Tessellations coming from long geodesics on surfaces

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April 14, 2020

Our setting

Joint work with Jayadev Athreya, Steve Lalley and Matt Wroten.

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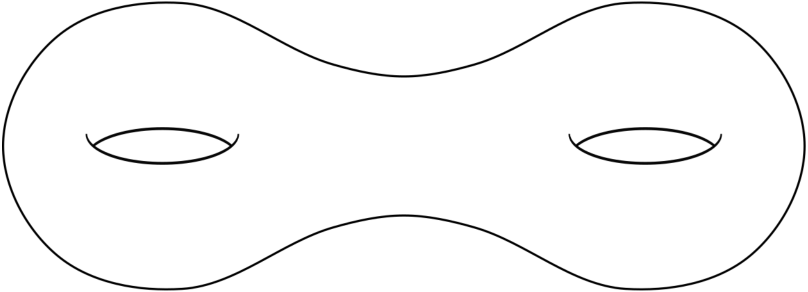

S - closed, hyperbolic surface

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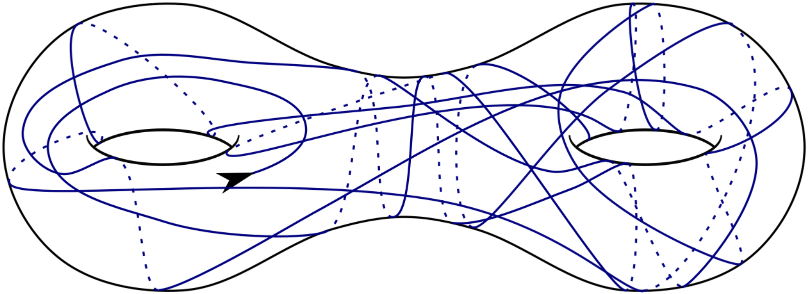
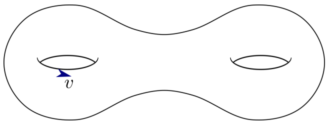


α - geodesic on S

How do typical geodesics look?



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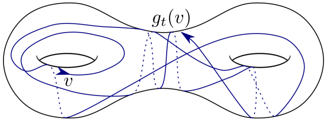

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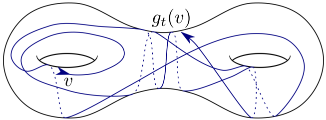

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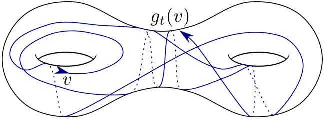
- g_t - geodesic flow is ergodic

Theorem (Birkhoff Ergodic theorem)

For almost every $v \in T_1S$, and any integrable $f : T_1S \to \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(g_t(v))dt = \frac{1}{2\pi Area(S)}\int_{T_1S} f(v)d\lambda$$

How do typical geodesics fill?



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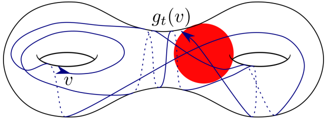

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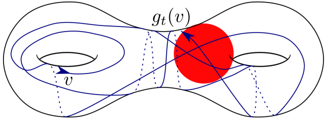

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$$\text{Time spent in } U \asymp \text{volume of } U$$

First conclusion

Almost all long arcs eventually cut $\mathcal{S}$ into simply connected regions

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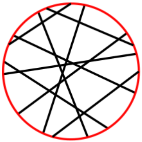
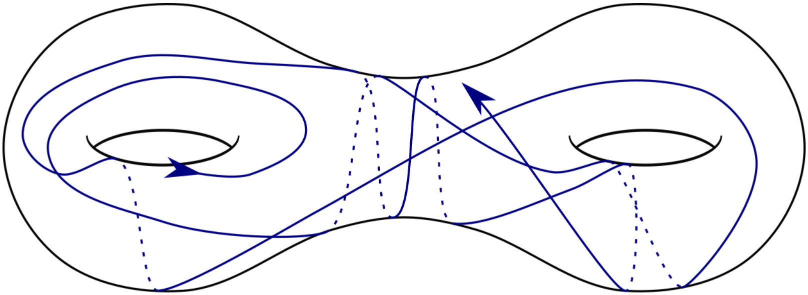
- $g_t(v)$ enters U from all directions equally



