Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas
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Tessellations coming from long geodesics on surfaces

Jenya Sapir

SUNY - Binghamton

April 14, 2020

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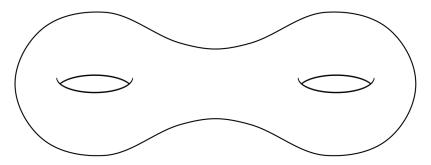
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Joint work with Jayadev Athreya, Steve Lalley and Matt Wroten.



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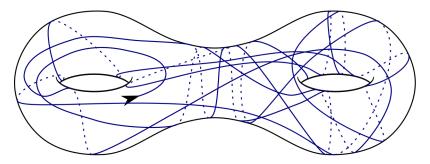
S - closed, hyperbolic surface

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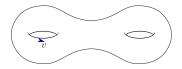


 α - geodesic on ${\it S}$

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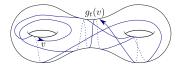
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• T_1S - unit tangent bundle

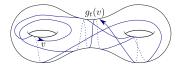
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- T_1S unit tangent bundle
- g_t geodesic flow

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- T_1S unit tangent bundle
- g_t geodesic flow is ergodic

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- T_1S unit tangent bundle
- g_t geodesic flow is ergodic

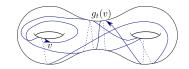
Theorem (Birkhoff Ergodic theorem)

For almost every $v \in T_1S$, and any integrable $f : T_1S \to \mathbb{R}$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(g_t(v))dt = \frac{1}{2\pi Area(S)}\int_{T_1S}f(v)d\lambda$$

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How do typical geodesics fill?

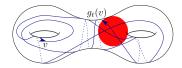


$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(g_t(v))dt = \frac{1}{2\pi Area(S)}\int_{T_1S}f(v)d\lambda$$

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How do typical geodesics fill?

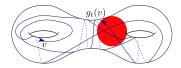


$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(g_t(v))dt = \frac{1}{2\pi Area(S)}\int_{T_1S}f(v)d\lambda$$

Choose f - indicator function of $U \subset T_1 S$.

Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas
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How do typical geodesics fill?



$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(g_t(v))dt = \frac{1}{2\pi Area(S)}\int_{T_1S}f(v)d\lambda$$

Choose f - indicator function of $U \subset T_1S$. Then,

Time spent in $U \asymp$ volume of U

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First con	clusion				

Almost all long arcs eventually cut ${\mathcal S}$ into simply connected regions



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First con	clusion				

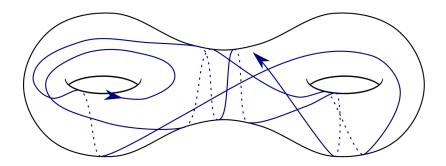
Almost all long arcs eventually cut ${\mathcal S}$ into simply connected regions

• $g_t(v)$ enters U from all directions equally



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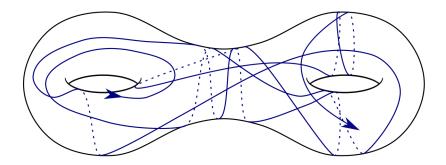
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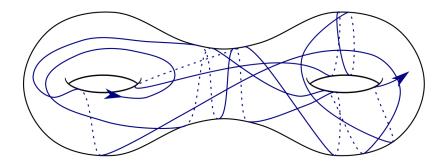
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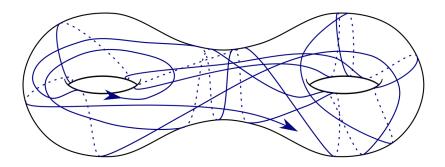
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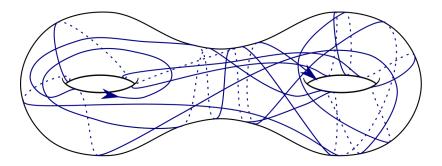
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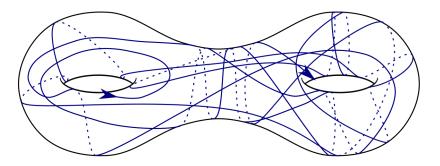


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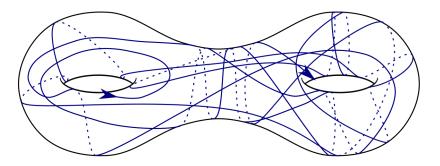
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Question

Arc α gives tessellation on S. How does it cut up the surface?

