Kissing number in non-Euclidean space

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joint work with

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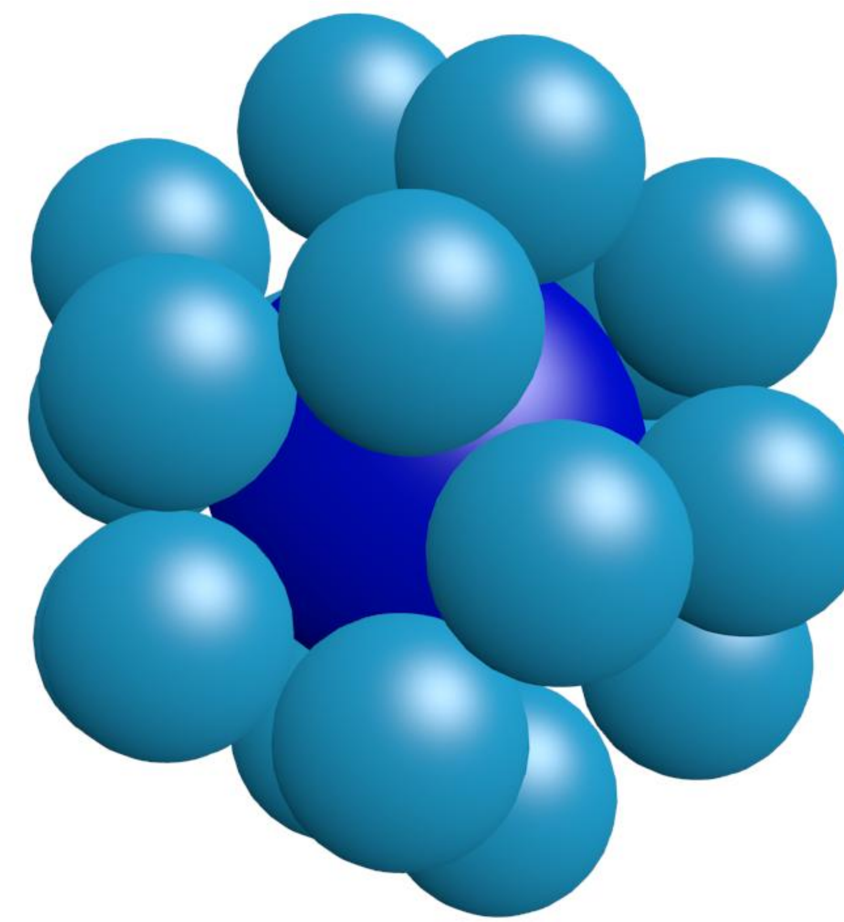
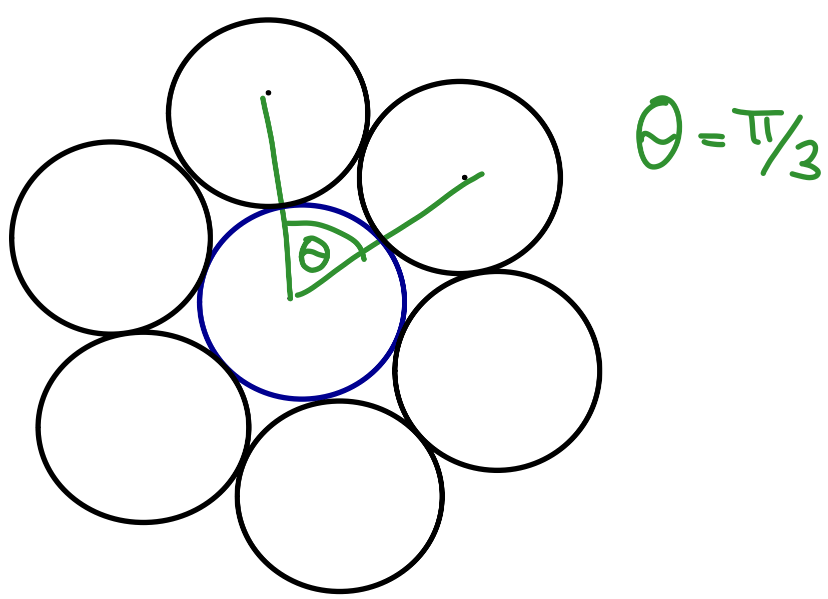


