Multipolytopes, volume polynomials, duality algebras

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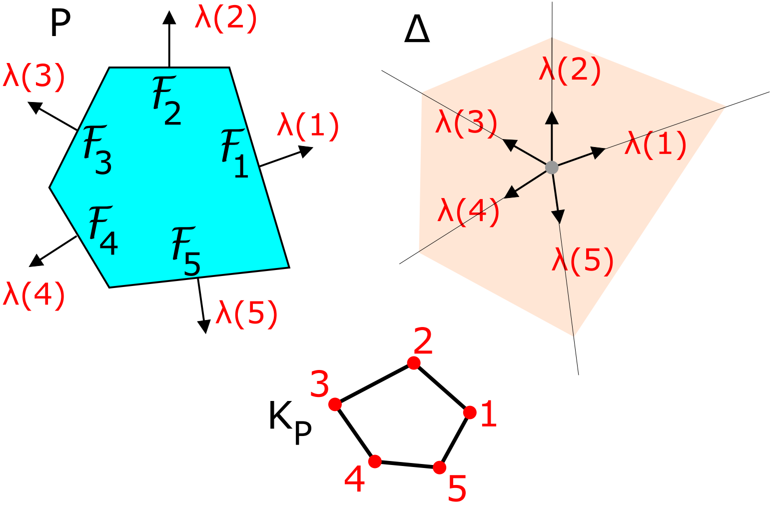
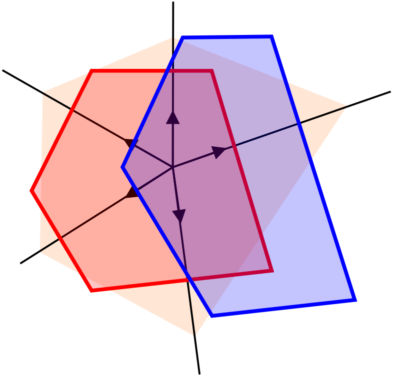
- Normal fan has information on normal vectors and combinatorics, but forgets support parameters.

Polytopes and fans



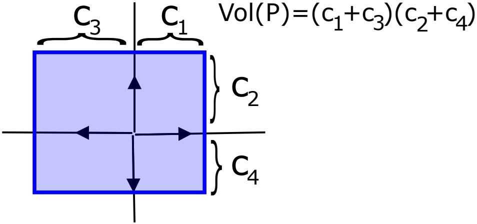
Analogous polytopes

- $\mathrm{Poly}(\Delta)$: space of all polytopes with normal fan Δ;
- $\mathrm{Poly}(\Delta) \subset \mathbb{R}^m$, since every polytope $P \in \mathrm{Poly}(\Delta)$ is encoded by its support parameters $(c_1, \ldots, c_m)$. Support parameters are natural coordinates on $\mathrm{Poly}(\Delta)$.



Volume polynomial

- $V_\Delta\colon \operatorname{Poly}(\Delta) \to \mathbb{R}$, $P \mapsto \operatorname{Vol}(P)$: the volume function.
- Claim (Khovanskii–Pukhlikov'92): V_Δ is a homogeneous polynomial, $V_\Delta \in \mathbb{R}[c_1, \ldots, c_m]_n$. It is called the volume polynomial of Δ.



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- Claim: $A^* = \mathcal{D}^* / \operatorname{Ann}\Psi$ is a Poincare duality algebra (i.e. $A^n \cong \mathbb{R}$, $A^j \otimes A^{n-j} \xrightarrow{\times} A^n$ — non-degenerate).

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- In particular, $\dim A^j = \dim A^{n-j}$.

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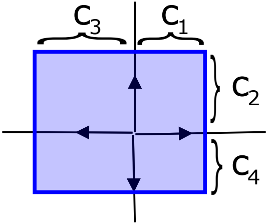


$$V=(c_1+c_3)(c_2+c_4)$$

$$A=R[\partial_1,\partial_2,\partial_3,\partial_4] \big/ \partial_1\partial_3, \partial_2\partial_4, \partial_1-\partial_3, \partial_2-\partial_4$$

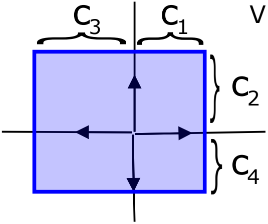
$A^0= R^1$ $\quad A^1= R^2$ $\quad A^2= R^1$