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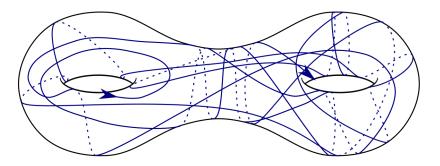


Question

Arc α gives tessellation on S. How does it cut up the surface?

• Proportions of triangles, quadrilaterals, ... , n-gons

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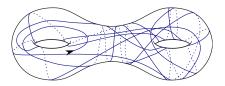


Question

Arc α gives tessellation on S. How does it cut up the surface?

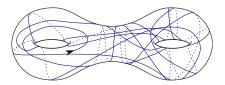
- Proportions of triangles, quadrilaterals, ... , n-gons
- Distribution of edge lengths, angles

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Vertices					



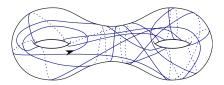
 α_ℓ - random arc, length ℓ

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Vertices					



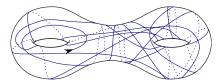
 $lpha_\ell$ - random arc, length ℓ with tessellation \mathcal{T}_ℓ

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Vertices					



 α_{ℓ} - random arc, length ℓ with tessellation T_{ℓ} with $v(\ell)$ vertices.

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Vertices					



 α_{ℓ} - random arc, length ℓ with tessellation T_{ℓ} with $v(\ell)$ vertices. Then,

$$u(\ell) \sim rac{1}{4\pi^2(g-1)}\ell^2$$

where $A(\ell) \sim B(\ell)$ if $A/B \rightarrow 1$, g = genus.

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Why ℓ ² s	elf_interse	ections?			



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Why ℓ^2 s	elf-interse	ections?			



all different crossing angles

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Why ℓ^2 s	elf-interse	ections?			



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all different crossing angles \implies most pairs intersect

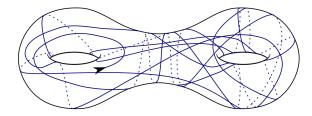
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Why ℓ^2 s	elf-interse	ections?			



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 $\begin{array}{l} \mbox{all different crossing angles} \\ \implies \mbox{most pairs intersect} \\ \implies \approx \ell^2 \mbox{ intersections} \end{array}$

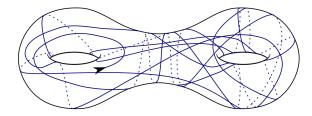
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Edges an	d faces				



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If T_{ℓ} has $v(\ell)$ vertices, $e(\ell)$ edges, $f(\ell)$ faces,

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Edges an	d faces				

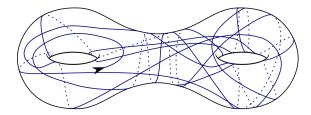


If T_{ℓ} has $v(\ell)$ vertices, $e(\ell)$ edges, $f(\ell)$ faces, then

$$v(\ell) \sim \frac{1}{2}e(\ell) \sim f(\ell)$$

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Edges an	d faces				

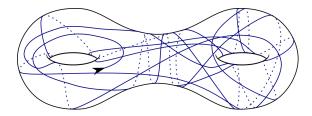


If T_ℓ has $v(\ell)$ vertices, $e(\ell)$ edges, $f(\ell)$ faces, then

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By Euler characteristic: $v - e + f = \chi(S)$; and 2e = 4v.

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Geometry of the tessellation

Theorem (Athreya-Lalley-S-Wroten)

W.p.1, the statistics of T_{ℓ} approach those of a Poisson line process on \mathbb{R}^2 .

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Geometry of the tessellation

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- E.g. w.p.1,
 - The proportion of *n*-gons in T_{ℓ} approaches $\tau_n > 0$ as $\ell \to \infty$.

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$$\tau_3 \to 2 - \pi^2/6 \approx .355$$

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- 4 = expected number of sides

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 - Edge lengths and intersection angles \rightarrow limiting distributions.

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• Angles mutually independent with prob. density $\frac{1}{2}\sin\theta$

Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas
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• Self-intersection times \rightarrow Poisson point process of intensity $\frac{1}{\textit{Area}(S)}$

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Tessellations on the plane

How to model a long geodesic

• Birkhoff: geodesics look locally like random collections of lines

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Tessellations on the plane

How to model a long geodesic

• Birkhoff: geodesics look locally like random collections of lines

• Rotation invariant

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Tessellat	ions on th	e plane			

How to model a long geodesic

• Birkhoff: geodesics look locally like random collections of lines

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- Rotation invariant
- Independent of choice of ball ("translation invariant")

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Tessellat	ions on th	e plane			

How to model a long geodesic

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Tessellations on the plane

How to model a long geodesic

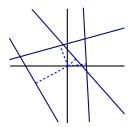
• Birkhoff: geodesics look locally like random collections of lines

- Rotation invariant
- Independent of choice of ball ("translation invariant")
- Let's look at random collections of lines in the plane.

