Using the folded apartment to deduce cutoff

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BASED ON JOINT WORK WITH ORI PARZANCHEVSKI

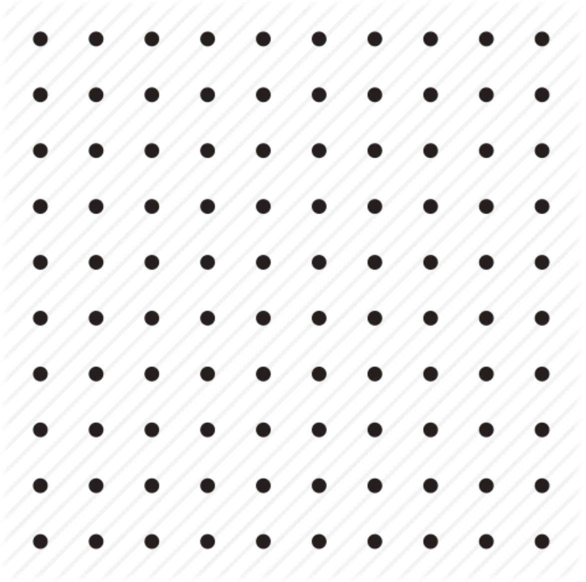
Talk Plan

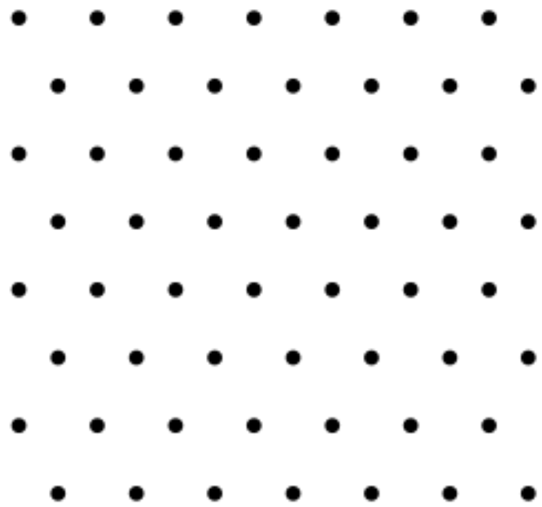
- Introduce the Bruhat-Tits buildings of type $\tilde{A}_d$ and their fundumental apartments.
- Discuss random walks on cells of a simplicial complex and the cutoff phenomenon.
- Define Ramanujan graphs and Ramanujan complexes.
- Review the proof from Lubetzky-Peres' work via the folded apartment of $\tilde{A}_1$.
- Our work on the higher dimensional cases.

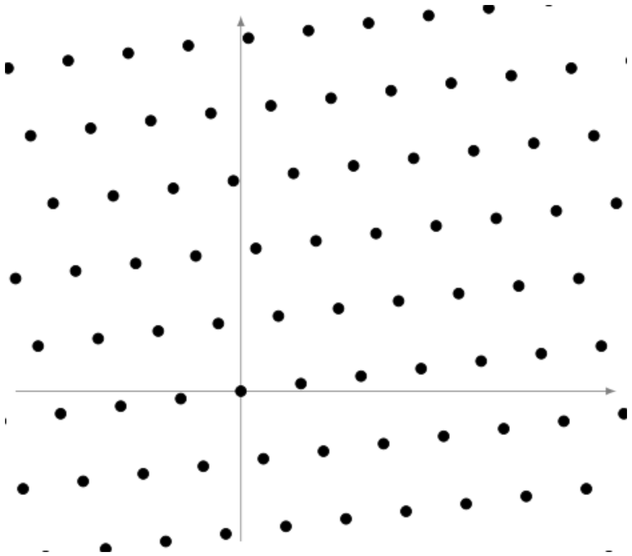
The Bruhat Tits building

- Integral lattices: $L = A \cdot \mathbb{Z}^d$ for $A \in \mathbb{Z}^{d \times d}$ invertible over $\mathbb{Q}$.

