RAAGs & expanders

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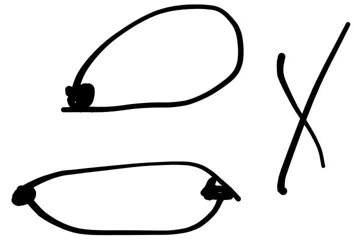
Expander graphs

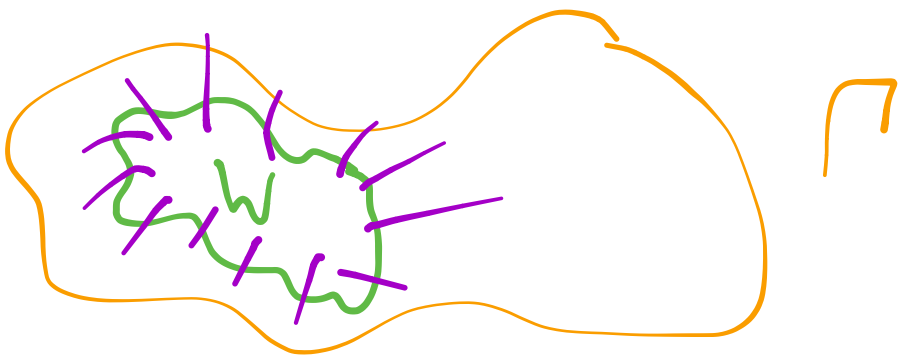
graphs are finite & simplicial



$\{\Gamma_i\}_{i \in \mathbb{N}}$ graphs

① connected, ② bounded valence, ③ difficult to disconnect.

③ robustness of network condition.
difficult to cut out large pieces of the network



Cheeger Constant (expansion constant)

$W \subset V(\Gamma) \qquad |W| \leq |V|/2$

$\partial W = \{ v \in V \mid v \notin W, \ v \text{ adjacent to } W \}$

$$h_W(\Gamma) = \frac{|\partial W|}{|W|}$$

$$h(\Gamma) = \min_{W \subset V,\ |W| \leq |V|/2} h_W(\Gamma).$$

robustness condition for an expander family is just $\inf_i h(\Gamma_i) > 0$.

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expansion versus diameter

if Γ is a conn. graph, valence $\leq k$

$$\operatorname{diam}(\Gamma) \geq \frac{\log |V|}{\log k}$$

$$\operatorname{diam} \Gamma \leq 2 \frac{\log \frac{|V|}{2}}{\log\left(1 + \frac{h(\Gamma)}{k}\right)} + 3$$

expander graphs are optimized for minimal diameter subject to resource constraint.

Barzdin - Kolmogorov

thick embeddings of graphs into $\mathbb{R}^3$.

nodes are separated & have radius 1

edges stay at least 1/2 apart unless they meet a common node.

Lemma: any finite of valence ≤ 6 admits a thick embedding into $\mathbb{R}^3$.

Q: what is the minimal radius of a ball needed to accommodate a thick embedding?

Thm (Barzdin-Kolmogorov): $\geq c\sqrt{h(\Gamma)|V|}$

$\exists \{\Gamma_i\}$ bdd valence ≤ 6 with

$|V_i| \longrightarrow \infty$ $\inf_i h(\Gamma_i) > 0$.

other applications: complex geometry

(gonality)

spectral geometry of R.S.

distortion of knots, diophantine problems, sieves, error correction in prob. computation

...

RAAGs $\quad \Gamma$ graph

$$A(\Gamma) = \langle V(\Gamma) \mid [v_i, v_j] = 1 \iff \{v_i, v_j\} \in E(\Gamma) \rangle$$

iso. type of $A(\Gamma)$ determines the iso. type of Γ & vice versa.

(Droms, Sabalka, ...)

$$\{\text{gp thy}\} \xleftrightarrow[\text{RAAGs}]{} \{\text{combinatorics}\}$$

Q: given a comb. property of graphs, how is this reflected algebraically in RAAGs?

Ex: ① Γ contains a square $\iff$ $A(\Gamma)$ contains a copy of $F_2 \times F_2$

(Kambites)

② Γ admits a nontrivial automorphism $\iff$ $Out(A(\Gamma))$ contains a finite nonab. subgroup.

(Flores - Kahrobaei - K)

③ Γ is k-colorable $\iff$

$$A(\Gamma) \twoheadrightarrow \prod_{i=1}^{k} F_{n_i}$$

with $\sum_{i=1}^{k} n_i = \underbrace{|V(\Gamma)|}_{\text{rank of abelianization of } A(\Gamma)}$.