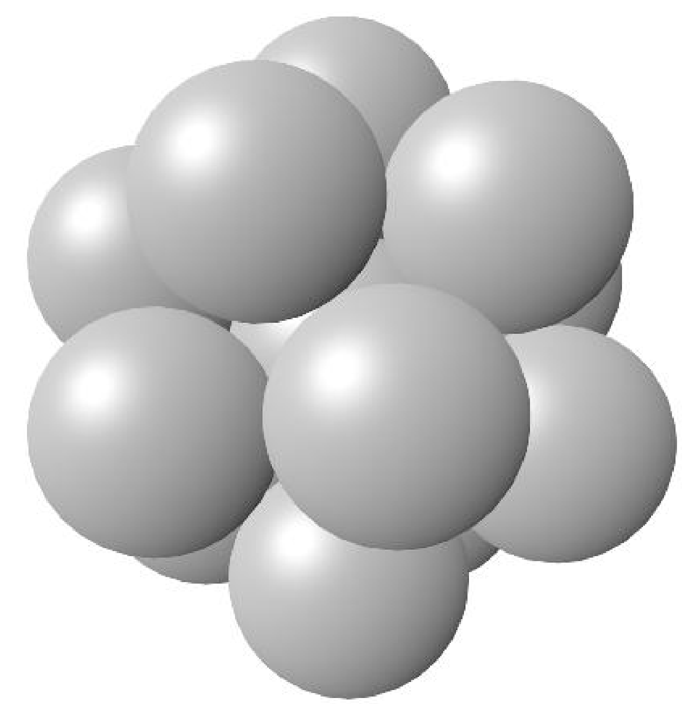
Kissing number in $\mathbb{R}^n$

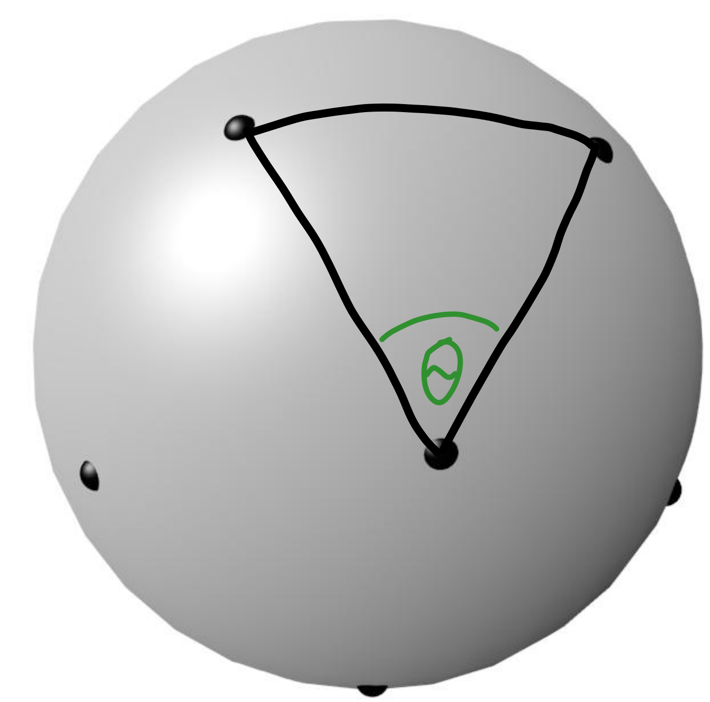
Kissing number of sphere = max number of pairwise non-overlapping unit spheres that can touch simultaneously a central sphere

Dimension 2: kissing number is 6



Dimension 3: kissing number is 12 (Schütte, van der Waerden, 1953)





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in $\mathbb{R}^n$: Just solved for $n = 1, 2, 3, 4, 8, 24$

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What is the kissing number in hyperbolic space $\mathbb{H}^n$ and in spherical space S^n?

Note: Unlike in $\mathbb{R}^n$: the kissing number in $\mathbb{H}^n$ and S^n depends on the radius.

