Cobordism Groups

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Goal: Classify Lagrangian submanifolds in a symplectic mfd (M, ω)

- up to Hamiltonian isotopy
- up to Lagrangian isotopy
- up to Lagrangian cobordism

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Definition: Let $* \subset \{$embedded, immersed, compact, exact, monotone, unobstructed, oriented, ...$\}$ be a subset of properties for Lagrangian submanifolds.

$$\mathrm{Cob}^*(M) = \mathrm{Lag}^*(M) \,/\, \mathcal{R} \Leftarrow \sum L_i = \sum L_j' ,$$

whenever $\exists$ cobordism V with property $*$ and ends L_i, L_j'

$\mathrm{Lag}^*(M)$: free abelian group generated by Lagrangians with property $*$

e.g. · $L_0 \overset{\text{Ham isotopic}}{\approx} L_1 \Rightarrow L_0 = L_1$

· $L = L_1 \#_p L_2 \Rightarrow L = L_1 + L_2$

in $\mathrm{Cob}^*(M)$

· see later for one more

Liouville domains

Definition A Liouville domain is a compact exact symplectic manifold (M,ω) with contact boundary ∂M.

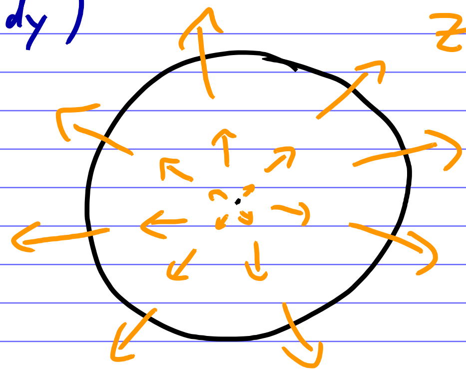
$\omega = d\lambda$, λ is called Liouville form

the vector field Z defined by $\omega(Z,-) = \lambda$ is outward pointing on ∂M

Ex $\left(M = \{(x,y) \in \mathbb{R}^{2n} \mid \|(x,y)\| \leq 1\},\ \omega = dx \wedge dy\right)$

$\lambda = \frac{1}{2}(x\,dy - y\,dx)$

$Z = \frac{1}{2}(x\,\partial_x + y\,\partial_y)$



$n = 1$

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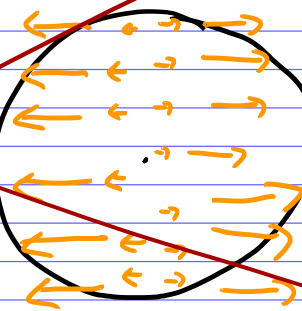

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no Liouville domain

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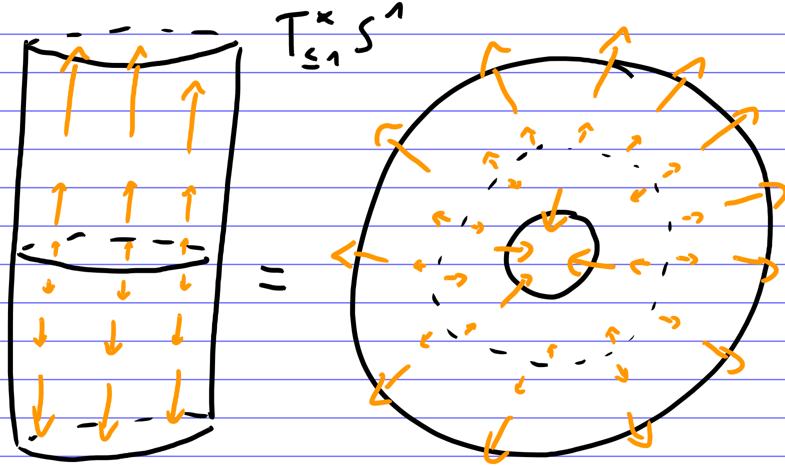
Ex $\left(M = T^*_{\leq 1} N, \ \omega = dp \wedge dq\right)$ ← compct Riem. mfd

$\lambda = p\,dq$

$Z = p\,\partial p$



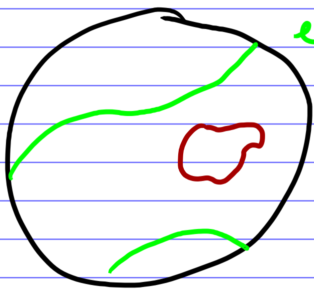
Exact conical Lagrangian submanifolds

Definition · A Lagrangian $L \subset \underbrace{(M, d\lambda)}_{\text{Liouville domain}}$ is **exact** if $\lambda|_L$ is exact

· $L \subset (M, d\lambda)$ is called **conical** if $\lambda|_{\partial L} = 0$.