- Examples:







Google images

Facts about integral lattices

- $A \cdot \mathbb{Z}^d \subseteq \mathbb{Z}^d$ and $A \cdot \mathbb{Z}^d = \mathbb{Z}^d$ iff $A \in GL_d\mathbb{Z}$
- If $A \cdot \mathbb{Z}^d = B \cdot \mathbb{Z}^d$ then $A^{-1}B \in GL_d\mathbb{Z}$
- co-volume of L is $|\det(A)|$
- L and L' are homothetic if $\exists\alpha \in \mathbb{Z} : L = \alpha L' \;\; or \;\; L' = \alpha L$. Denote by $L \approx L'$.
- L is primitive if $\forall\alpha \in \mathbb{Z} : L/\alpha \nsubseteq \mathbb{Z}^d$

Vertices of the building

- Fix a prime p for the rest of the construction.
- $\left(X_p^d\right)^0 =$ integral lattices with a p-power co-volume up to homothety $=$ primitive integral lattices with a p-power co-volume =

$$\left\{ \begin{pmatrix} p^{n_1} & & & \\ & \ddots & & \boldsymbol{a_{i,j}} \\ & & \ddots & \\ & \mathbf{0} & & \ddots \\ & & & & p^{n_d} \end{pmatrix} \middle| \begin{array}{l} 0 \leq a_{i,j} < p^{n_i}, \quad n_i \geq 0 \\ \gcd\left(a_{i,j}, p^{n_k}\right) = 1 \end{array} \right\}$$