Kissing number in hyperbolic space

$K_H(n,r)$: Kissing number in hyperbolic space $\mathbb{H}^n$ of spheres of radius r

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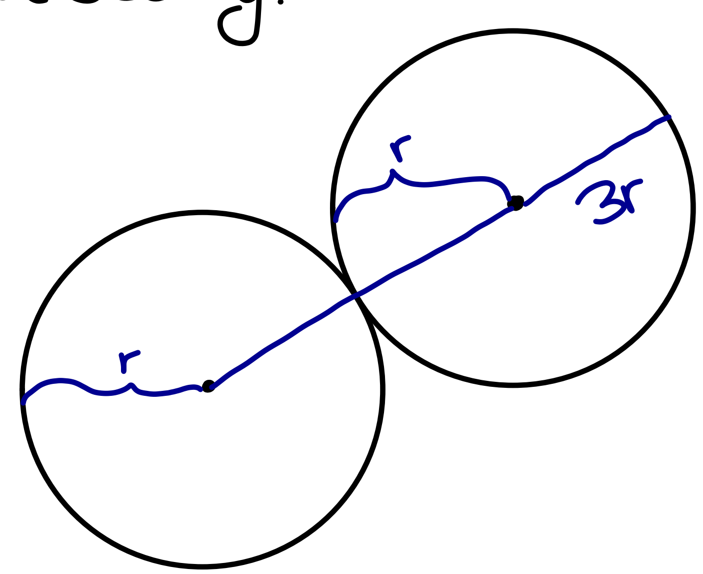
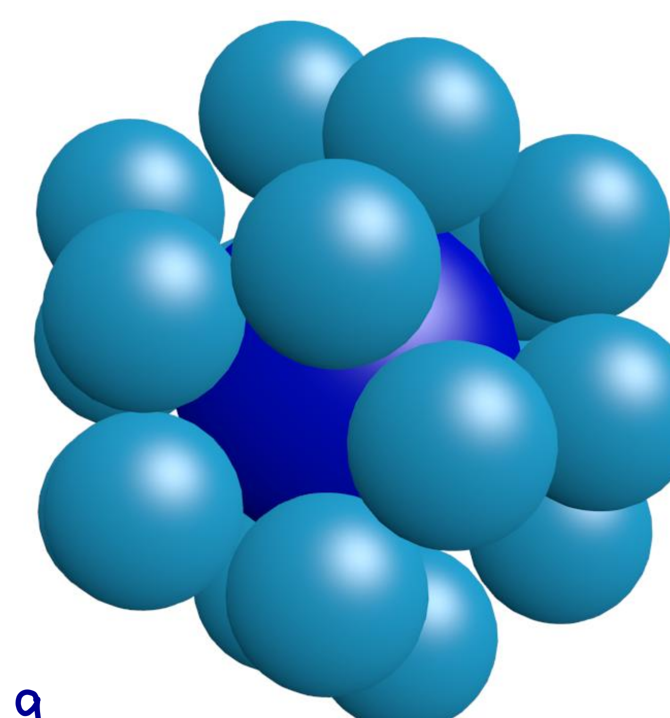
Use purely Euclidean picture of kissing configuration seen in the Poincaré ball model

$\Rightarrow K_H(n,r) =$ max number of spheres of radius $\frac{1}{2}\left(\tanh\frac{3r}{2} - \tanh\frac{r}{2}\right)$ that can simultaneously touch a central sphere of radius $\tanh\frac{r}{2}$ in $\mathbb{R}^n$ without pairwise intersecting.





$r = 0.9$

Kissing number in hyperbolic space

<u>Theorem</u>: For any $n \geq 2$ and a non-negative number $r > 0$

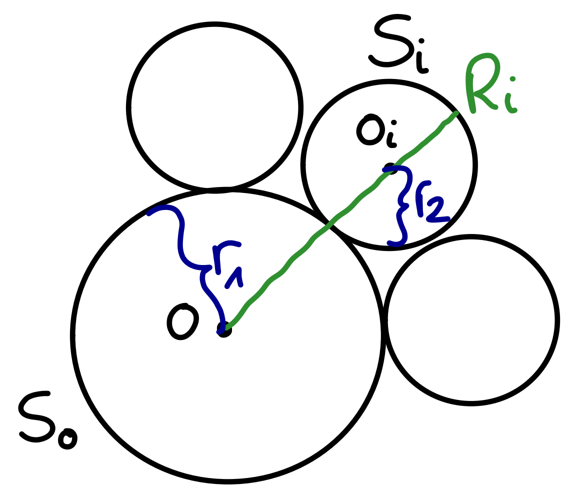
$$K_H(n,r) \leq \frac{2\,B\left(\frac{n-1}{2},\frac{1}{2}\right)}{B\left(\frac{\operatorname{sech}^2 r}{4};\frac{n-1}{2},\frac{1}{2}\right)}$$

where $B(x;y,z) = \int_0^x t^{y-1}(1-t)^{z-1}\,dt$ is the incomplete beta-function

and $B(y,z) = B(1;y,z)$ for all $x \in [0,1]$ and $y,z > 0$.