Sequence of tessellations



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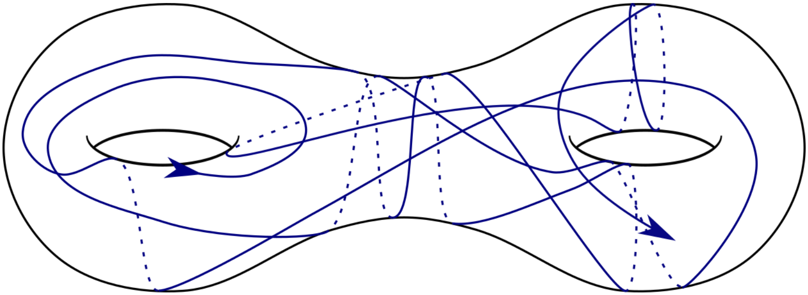

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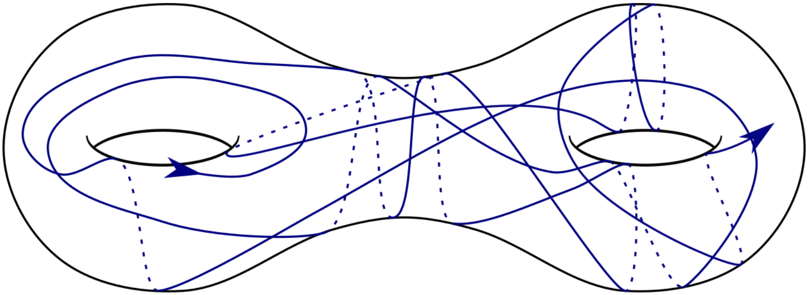

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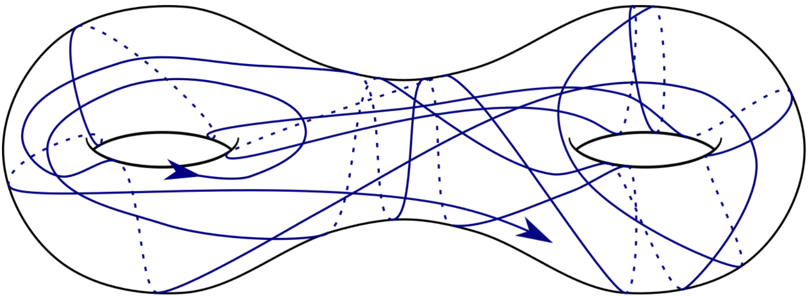

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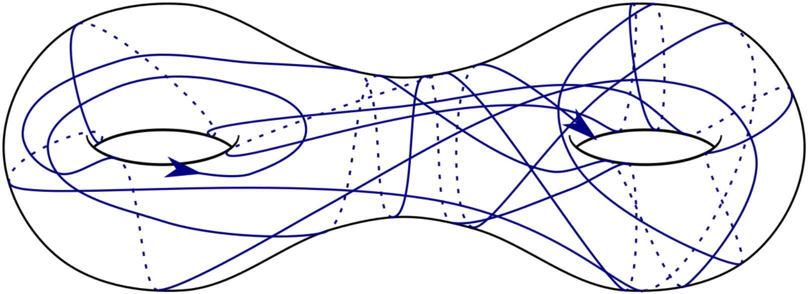

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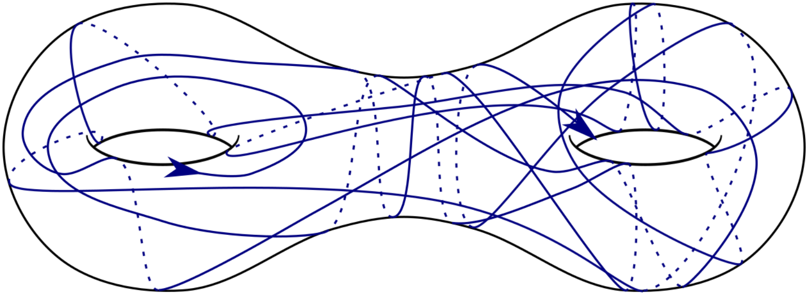

Question

Arc α gives tessellation on S. How does it cut up the surface?