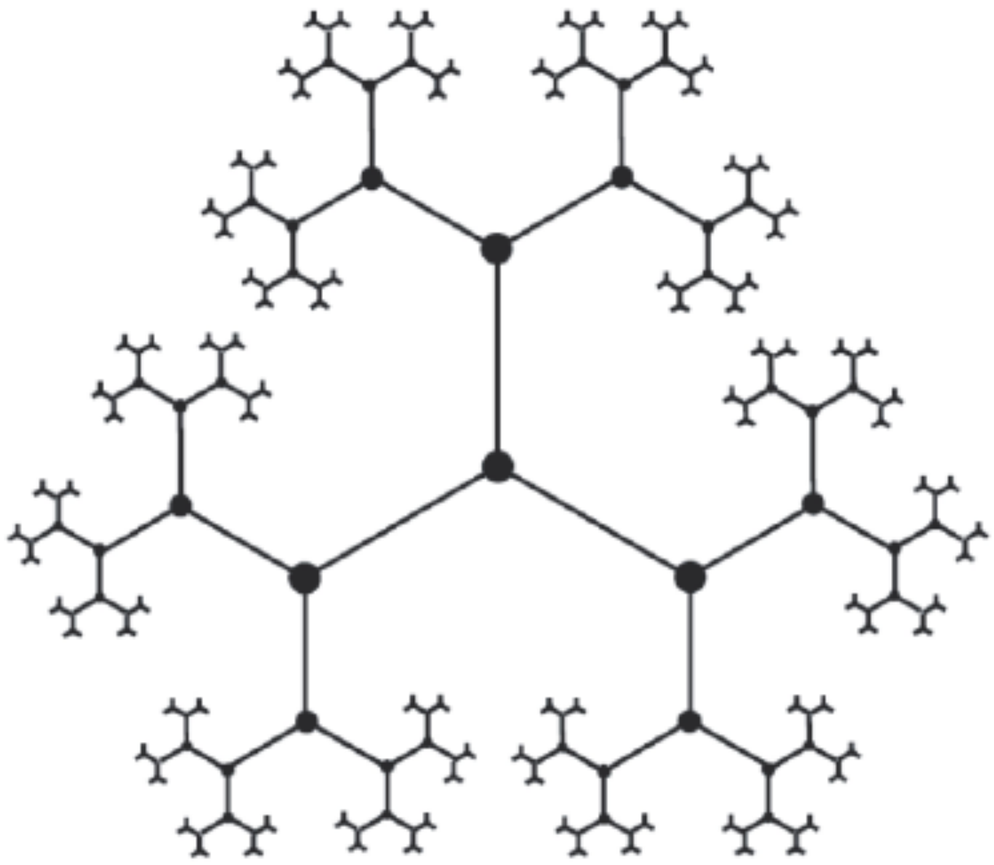
Adjacency of lattices

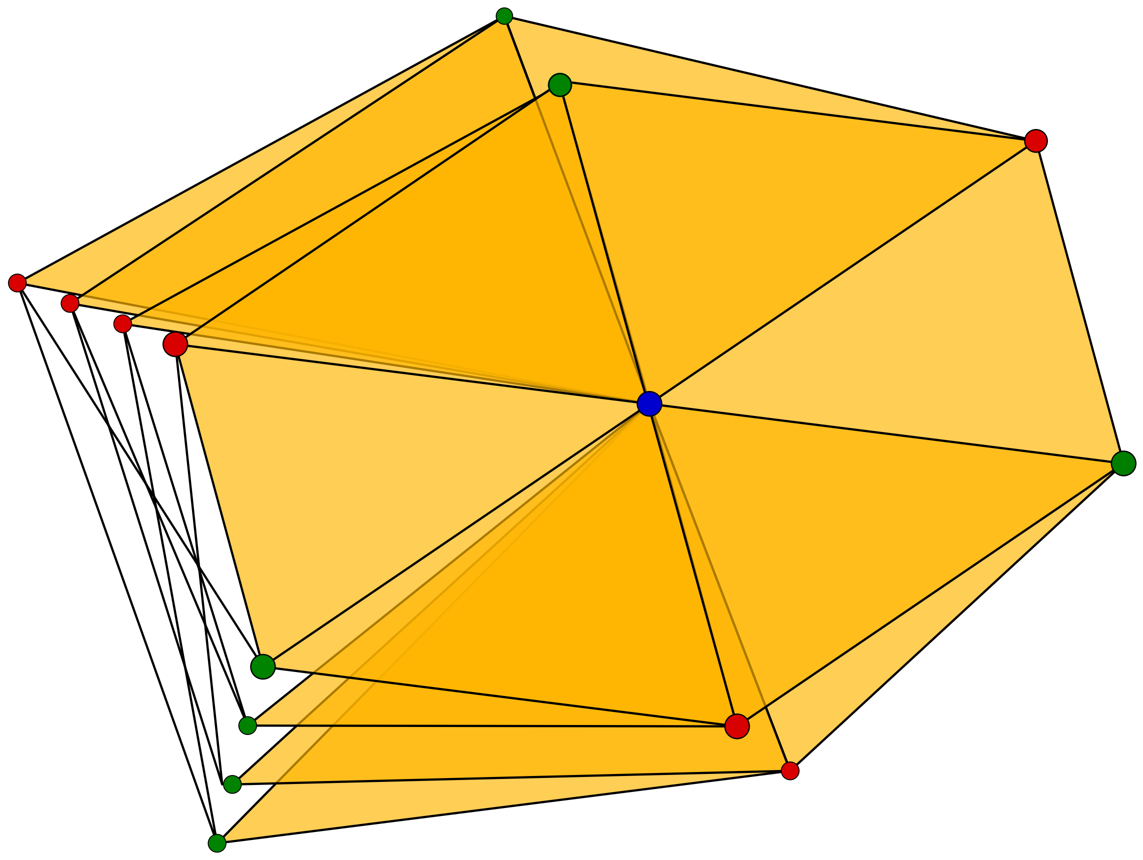
- $L, L' \in \left(X_p^d\right)^0$ are adjacent if $pL \subseteq L' \subseteq L$ or $pL' \subseteq L \subseteq L'$.

- Examples: Let $p = 3$, $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 9 & 1 \\ 0 & 1 \end{pmatrix}$

- X_p^d is the clique complex of the graph we got

Examples by Web and OP





Algebraic description of adjacency

- Given A, B in primitive form, $A \sim B$ if there exists N of the form

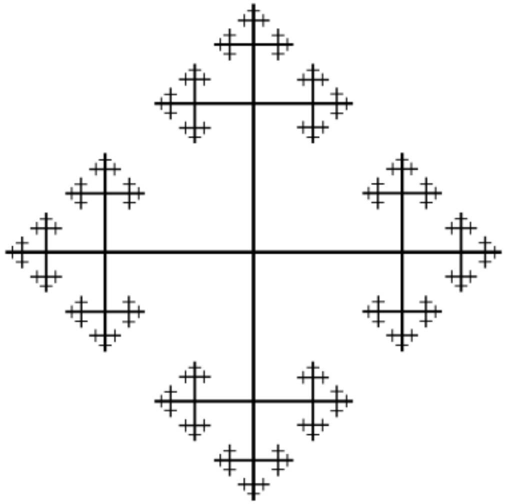
$$\begin{pmatrix} p & * & 0 & * & 0 \\ & 1 & 0 & 0 & 0 \\ & & p & * & * \\ & \mathbf{0} & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