Upper Bounds for $K_H(n,r)$

Proof:



S_0 central sphere

$S_1, \dots, S_k$ spheres touching S_0

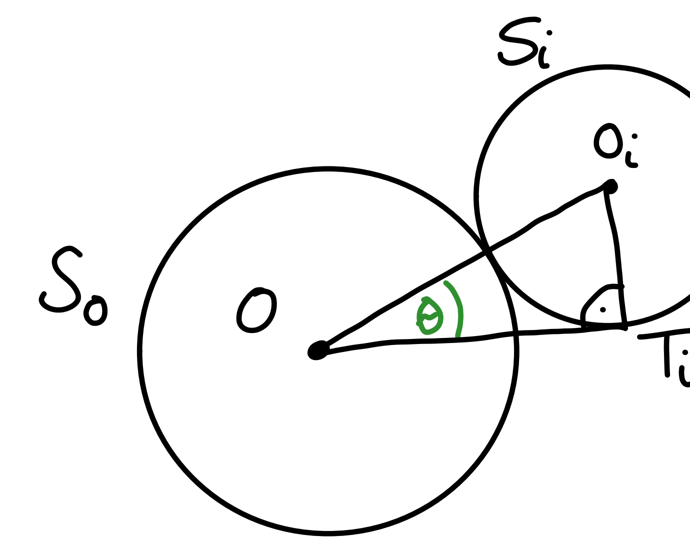
Euclidean distance from O of the ball model to any point at hyperbolic distance r from O equals $\tanh \frac{r}{2}$

$\Rightarrow$ Euclidean radius r_1 of S_0 is $\tanh \frac{r}{2}$

Euclidean radius r_2 of S_i is $\frac{1}{2}\left(\tanh \frac{3r}{2} - \tanh \frac{r}{2}\right)$

Upper Bounds for $K_H(n,r)$

Project S_i to S_0 along the rays emanating from O $\Rightarrow$ obtain spherical cap C_i



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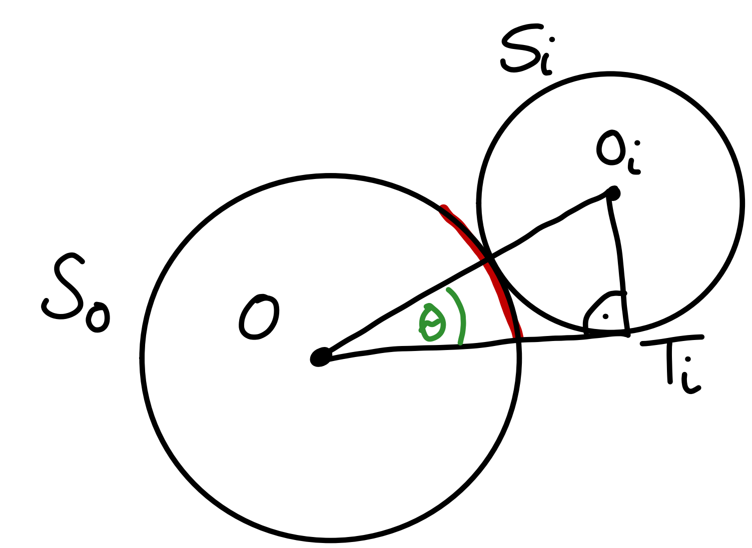

$$K_H(n,r) \cdot \text{Area } C_i = \sum_{i=1}^{k} \text{Area } C_i \leq \text{Area } S_0$$