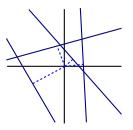
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Tessellat	ions on th	e plane			

How to model a long geodesic

- Birkhoff: geodesics look locally like random collections of lines
- Let's look at random collections of lines in the plane. Want: rotation and translation invariant

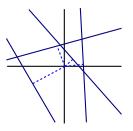


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• Draw lines at angle θ_i , distance r_i from (0, 0).

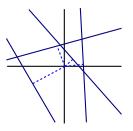
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- Draw lines at angle θ_i , distance r_i from (0, 0).
- Choose sequence $\theta_1, \ldots, \theta_n$ at random in $[0, 2\pi]$

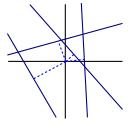
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- Draw lines at angle θ_i , distance r_i from (0, 0).
- Choose sequence $\theta_1, \ldots, \theta_n$ at random in $[0, 2\pi]$
- Choose $r_1 < \cdots < r_n \in \mathbb{R}$ with Poisson distribution with intensity λ

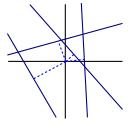
$$P(\#\{r_i \in [0,\ell]\} = n) = \frac{(\lambda\ell)^n}{n!} e^{-\lambda\ell}$$

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Get tesselation of \mathbb{R}^2 .

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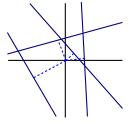


Get tesselation of \mathbb{R}^2 .

Miles ('64): statistics for frequencies of *n*-gons, side lengths, angles, etc in Poisson line process.

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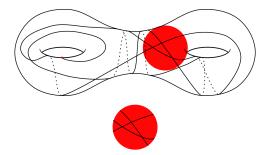


Get tesselation of \mathbb{R}^2 .

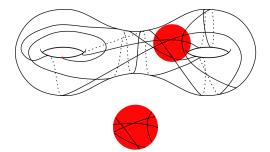
Miles ('64): statistics for frequencies of *n*-gons, side lengths, angles, etc in Poisson line process.

We show: statistics on surface same as for Poisson line process on $\ensuremath{\mathbb{R}}^2.$

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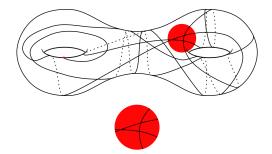
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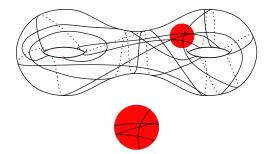
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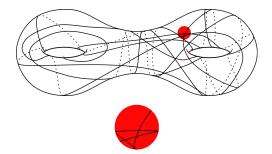
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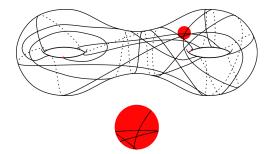
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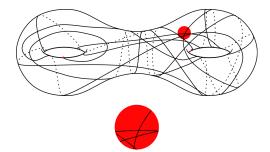
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By Birkhoff, α_{ℓ} expected to cross $B(x, A/\ell) \approx A$ times

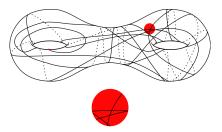
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By Birkhoff, α_{ℓ} expected to cross $B(x, A/\ell) \approx A$ times As $\ell \to \infty$, looks like Euclidean lines in Euclidean disk!

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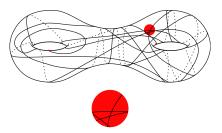


Theorem (Athreya-Lalley-S-Wroten)

As $\ell \to \infty,$ the tesselations of B(x,A/\ell) approach a Poisson line process.

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Theorem (Athreya-Lalley-S-Wroten)

As $\ell \to \infty$, the tesselations of $B(x, A/\ell)$ approach a Poisson line process.

Theorem (Athreya-Lalley-S-Wroten)

Given two points x, x', tesselations in $B(x, A/\ell) \cup B(x', A/\ell)$ approach independent pair of Poisson line processes.

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An incor	rect (but	illuminating)	idea.		