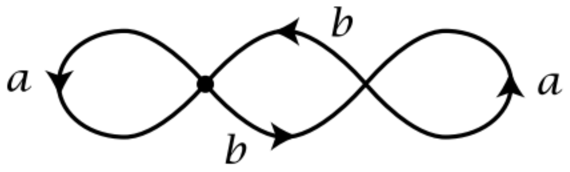
such that either $A\mathbb{Z}^d \approx BN\mathbb{Z}^d$ or $AN\mathbb{Z}^d \approx B\mathbb{Z}^d$

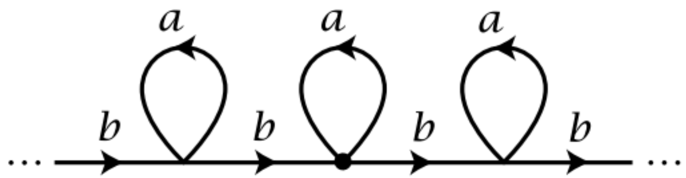
Action of $PGL_d(\mathbb{Q}_p)$

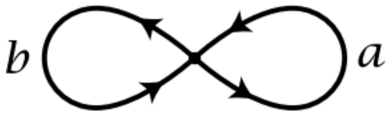
- Alternative definition of $\left(X_p^d\right)^0$ is $PGL_d(\mathbb{Q}_p)/PGL_d(\mathbb{Z}_p)$
- If we define the left action of $PGL_d(\mathbb{Q}_p)$ on its cosets we get a **simplicial** action on the building.

Interlude – Covering spaces









From Hatcher

Interlude – Quotient spaces

- G a group and X a topological space
- $G \curvearrowright X$ nicely we can define X/G
- If X is a simplicial complex and G acts nicely then X/G is also a simplicial complex
- <u>Example</u>: Every $2d$-reg graph is a quotient of the $2d$-reg tree via the action of the rank d free group on it

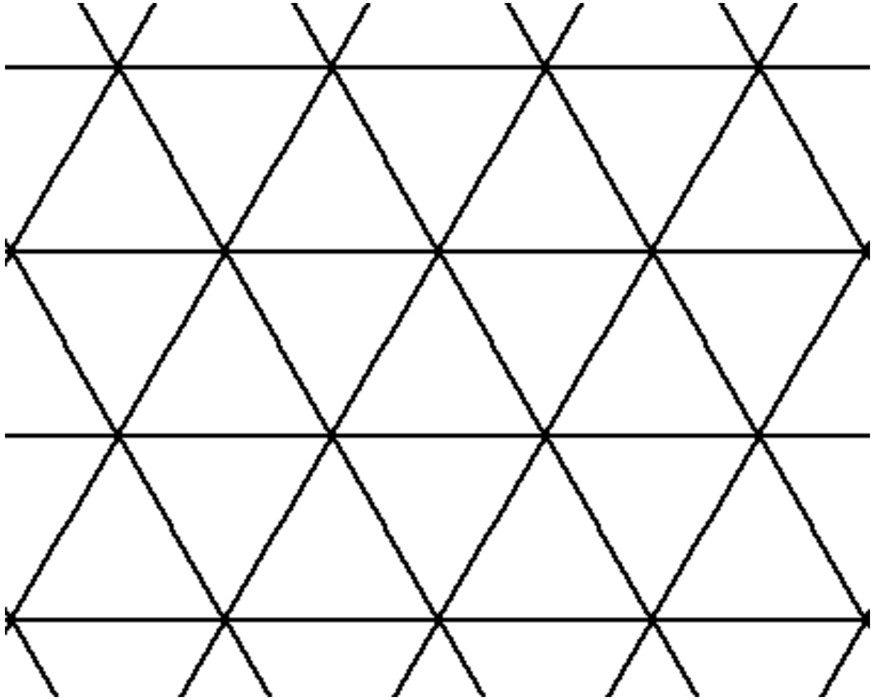
The apartment and folded apartment

- Since $PGL_d(\mathbb{Q}_p) \curvearrowright X_p^d$, we can ask what is the quotient space w.r.t the stabelizer of a cell.