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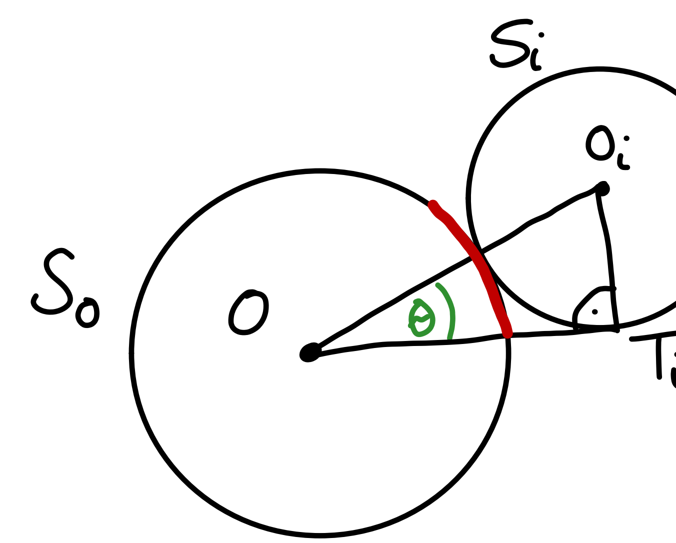

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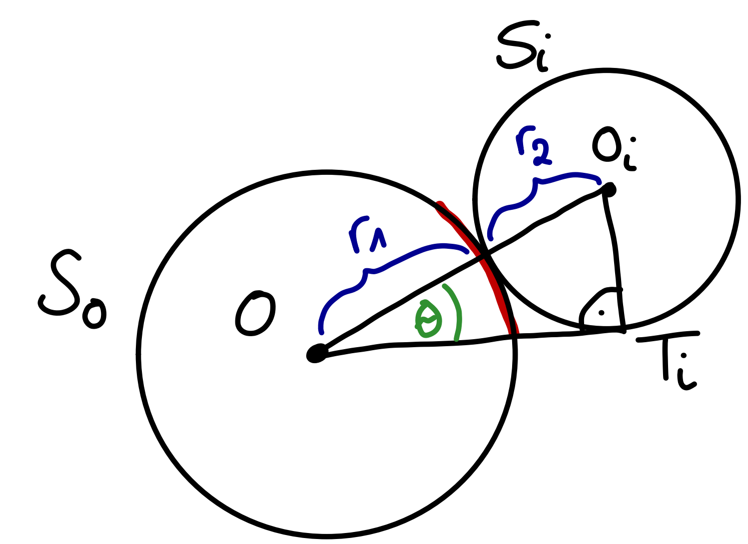

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$$\sin\theta = \frac{r_2}{r_1+r_2}\,,\quad \text{where } r_1 = \tanh\frac{r}{2} \text{ and } r_2 = \frac{1}{2}\left(\tanh\frac{3r}{2} - \tanh\frac{r}{2}\right)$$

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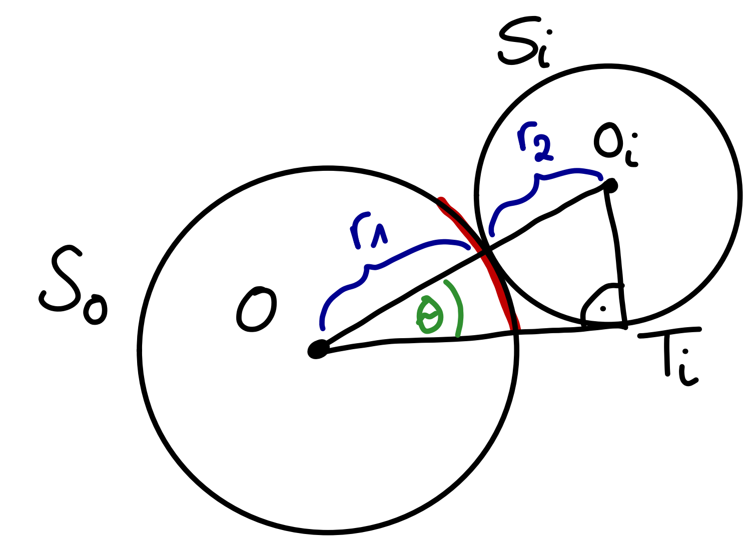

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$$\Rightarrow K_H(n,r) \leq \frac{\text{Area } S_0}{\text{Area } C_i} = \text{Area } S_0 \cdot 2 \cdot \frac{1}{\text{Area } S_0}\,\frac{B(\frac{n-1}{2}, \frac{1}{2})}{B(\frac{\operatorname{sech}^2 r}{4}, \frac{n-1}{2}, \frac{1}{2})} = \frac{2B(\frac{n-1}{2}, \frac{1}{2})}{B(\frac{\operatorname{sech}^2 r}{4}; \frac{n-1}{2}, \frac{1}{2})} \quad \checkmark$$

Lower Bounds for $K_H(n,r)$

Lemma 1) Let $\mathbb{S}^n$ be packed by closed metric balls B_i, $i=1,2,\dots$ of equal angular radius r, and let, and let such packing be maximal.
Then $\mathbb{S}^n$ is covered by closed metric balls B_i', $i=1,2,\dots$ concentric to B_i, of radius $2r$.