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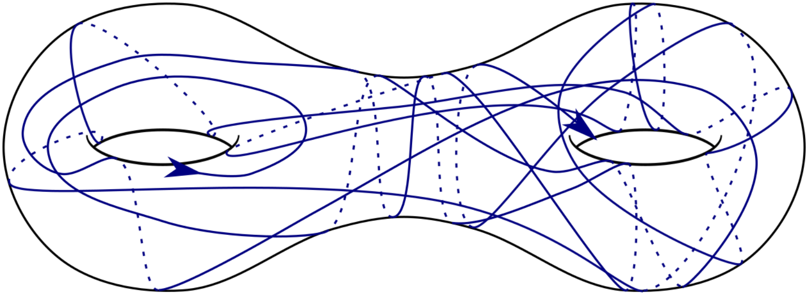

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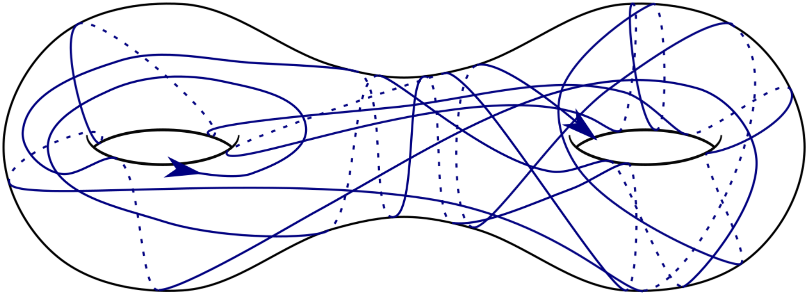
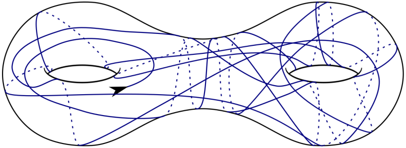


Question

Arc α gives tessellation on S. How does it cut up the surface?

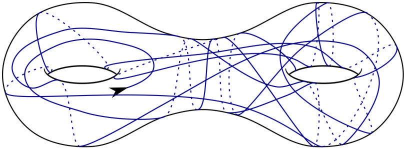
- *Proportions of triangles, quadrilaterals, ... , n-gons*
- *Distribution of edge lengths, angles*

Vertices



α_ℓ - random arc, length ℓ

Vertices



α_ℓ - random arc, length ℓ with tessellation T_ℓ

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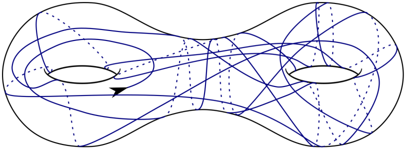

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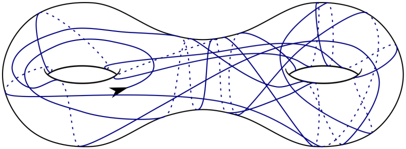


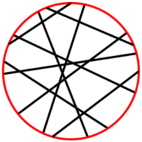
α_ℓ - random arc, length ℓ with tessellation T_ℓ with $v(\ell)$ vertices. Then,

$$v(\ell) \sim \frac{1}{4\pi^2(g-1)}\ell^2$$

where $A(\ell) \sim B(\ell)$ if $A/B \to 1$, g = genus.

Why ℓ^2 self-intersections?

A length ℓ arc will cross any region $\approx \ell$ times:



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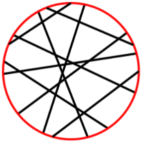

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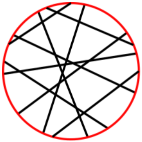

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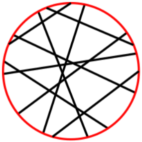

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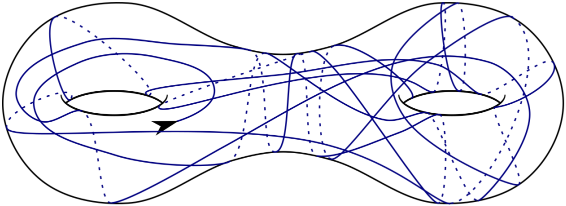
Edges and faces

All complementary regions are polygons w.p.1!