Structure of duality algebra of a fan

Theorem (Pukhlikov–Khovanskii'92, Timorin'03)

1. If Δ corresponds to smooth toric variety X, then $\mathcal{A}^*(\Delta) \cong H^{2*}(X;\mathbb{R})$.
2. $\mathcal{A}^*(\Delta)$ is a Lefschetz algebra.
3. $\mathcal{A}^*(\Delta) \cong \mathbb{R}[K]/(l.s.o.p.)$, where K is the underlying simplicial sphere of Δ (sphere dual to P), $\mathbb{R}[K]$: its Stanley–Reisner algebra, and $(l.s.o.p.)$: the linear system of parameters determined by the fan Δ.
4. $\dim \mathcal{A}^j(\Delta) = h_j$, h-numbers of K.

Example



$V = (c_1 + c_3)(c_2 + c_4)$

Stanley-Reisner rel's correspond to nonintersecting collections of facets

$A(\Delta) = R[\partial_1, \partial_2, \partial_3, \partial_4] / \partial_1\partial_3, \partial_2\partial_4, \partial_1 - \partial_3, \partial_2 - \partial_4$

Linear rel's

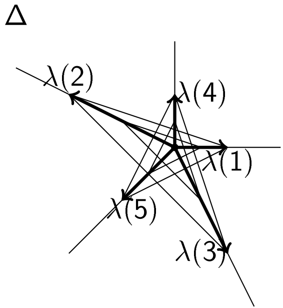
$= H^*(\mathbb{C}P^1 \times \mathbb{C}P^1)$

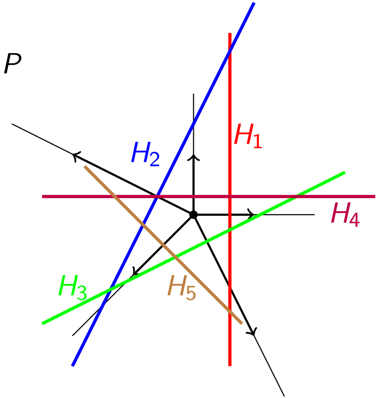
Combinatorial consequences

Recall that $\sum_{j=0}^{n} h_j t^{n-j} = \sum_{j=0}^{n} f_{j-1}(t-1)^{n-j}$, where $f_{j-1} = \#$ of $(j-1)$-simplices of $K = \#$ of $(n-j)$-faces of P.

- Poincare duality implies $h_j = h_{n-j}$ (Dehn–Sommerville rel's)
- Lefschetz property implies $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{[n/2]}$ (Necessity in g-theorem proved by Stanley).

Multi-fans and multi-polytopes





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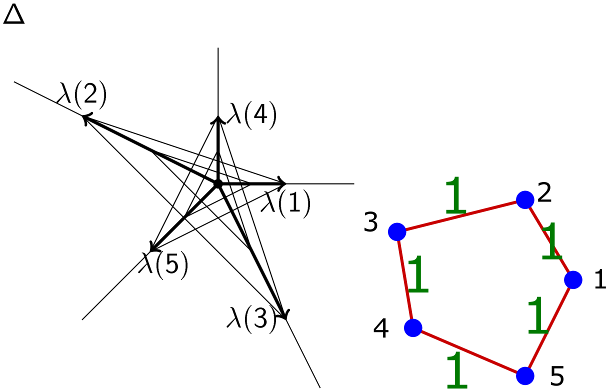
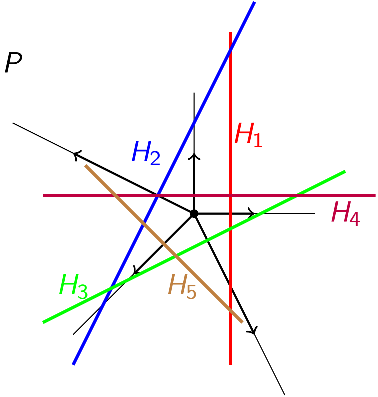