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<u>Theorem</u>: For any integer $n\geq 2$ and a non-negative number $r\geq 0$

$$K_H(n,r)\geq \frac{2B(\frac{n-1}{2},\frac{1}{2})}{B(\operatorname{sech}^2 r-\frac{\operatorname{sech}^4 r}{4};\frac{n-1}{2},\frac{1}{2})}$$

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<u>Proof</u>: By lemma 2 the packing of S_0 by $k = K_H(n,r)$ spherical caps C_i is maximal.

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Asymptotic Behaviour

Corollary: Let $n \geq 2$ be a fixed natural number, then

$$\frac{n-1}{2^{n-1}} B\left(\frac{n-1}{2}, \frac{1}{2}\right) e^{(n-1)r} \lesssim K_H(n,r) \lesssim (n-1) B\left(\frac{n-1}{2}, \frac{1}{2}\right) e^{(n-1)r}$$

for $r \to \infty$.

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<u>Corollary</u>: For the kissing number $k_H(n,r)$ of spheres in $\mathbb{H}$ of radius r, $n \geq 2$

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Exponential lower bound for $k_H(2,r)$ as $r \to \infty$ follows from work by Bowen.

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Theorem: For any integer $n \geq 2$ and a non-negative number $r \leq \frac{\pi}{3}$,

$$K_S(n,r) \leq \frac{2\,B\left(\frac{n-1}{2},\frac{1}{2}\right)}{B\left(\frac{\sec^2 r}{4};\frac{n-1}{2},\frac{1}{2}\right)}$$

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Limiting values of K_s

- For $r=0$, we obtain $\sin \quad = \frac{1}{2}$ or $\theta = \frac{\pi}{3}$ $\Rightarrow$ we consider K in $\mathbb{R}^n$
- $K_s(n,r)$ decreasing function in r, $0 \leq r \leq \frac{\pi}{3}$ for any $n \geq 2$

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- $\lim_{r\to 0} K(2,r) = 5$, $K(2)=6$
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- Conjecture $\lim_{r\to 0} K(4,r) = 22$, it is known $K(4)=24$

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- $K_s(n, \frac{\pi}{3}) = 2$ for all $n \geq 2$

Semidefinite Programming Bound

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$C, A_1,\dots,A_m \in \mathbb{R}^{n\times n}$ given symmetric matrices, $b \in \mathbb{R}^m$ given vector

$X \succcurlyeq 0$: positive semidefinite (all eigenvalues non-negative)

$\langle X,Y\rangle = tr(X^T Y)$

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- Indepence number of a graph $G=(V,E)$:

  $\alpha(G)$ = max number of vertices such that no two vertices are connected by an edge.

Semidefinite Programming Bound

- $\alpha(G) = \max \sum_{v \in V} x_v^2$

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$\Rightarrow$ infinite graph

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- Take the dual problem

Semidefinite Programming Bound

$\Rightarrow$ 2 point bound for kissing problem

Bachor, Vallentin formulated 3 point bound:

Semidefinite Programming Bound

$\Rightarrow$ 2 point bound for kissing problem

Bachoc, Vallentin formulated 3 point bound:

$$\rho^* = \min \quad 1 + \sum_{k=1}^{d} a_k + b_{11} + \langle J, F_0 \rangle$$

$$a_k \geq 0 \quad \text{for} \quad k=1,\dots,d, \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \succeq 0$$

$$F_k \in \mathbb{R}^{(d-k+1)\times(d-k+1)}, \quad F_k \succeq 0 \quad \text{for } k=0,\dots,d$$

$$\sum_{k=0}^{d} a_k P_k^n(u) + 2b_{12} + b_{22} + 3\sum_{k=0}^{d} \langle S_k^n(u,u,1), F_k \rangle \leq -1 \qquad \text{for} \quad (u,u,1) \in \Delta_0$$

$$b_{22} + \sum_{k=0}^{d} \langle S_k^n(u,v,t), F_k \rangle \leq 0 \qquad \text{for } (u,v,t) \in \Delta$$

where

$$\Delta_0 = \{(u,u,1) : -1 \leq u \leq \cos\theta\}, \quad \Delta = \{(u,v,t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq \cos\theta,\ 1 + 2uvt - u^2 - v^2 - t^2 \geq 0\}$$