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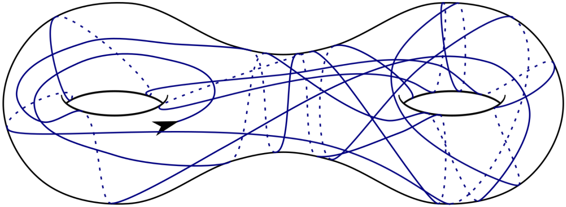

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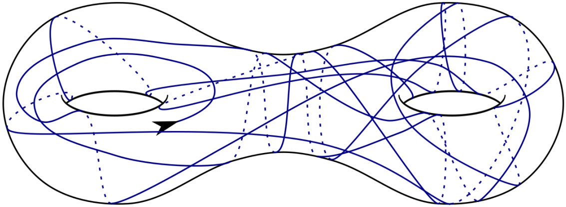

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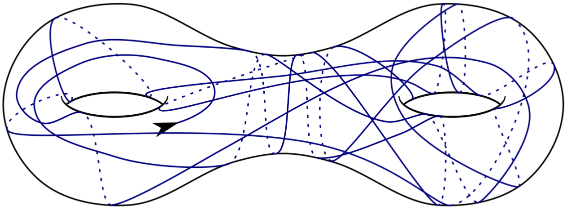

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  - Self-intersection times $\to$ Poisson point process of intensity $\frac{1}{Area(S)}$

Tessellations on the plane

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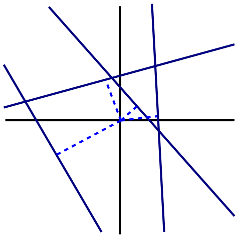
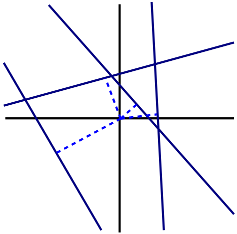
- Let's look at random collections of lines in the plane.
  Want: rotation and translation invariant



Poisson line process



- Draw lines at angle θ_i, distance r_i from $(0, 0)$.

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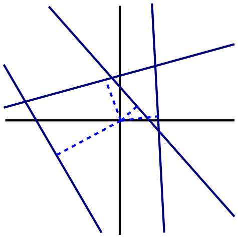

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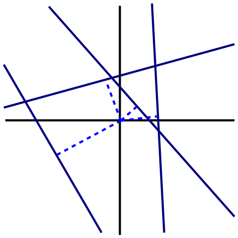
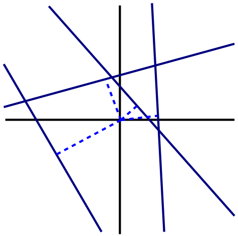


- Draw lines at angle θ_i, distance r_i from $(0,0)$.
- Choose sequence $\theta_1, \ldots, \theta_n$ at random in $[0, 2\pi]$
- Choose $r_1 < \cdots < r_n \in \mathbb{R}$ with Poisson distribution with intensity λ

$$P(\#\{r_i \in [0, \ell]\} = n) = \frac{(\lambda\ell)^n}{n!} e^{-\lambda\ell}$$

Poisson line process



Get tesselation of $\mathbb{R}^2$.

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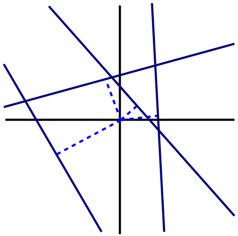

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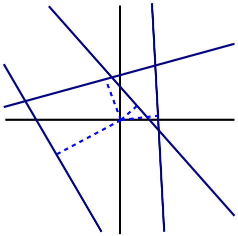

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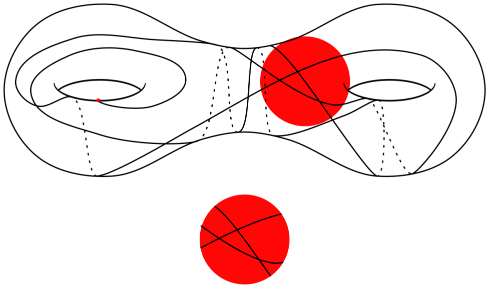
We show: statistics on surface same as for Poisson line process on $\mathbb{R}^2$.

