{M}) + 1)$.

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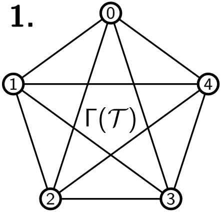

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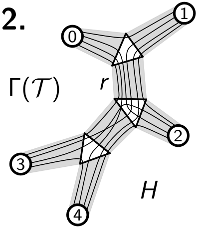

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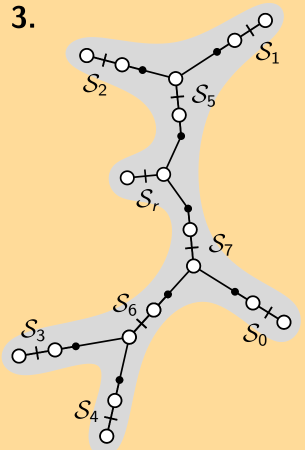

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⇓ [Bienstock 1990]

2. Low-congestion layout

⇓ gives a template for a

3. Gen. Heegaard splitting

$g(\mathcal{S}_i) \leqslant 18\,(\mathrm{tw}\,(\Gamma(\mathcal{T})) + 1)$

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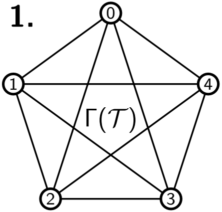

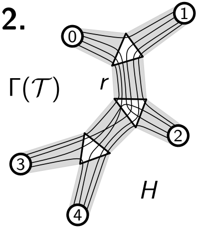

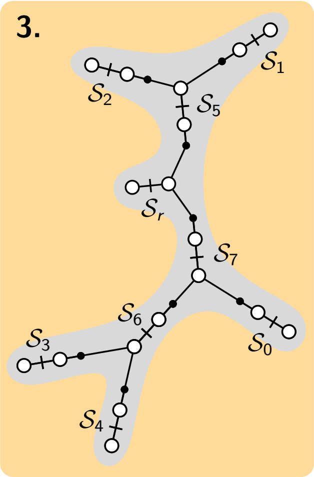

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$\Downarrow$ [Bienstock 1990]

2. Low-congestion layout

$\Downarrow$ gives a template for a

3. Gen. Heegaard splitting

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[Scharlemann–Thompson '94]

4. Bring into *thin position*

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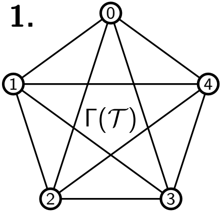

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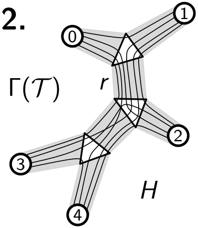

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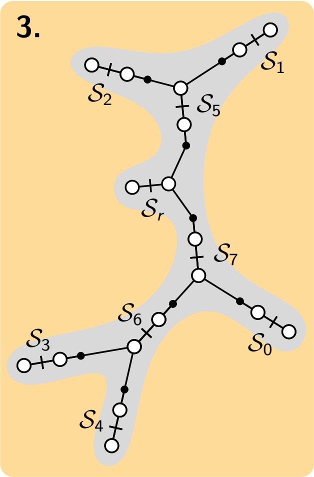

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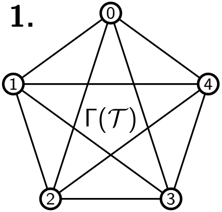

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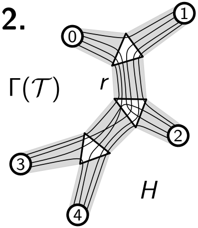

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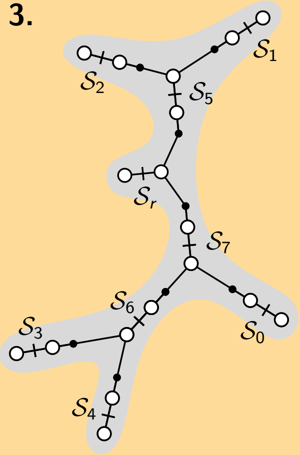

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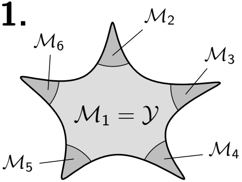

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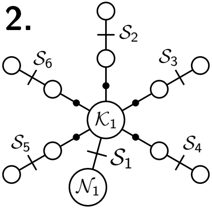
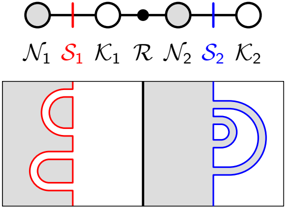
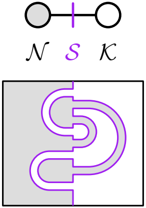




3. $\rightsquigarrow$



0. Hyperbolic 3-manifold $\mathcal{M}$

$\Downarrow$ [Kazhdan–Margulis 1968]

1. Thick-thin decomposition

$\Downarrow$ [Jørgensen–Thurston 1979] [Kobayashi–Rieck 2011]

2. Gen. Heegaard splitting

$\Downarrow$ [Schultens 1993] [Bachman et al. 2017]

3. Heegaard splitting

$\Downarrow$ **Theorem 2**

4. Triangulation $\mathcal{T}$ □

Thank you for your attention!