(Flores-Kahrobaei-K).

Now: how to characterize expander families?

<u>Vector space expanders</u> $\{V_i, W_i, g_i\}_{i \in \mathbb{N}}$

V_i, W_i are vector spaces over the

same field. $\quad q_i : V_i \times V_i \longrightarrow W_i$

bilinear map, (anti)-symmetric.

$\{V_i, W_i, q_i\}$ is a vector space expander. $\exists$

① $\dim V_i \longrightarrow \infty$

② ~~pass~~ each V_i is pairing-connected.

$V_i = V \neq V^0 \oplus V'$ $\quad V^0, V' \neq 0$ $\quad$ & $q(v, w) = 0$ $\quad v \in V^0, w \in V'$

③ finite q-valence (indep. of i)

$\emptyset \neq S \subset V$ $\qquad B$ a basis for V

$$d_B(S) = \max_{s \in S} \left| \{ b \in B \mid q(s, b) \neq 0 \} \right|$$

$$d(S) = \min_{B \text{ a basis}} d_B(S)$$

$$d(V) = \min_{S \text{ spans } V} d(S) \Leftarrow q\text{-valence of } V$$

4. unif. bounded expansion constant.

$F \subset V$ subspace of dim $\leq \frac{1}{2}\dim V$

C = complement of F in V wrt q

$$h_F = \frac{\dim V - \dim F - \dim C + \dim(C \cap F)}{\dim F}$$

$$h_V = \inf_{\dim F \leq \frac{\dim V}{2}} h_F .$$

Thm (Flores – Kahrobaei – K)

A sequence of finite simp. graphs $\{\Gamma_i\}_{i \in \mathbb{N}}$ forms an expander family

$\iff \{V_i, W_i, q_i\}_{i \in \mathbb{N}}$ is a V.S. expander, $V_i = H^1(A(\Gamma_i))$

$W_i = H^2(A(\Gamma_i))$

q_i = cup product.

Remarks: $\operatorname{rk} A(\Gamma_i) = |V(\Gamma_i)| =$
$\dim H^1(A(\Gamma_i))$
$|V(\Gamma_i)| \to \infty \Leftrightarrow \dim V_i \to \infty$

Lemma: Γ_i is connected $\Leftrightarrow$
$H^1(A(\Gamma_i))$ is $\cup$-connected.

Lemma: $\cup$-valence of $H^1(A(\Gamma_i))$ differs from the max. valence of Γ_i by ≤ 1.

$H^1(A(\Gamma))$ & $H^2(A(\Gamma))$ admit nice bases $v_1^*, \dots, v_k^*, e_1^*, \dots, e_\ell^*$ in bijection with $V = \{v_1, \dots, v_k\}$
$E = \{e_1, \dots, e_\ell\}$

$$v_i^* \cup v_j^* = \begin{cases} \pm e_k^* & \text{if } e_k = \{v_i, v_j\} \\ 0 & \text{ow.} \end{cases}$$

if $F \subset H^1(A(\Gamma))$ is gen by.
$v_1^*, \dots, v_s^*$ $\quad W = \{v_1, \dots, v_s\}$

$$h_F = \frac{|\partial W|}{|W|} \quad \text{(easy lemma)}$$

Lemma (harder): given $F \subset V$ arb. $\dim F \leq \frac{1}{2} \dim V$

$$h_F \geq \frac{|\partial W|}{|W|} \quad \text{for some } W \subset V \quad |W| \leq \tfrac{1}{2}|V|$$

tedious sorting argument.

Final remarks

1. V.S. expanders are more general than graph expanders.

   ∴ $\exists$ sequence of V.S. expanders not arising from cohomology of RAAGs from an expander family.

2. no good higher dimensional version.

3. no relation to dimension-expanders.

   Lubotzky–Zelmanov

Bourgain – Yehudayoff

$\{V_i, W_i, q_i\}$ coming from $\{\Gamma_i\}$

fixed i choose $v_1 \in \Gamma = \Gamma_i$

$I = I_i$ $\quad q_0(v_i, v_j) = \begin{cases} 1 & \text{if } i=j=1 \\ 0 & \text{ow.} \end{cases}$

$V_i' = V_i \qquad W_i' = W_i \oplus F$

$q_i' = q_i \oplus q_0$

in coho. of a RAAG

$v \cup v = 0 \quad v \in H^1$

not true for q_i'

expansion constant unchanged.