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• Divide length $\ell \in \mathbb{N}$ arc into length 1 segments $\alpha_1, \ldots, \alpha_\ell$

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- Divide length $\ell \in \mathbb{N}$ arc into length 1 segments $lpha_1, \dots, lpha_\ell$
- Birkhoff: expect c(A) total crossings



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- Birkhoff: expect c(A) total crossings
- Incorrect assumption: Segments all independent



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- Thus, $P(\alpha_i \text{ crosses } B(x, A/\ell)) \approx c(A)/\ell$

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An incor	rect (but	illuminating)	idea.		

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- Birkhoff: expect c(A) total crossings
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- Thus, $P(\alpha_i \text{ crosses } B(x, A/\ell)) \approx c(A)/\ell$
- Binomial distribution of crossings:

$$P(\#\{\alpha_i \text{ crosses } B(x, A/\ell)\} = n) = {}_{\ell}C_n \left(\frac{c(A)}{\ell}\right)^n \left(1 - \frac{c(A)}{\ell}\right)^{\ell-n}$$

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• Binomial distribution of crossings:

$$P(\#\{\alpha_i \text{ crosses } B(x, A/\ell)\} = n) = \mathcal{C}_n \left(\frac{c(A)}{\ell}\right)^n \left(1 - \frac{c(A)}{\ell}\right)^{\ell-n}$$

• As $\ell \to \infty$, this approaches Poisson distribution of intensity c(A) in ball of radius 1!

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• Binomial distribution of crossings:

$$P(\#\{\alpha_i \text{ crosses } B(x, A/\ell)\} = n) = \mathcal{L}_n\left(\frac{c(A)}{\ell}\right)^n \left(1 - \frac{c(A)}{\ell}\right)^{\ell-n}$$

• As $\ell \to \infty$, this approaches Poisson distribution of intensity c(A) in ball of radius 1!

$$\lim_{\ell \to \infty} P(\#\{\alpha_i \text{ crosses } B(x, A/\ell)\} = n) = \frac{(c(A))^n}{n!} e^{-c(A)}$$

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Problems and solution	ons				

The problem:

• Length 1 subarcs aren't really independent

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Problems and solution	ons				

The problem:

• Length 1 subarcs aren't really independent

The solution:

• Two subarcs "far enough apart" are close to independent.

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Problems and solution	ons				

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Problems and solution	ons				

The problem:

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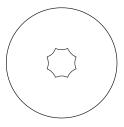
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• Unlikely to return to very small ball quickly.

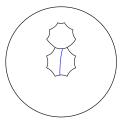
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Geodesic trajectories					
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- Choose fundamental domain for S
- Tile universal cover, \tilde{S}

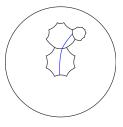
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Geodesic traject	ories				
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- Choose fundamental domain for S
- Tile universal cover, \tilde{S}
- \bullet Geodesic trajectory \rightarrow sequence of f.d. edge crossings

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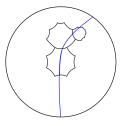


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• Most sequences \rightarrow (unique) geodesic

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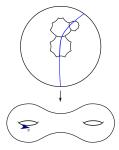


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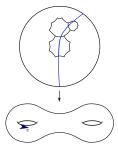


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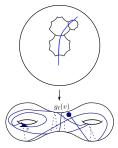
Given $v \in T_1S$, get g_v in \tilde{S}

Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas
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Geodesic trajectories					



Given $v \in T_1S$, get g_v in \tilde{S} \rightsquigarrow bi-infinite word $\ldots x_{-1}x_0x_1\ldots$

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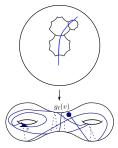


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Given $v \in T_1S$, get g_v in \tilde{S} \rightsquigarrow bi-infinite word $\dots x_{-1}x_0x_1\dots$ Length *I* trajectory from v

Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas
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Geodesic trajectories					

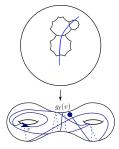


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Given $v \in T_1S$, get g_v in \tilde{S} \rightsquigarrow bi-infinite word $\dots x_{-1}x_0x_1\dots$ Length *I* trajectory from v \rightsquigarrow subword $x_0x_1\dots x_{n(\ell)}$

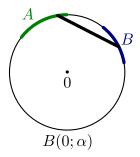
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Geodesic trajectories					



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Given $v \in T_1S$, get g_v in \tilde{S} \rightsquigarrow bi-infinite word $\ldots x_{-1}x_0x_1\ldots$ Length / trajectory from v \rightsquigarrow subword $x_0x_1\ldots x_{n(\ell)}$ Next: Encode crossing of ball with subwords

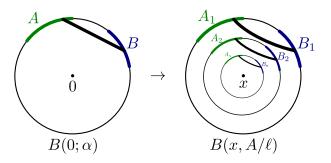
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Fix crossing direction in \mathbb{R}^2 .