Back to the surface

Look at length ℓ arc in ball of size A/ℓ (+ rescale, map to plane.)



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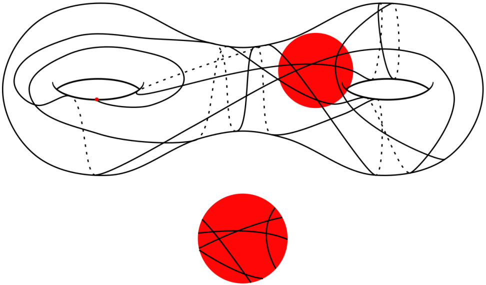

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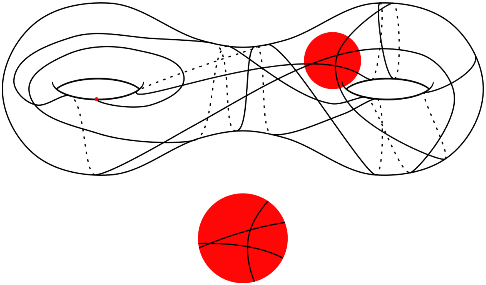

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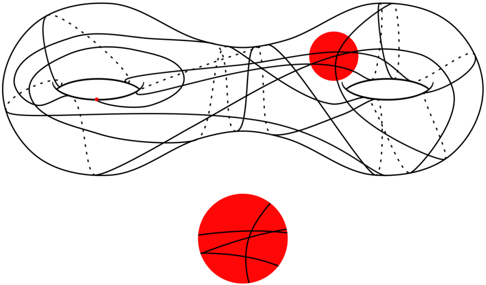

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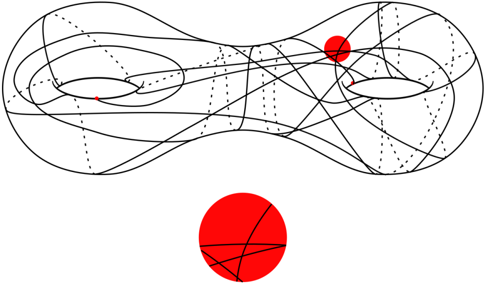

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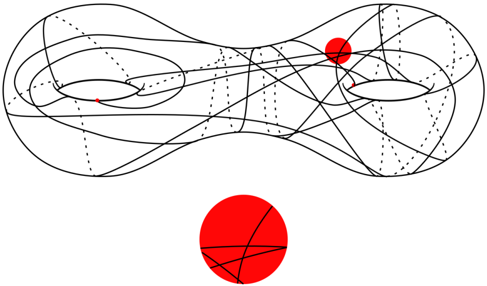
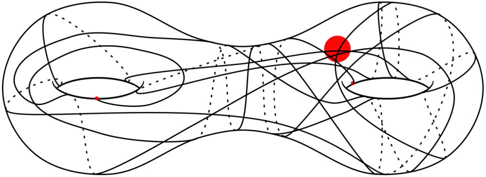


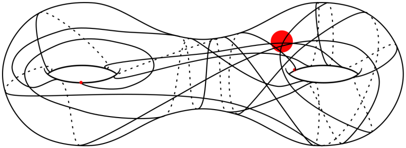
By Birkhoff, α_ℓ expected to cross $B(x, A/\ell) \approx A$ times

Back to the surface

Look at length ℓ arc in ball of size A/ℓ (+ rescale, map to plane.)

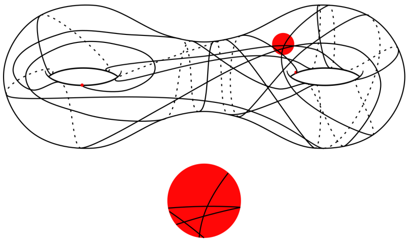


By Birkhoff, α_ℓ expected to cross $B(x, A/\ell) \approx A$ times
As $\ell \to \infty$, looks like Euclidean lines in Euclidean disk!