- If we take any cell of dim $d-1$ we get an apartment.

- If we take a vertex we get a folded apartment

Examples



Fundumental apartment – Algebraic viewpoint

$$X_1^d = \left\{ \begin{pmatrix} p^{n_1} & & \mathbf{0} \\ & \ddots & \\ & & p^{n_d} \end{pmatrix} \mid \min\{n_1, \ldots, n_d\} = 0 \right\}$$

$$\mathcal{S} = \left\{ \begin{pmatrix} p^{\alpha_1} & & & & 0 \\ & p^{\alpha_2} & & & \\ & & \ddots & & \\ & & & p^{\alpha_{d-1}} & \\ & & & & 1 \end{pmatrix} \middle| \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{d-1} \right\}$$

Random walks and the cutoff phenomenon

- Given a d-reg directed graph D we can define the *random walk operator* P in the following way: if μ is a probability measure on the vertices of the graph, then

$$\mu P(x) = \frac{1}{d} \sum_{y \to x} \mu(y) \quad \forall x \in V(D)$$

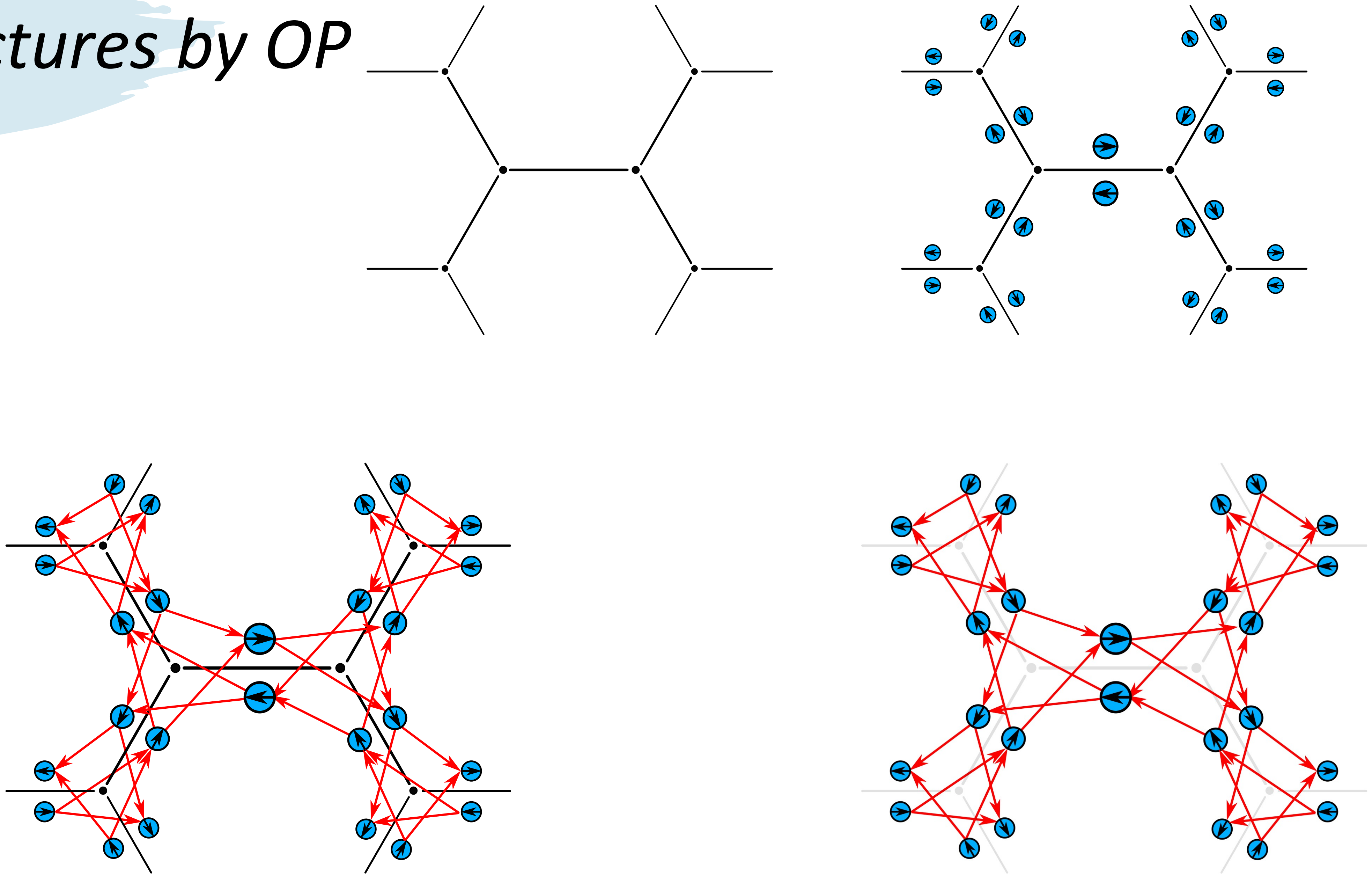
- Note that P is a bounded operator on $L^2(D)$