Semidefinite Programming Bound

$p^* \geq A(n,\theta) = \max \{ |C| : C \subset S^{n-1} \text{ and } x \cdot y \leq \cos\theta \text{ for all distinct } x,y \in C \}$

Semidefinite Programming Bound

$p^* \geq A(n,\Theta) = \max\{|C| : C \subset S^{n-1} \text{ and } x\cdot y \leq \cos\Theta \text{ for all distinct } x,y \in C\}$

$K_H(n,r) = \max\{|C| : C \subset S^{n-1} \text{ and } x\cdot y \leq 1 - \frac{1}{\cosh(2r)} \text{ for all distinct } x,y \in C\}$

Semidefinite Programming Bound

$$p^* \geq A(n,\Theta) = \max\{|C| : C \subset S^{n-1} \text{ and } x\cdot y \leq \cos\Theta \text{ for all distinct } x,y \in C\}$$

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The optimal solution of the SDP with

$$\Delta_0 = \{(u,u,1) : -1 \leq u \leq 1 - \frac{1}{1+\cosh(2r)}\},$$

$$\Delta = \{(u,v,t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq \frac{1}{1+\cosh(2r)} \text{ and } 1 + 2uvt - u^2 - v^2 - t^2 \geq 0\}$$

is an upper bound on $K_H(n,r)$.

Semidefinite Programming Bound

$p^* \geq A(n,\Theta) = \max\{|C| : C \subset S^{n-1}$ and $x \cdot y \leq \cos\Theta$ for all distinct $x,y \in C\}$

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The optimal solution of the SDP with

$$\Delta_0 = \left\{(u,u,1) : -1 \leq u \leq 1 - \frac{1}{1+\cosh(2r)}\right\},$$

$$\Delta = \left\{(u,v,t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq \frac{1}{1+\cosh(2r)} \text{ and } 1 + 2uvt - u^2 - v^2 - t^2 \geq 0\right\}$$

is an upper bound on $K_H(n,r)$.

$K_S(n,r) = \max\{|C| : C \subset S^{n-1}$ and $x \cdot y \leq 1 - \frac{1}{\cos(2r)}$ for all distinct $x,y \in C\}$