Theorem (Athreya-Lalley-S-Wroten)

As $\ell \to \infty$, the tesselations of $B(x, A/\ell)$ approach a Poisson line process.



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Given two points x, x', tesselations in $B(x, A/\ell) \cup B(x', A/\ell)$ approach independent pair of Poisson line processes.

An incorrect (but illuminating) idea.

Random length ℓ arc in $B(x, A/\ell)$: distribution of crossing number

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$$P(\#\{\alpha_i \text{ crosses } B(x, A/\ell)\} = n) = {}_\ell C_n \left(\frac{c(A)}{\ell}\right)^n \left(1 - \frac{c(A)}{\ell}\right)^{\ell - n}$$

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$$\lim_{\ell \to \infty} P(\#\{\alpha_i \text{ crosses } B(x, A/\ell)\} = n) = \frac{(c(A))^n}{n!} e^{-c(A)}$$

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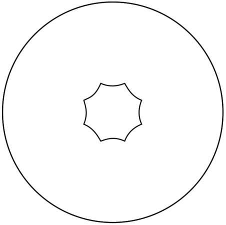
- Unlikely to return to very small ball quickly.

Bowen-Series symbolic dynamics



- Choose fundamental domain for S
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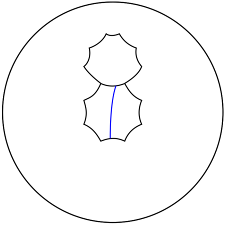
Bowen-Series symbolic dynamics



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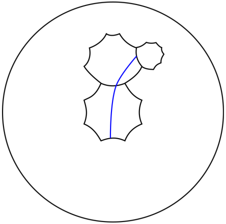

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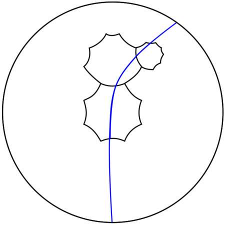

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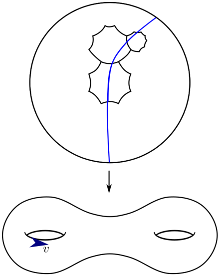
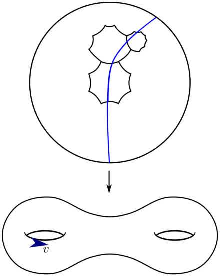


Given $v \in T_1S$, get g_v in $\tilde{S}$



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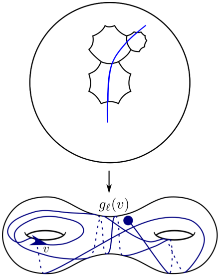
$\rightsquigarrow$ bi-infinite word $\ldots x_{-1} x_0 x_1 \ldots$



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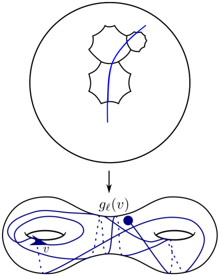

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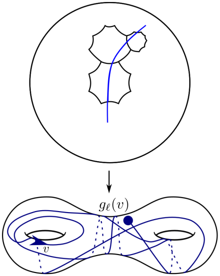

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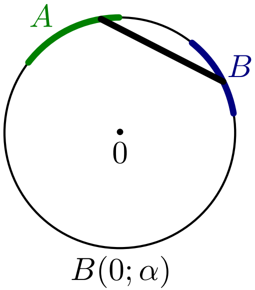
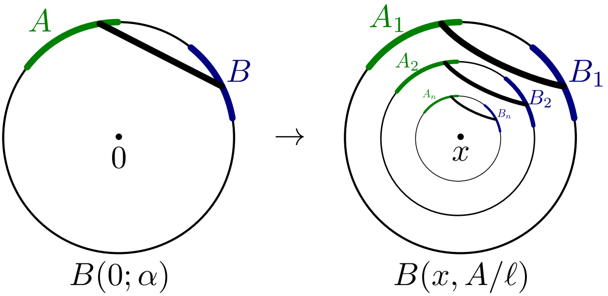
Next: Encode crossing of ball with subwords

Small disk crossings



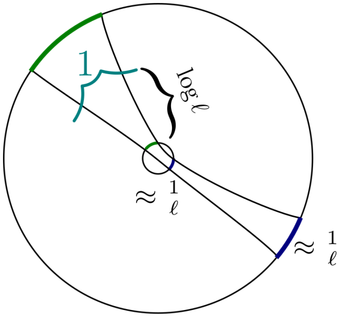
Fix crossing direction in $\mathbb{R}^2$.