SRW and NBRW

- If G is a non-directed reg graph we can define the simple random walk operator by replaceing every non-directed edge by two directed ones and using the previous definition

- For the same graph we can define the Non-backtracking random walk whose vertices are the directed edges we created before and you can move from uv to vw if $u \neq w$

Pictures by OP



Examples

Stationary distribution and mixing time

- If D is a d-reg graph with n vertices and P is ergodic (it has a strictly positive power) one can show that

$$\mu P^t \xrightarrow{t\to\infty} \frac{1}{n}$$

where $\frac{1}{n}$ is the uniform distribution on the vertices.

- We define for every $\varepsilon > 0$ the total variation ε –mixing time of P to be

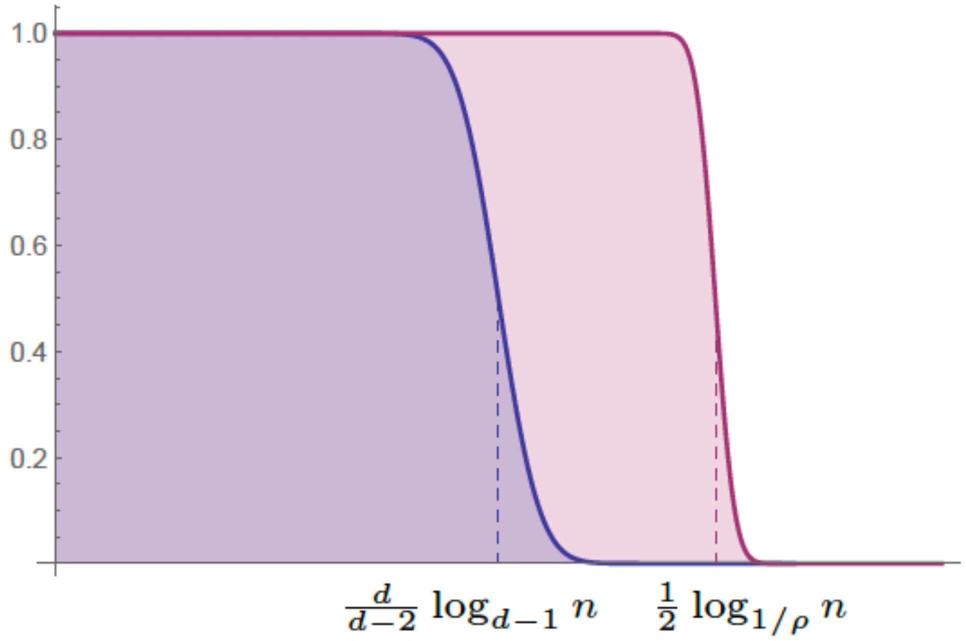
$$t_{mix}(\varepsilon) = \min\{t \geq 1 \mid \forall x \in D, ||\delta_x P^t - \pi||_1 \leq 2\varepsilon\}$$

The cutoff phenomenon

- A sequnece $\{D_n\}$ of d-reg directed graphs are said to exhibit cutoff if for every $\varepsilon > 0$ we have