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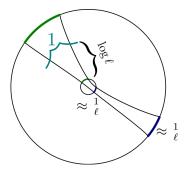
Small disk crossings



Fix crossing direction in \mathbb{R}^2 . Get crossing pattern of $B(x, \frac{A}{\ell})$ for each ℓ .

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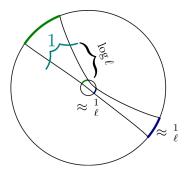
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Use edge crossing to encode disc crossings:

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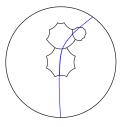


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Use edge crossing to encode disc crossings:

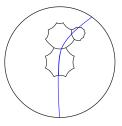
• Crossing $B(x, A/\ell) \leftrightarrow \text{length } \log^2(\ell)$ subwords

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Geodesic $\gamma \quad \leftrightarrow \quad \ldots x_{-1}x_0x_1 \ldots$ word in f.d. edges

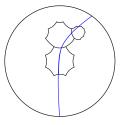
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Geodesic $\gamma \leftrightarrow \dots x_{-1}x_0x_1\dots$ word in f.d. edges cut into indep., ident. distr. subwords! (Lalley)

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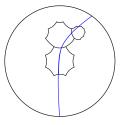
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Geodesic $\gamma \leftrightarrow \dots x_{-1}x_0x_1\dots$ word in f.d. edges Forgetting the past \leftarrow cut into indep., ident. distr. subwords! (Lalley)

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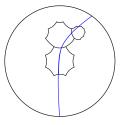


 $\begin{array}{rcl} \mbox{Geodesic } \gamma & \leftrightarrow & \dots x_{-1} x_0 x_1 \dots \mbox{ word in f.d. edges} \\ \mbox{Forgetting the past} & \leftarrow & \mbox{cut into indep., ident. distr. subwords!} \\ & & (Lalley) \end{array}$

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• Cut length ℓ word into i.i.d. chunks of length $\log^3(\ell)$

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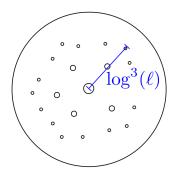


 $\begin{array}{rcl} \mbox{Geodesic } \gamma & \leftrightarrow & \dots x_{-1} x_0 x_1 \dots \mbox{ word in f.d. edges} \\ \mbox{Forgetting the past} & \leftarrow & \mbox{cut into indep., ident. distr. subwords!} \\ & & (Lalley) \end{array}$

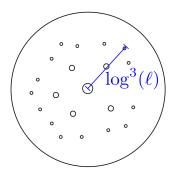
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- Cut length ℓ word into i.i.d. chunks of length $\log^3(\ell)$
- Look for length $\log^2(\ell)$ subwords

Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas
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Solutions to our pr	oblems				
One wor	d per chu	nk			



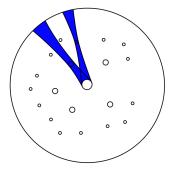
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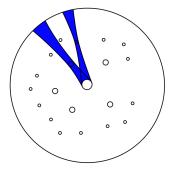
• Disc crossings occur far apart

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- Disc crossings occur far apart
- Look at nearby lifts of $B(x, A/\ell)$ (in $\approx \log^3 \ell$ radius)

Long Arcs 0000000000	On the plane	Back to the surface	An incorrect proof	Fixes ○○○○○○●	Future ideas			
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Solutions to our problems								
One word per chunk								



- Disc crossings occur far apart
- Look at nearby lifts of $B(x, A/\ell)$ (in $\approx \log^3 \ell$ radius)
- Measure of arcs returning quickly goes to 0

Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas		
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The same theorems should hold for



Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas		
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The same theorems should hold for

• Variable negative curvature

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The same theorems should hold for

- Variable negative curvature
- Closed geodesics

Long Arcs	On the plane	Back to the surface	An incorrect proof	Fixes	Future ideas
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Future id	eas				

The same theorems should hold for

- Variable negative curvature
- Closed geodesics

Questions:

• What about geodesics that don't equidistribute, but still fill?

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Future id	eas				

The same theorems should hold for

- Variable negative curvature
- Closed geodesics

Questions:

• What about geodesics that don't equidistribute, but still fill?

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• What about typical geodesics in flat metrics?