Small disk crossings



Fix crossing direction in $\mathbb{R}^2$. Get crossing pattern of $B(x, \frac{A}{\ell})$ for each ℓ.

Small disk crossings



Use edge crossing to encode disc crossings:

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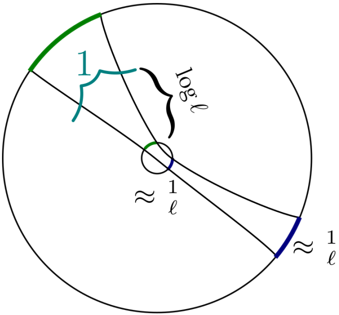
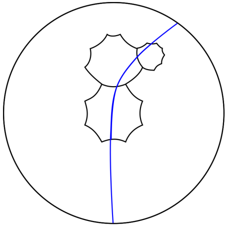


Use edge crossing to encode disc crossings:

- Crossing $B(x, A/\ell) \leftrightarrow$ length $\log^2(\ell)$ subwords

Why geodesics "forget their past"



Geodesic $\gamma \quad \leftrightarrow \quad \ldots x_{-1}x_0x_1\ldots$ word in f.d. edges

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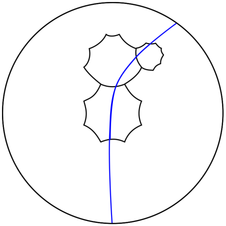

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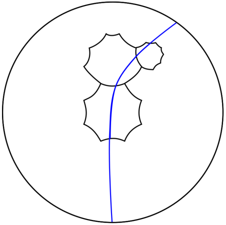

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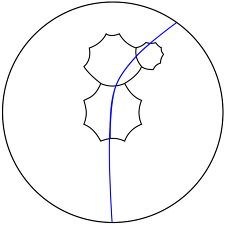

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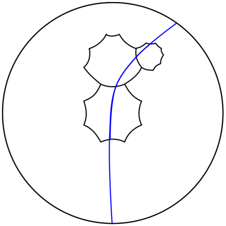

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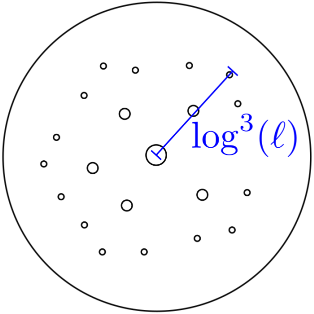
One word per chunk



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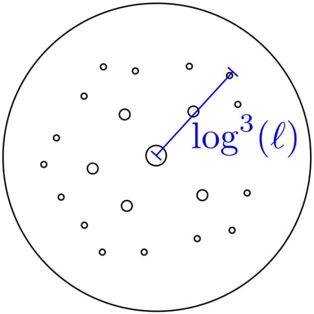

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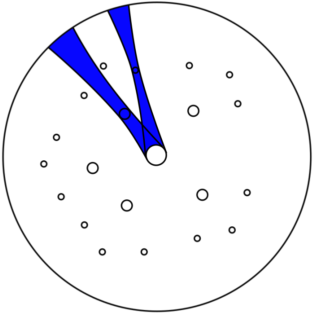

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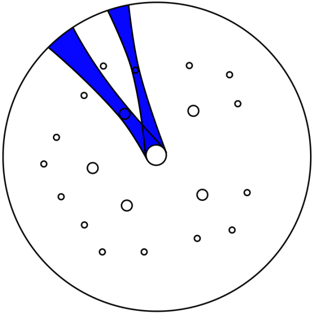

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Future ideas

The same theorems should hold for

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- Variable negative curvature

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- What about typical geodesics in flat metrics?