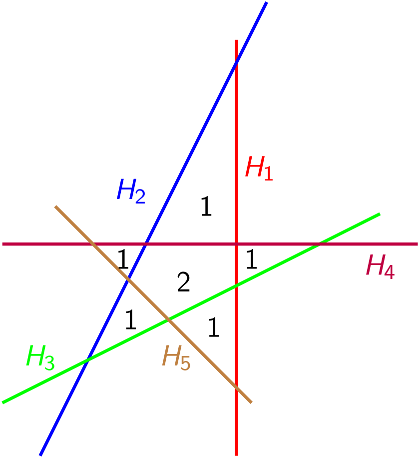
Multi-polytope

A multi-polytope based on a multi-fan Δ is a pair $(\Delta, \{H_1, \ldots, H_m\})$ where H_i is an affine hyperplane orthogonal to $\lambda(i)$. We have:

- $H_i = \{x \in \mathbb{R}^n \mid \langle \lambda(i), x \rangle = c_i\}$;
- $(c_1, \ldots, c_m)$: support parameters of mutli-polytope P.
- The space $\mathrm{MPoly}(\Delta)$ of all multi-polytopes based on a given multi-fan Δ can be identified with $\mathbb{R}^m$ (since every multi-polytope on Δ is uniquely determined by its support parameters).

Multi-fans and multi-polytopes





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- Claim (Hattori–Masuda): $V_\Delta(c_1, \ldots, c_m)$ is a homogeneous polynomial, called the volume polynomial of a multi-fan Δ.
- Similar to case of fans we introduce a duality algebra of a multi-fan Δ:

$$\mathcal{A}^*(\Delta) := \mathcal{D}^* / \operatorname{Ann} V_\Delta$$

Results: computational formula

Let $u \in \mathbb{R}^n$ be a generic vector. If $I = \{i_1, \cdots, i_n\} \in K$, then $w(I) \neq 0$ and therefore $\lambda(i_1), \cdots, \lambda(i_n)$ is a basis. Let $(\alpha_{I,1}, \cdots, \alpha_{I,n})$ be the coordinates of u in this basis.

Theorem

$$V_\Delta(c_1, \cdots, c_m) = \sum_{I \in K, |I| = n} \frac{w(I)}{n! |\det(\lambda(I))| \prod_{j=1}^n \alpha_{I,j}} (\alpha_{I,1} c_{i_1} + \cdots + \alpha_{I,n} c_{i_n})^n.$$

This formula generalizes Lawrence's formula for the volume of simple convex polytope.

Results: spheres and manifolds

Theorem

If the underlying simplicial complex K of Δ is a homology sphere, then $\mathcal{A}^*(\Delta) = \mathbb{R}[K]/(l.s.o.p.)$. In particular, $\dim \mathcal{A}^j(\Delta) = h_j(K)$.

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If the underlying simplicial complex K of Δ is an orientable connected homology manifold, then $\mathcal{A}^*(\Delta) = \mathbb{R}[K]/(l.s.o.p.)/I_{NS}$, where I_{NS} is the ideal introduced by Novik–Swartz ('09). In particular, $\dim \mathcal{A}^j(\Delta) = h''_j(K)$.

h''-numbers can be computed by the formula

$$h''_j = h_j + \binom{n}{j}\left(\sum_{k=1}^{j}(-1)^{j-k-1}\dim \widetilde{H}_{k-1}(K;\mathbb{R})\right)$$

for $j < n$, and $h''_n = 1$.

Results: g-theorem for multi-fans fails

Question:

Is $\mathcal{A}^*(\Delta)$ Lefschetz for all multi-fans Δ? This would imply g-conjecture for spheres and generalized g-conjecture (Kalai's conjecture) for manifolds.

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Answer: NO.

Theorem

Any Poincare duality algebra over $\mathbb{R}$ generated in degree 1 is isomorphic to $\mathcal{A}^*(\Delta)$ for some multi-fan Δ.

There exist PDA whose dimensions are not unimodal. Thus g-theorem fails for multi-fans.

Results: torus manifolds

In toric geometry there is a correspondence between toric varieties and rational fans. Hattori and Masuda introduced the correspondence

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Theorem

Under certain restrictions on X the algebra $\mathcal{A}^*(\Delta_X)$ is a subquotient of the cohomology algebra $H^*(X;\mathbb{R})$.

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