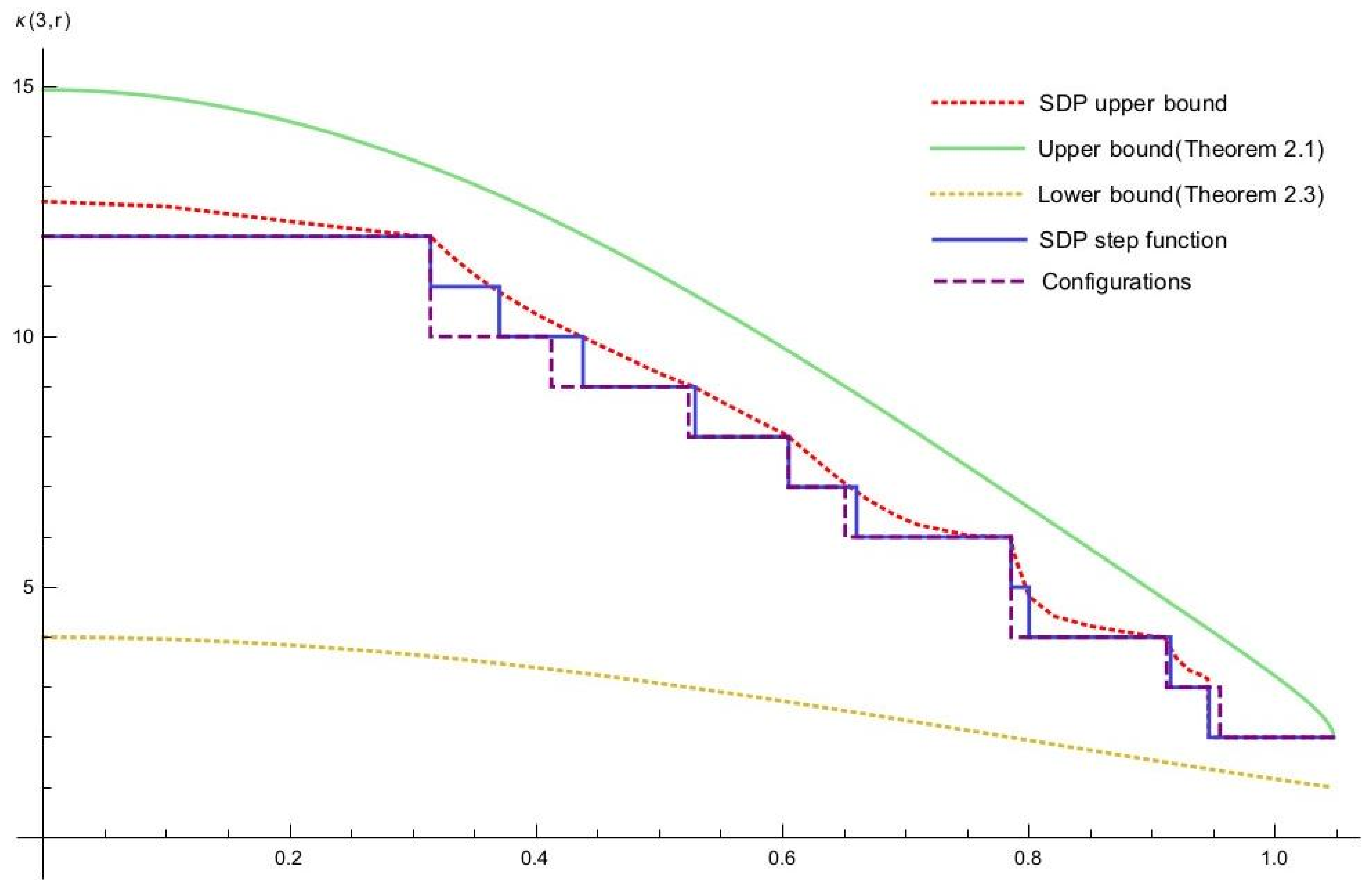
TABLE 1. Bounds for the kissing number in $\mathbb{H}^3$

r	theoretical lower bound	lower bound by construction	SDP upper bound	Levenshtein bound	Coxeter bound
0	4	12 [27]	12.368591[20]	13.2857	13.3973
0.3007680932244	4.37289	13	13.66695	14.6365	14.7591
0.3741678937820	4.58663	14	13.57930	15.4829	15.5389
0.4603413898301	4.90925	15	15.76145	16.6843	16.7150
0.5150988762761	5.15856	16	16.63748	17.5619	17.6233
0.5575414271933	5.37771	17	17.39631	18.3659	18.4214
0.6117193853329	5.69307	18	18.57836	19.5957	19.5694
0.6752402229782	6.1184	19	20.12475	21.1343	21.1170
0.6839781903772	6.18194	20	20.43374	21.3570	21.3482
0.7441766799717	6.65554	21	21.88751	23.0631	23.0705
0.7727858684533	6.90384	22	22.81495	24.0041	23.9732
0.8064065300517	7.21623	23	24.08326	25.2137	25.1087
0.8070321648835	7.22226	24	24.32215	25.2348	25.1306

TABLE 2. Bounds for the kissing number in $\mathbb{H}^4$

r	theoretical lower bound	lower bound by construction	SDP upper bound	Levenshtein bound	Coxeter bound
0	5.11506	24 [22]	24.05691[20]	26	26.4420
0.2803065634764	5.70802	25	28.36959	29.9154	29.9757
0.2937915284847	5.76935	26	28.54566	30.2755	30.3417
0.3533981811745	6.08306	27	30.35228	32.0432	32.2152
0.4029707622959	6.40115	29	32.37496	33.8969	34.1172
0.4361470369242	6.64597	30	33.73058	35.3805	35.5826

Plot of $\kappa_s(3,r)$



Plot of $\kappa_s(4,r)$