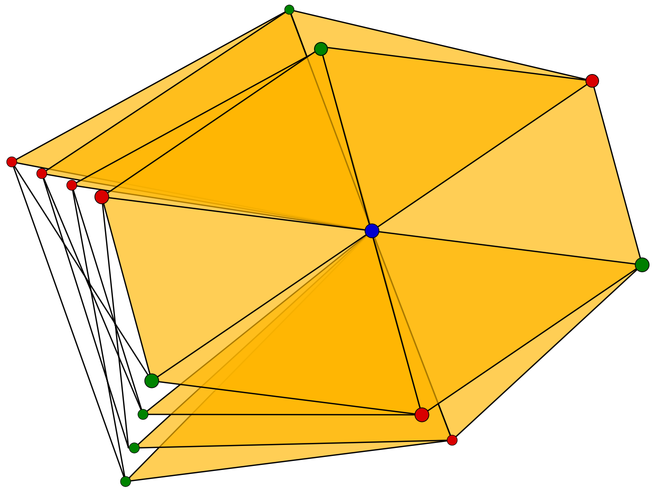
$$\frac{t_{mix}(\varepsilon,\mathcal{D}_n)}{t_{mix}(1-\varepsilon,\mathcal{D}_n)} \xrightarrow{n\to\infty} 1$$

Pictures from LP



Walks on simplicial complexes

- If X is a simplicial complex, we can define walks on its higher dim (ordered or unordered) cells.
- For example: Move uniformly from a triangle xyz to yzw such that $w \neq x$



Ramanujan graphs and complexes

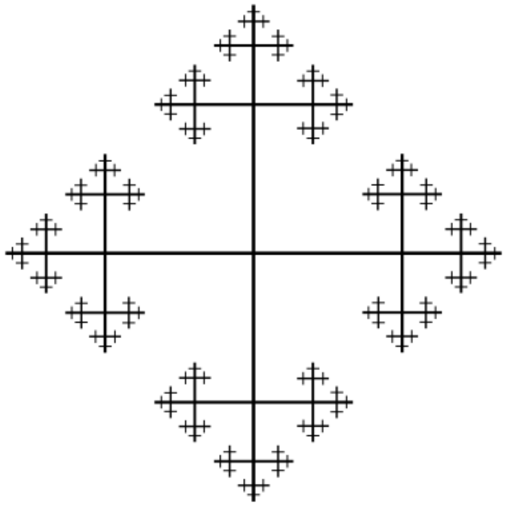
- "Ramanujan complexes mimic their universal cover spectrally"
- A d-reg graph is Ramanujan if the non-trivial spectrum of the SRW operator on it is contained in that of its covering tree

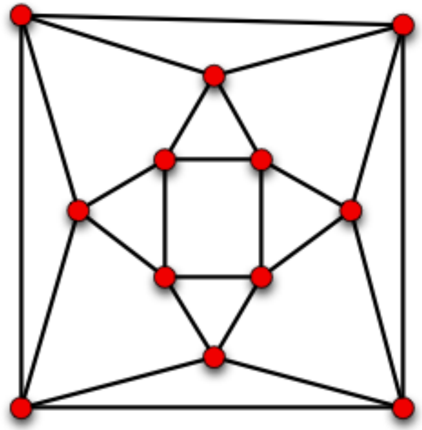
Ramanujan graphs and complexes

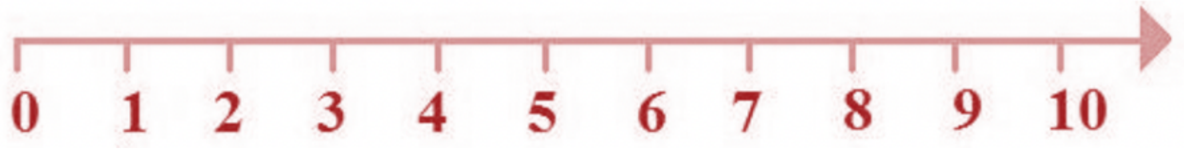
- A finite quotient of the building X_p^d is Ramanujan if the non-trivial spectrum of any "geometric" operator on cells of the complex is contained in the spectrum of this operator on the building

- Notes:
  - The operators we have shown are geometric
  - There exist inifinitely many Ramanujan complexes (Fir,Li04,LSV05,EP18) and they are useful and interesting

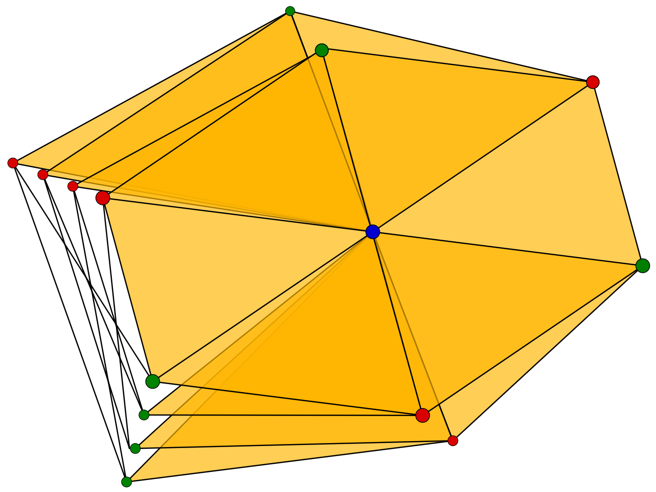
Apartment analysis – 1 dim case

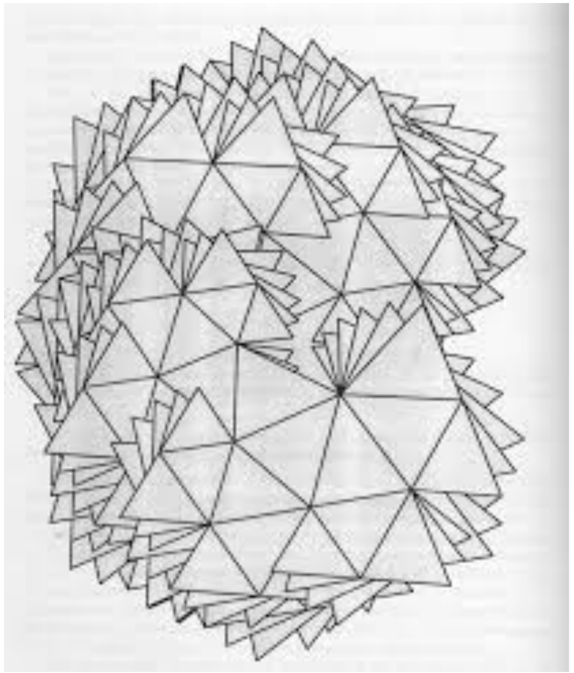


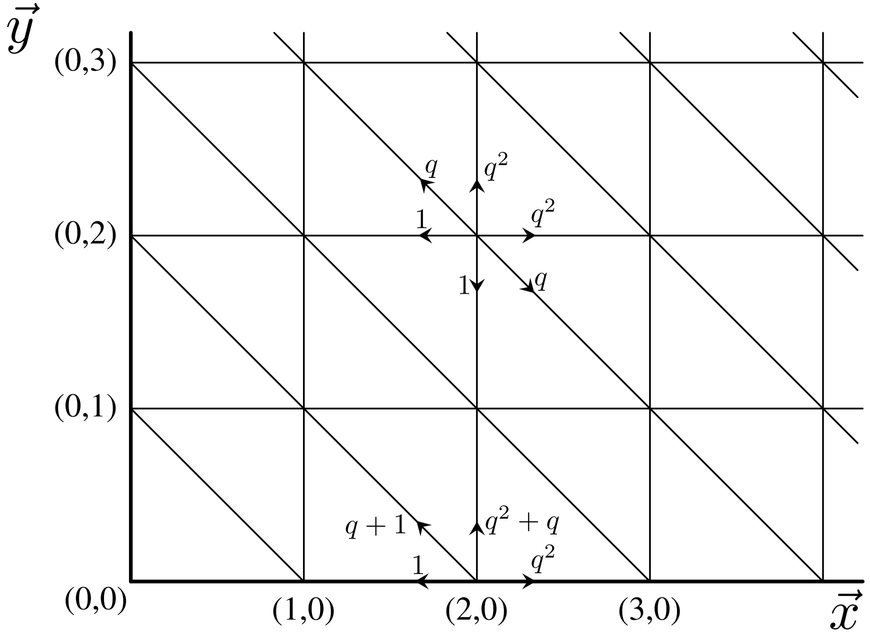




Apartment analysis – 2 dim case







Thank you