Rem. If $H^1(L) = 0$ then $\omega|_L = 0 \Leftrightarrow \lambda|_L$ closed $\Leftrightarrow \lambda|_L$ exact

Ex



not exact: If $L = \partial D$ and L exact

$$\mathrm{vol}(D) = \int_D d\lambda = \int_L \lambda = 0 \qquad ↯$$

Stokes

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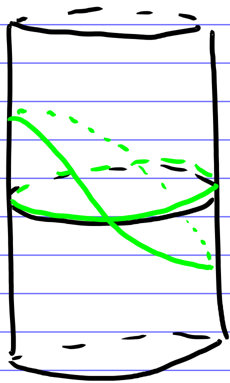
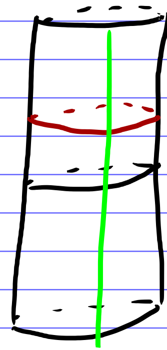


exact, conical



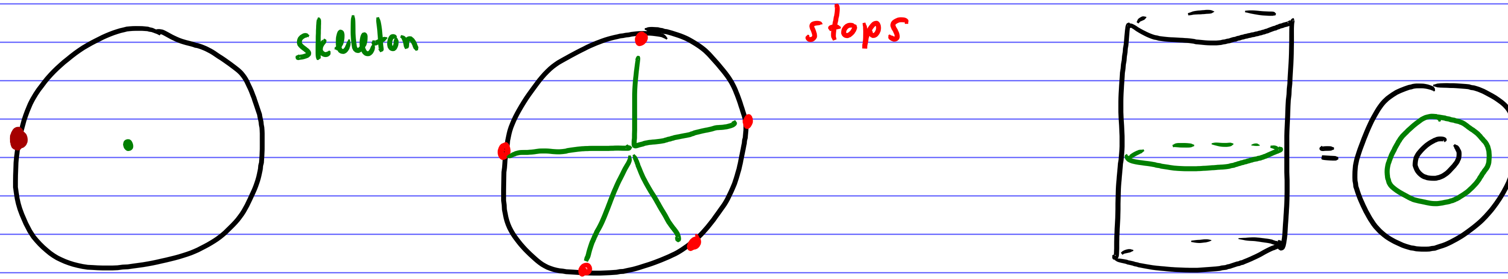
not exact

Skeleton, cocores and stops

Definition · A *stop* $S \subset \partial M$ is a closed subset

· The *skeleton* $sk(M)$ of $(M, d\lambda, S)$ is the set of points which flows to $S \cup int(M)$ under Z for ∞ time.

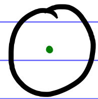
Ex

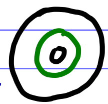
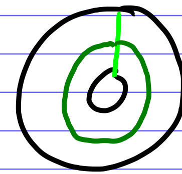


Lemma If $L \subset (M, d\lambda, S)$ exact and $L \cap sk(M) = \emptyset$ then $\exists$ cobordism $L \rightsquigarrow \emptyset$, i.e. $L = 0$ in $Cob^*(M, S)$

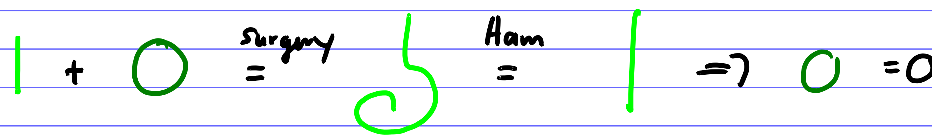
Calculation of groups

Consider $* = \{$embedded, exact, conical, compact$\}$

Ex $Cob^*($  $) = \{0\}$ because
- $L \approx S^1 \Rightarrow L$ not exact
- $L \approx [0,1] \Rightarrow$ displacable from skeleton

Ex $Cob^*($  $)$ generated by a fiber 

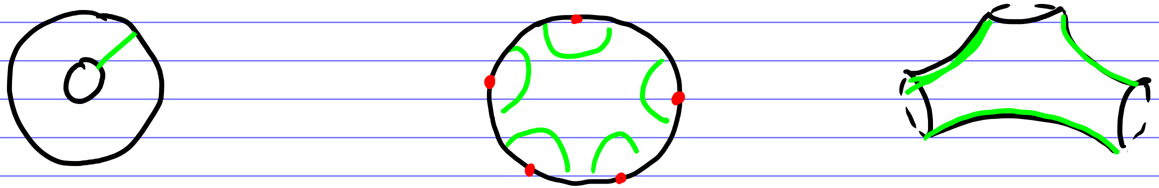
because
- $L \approx S^1 \Rightarrow L \overset{Ham}{\approx}$ zero section
- $L \approx [0,1]$:
  - If ∂L on same component of $\partial\Sigma$ then we can make L and $sk(\Sigma)$ disjoint $\Rightarrow L = 0$
  - otherwise $L =$ fiber

But also note 

Arc systems and Ribbon graphs

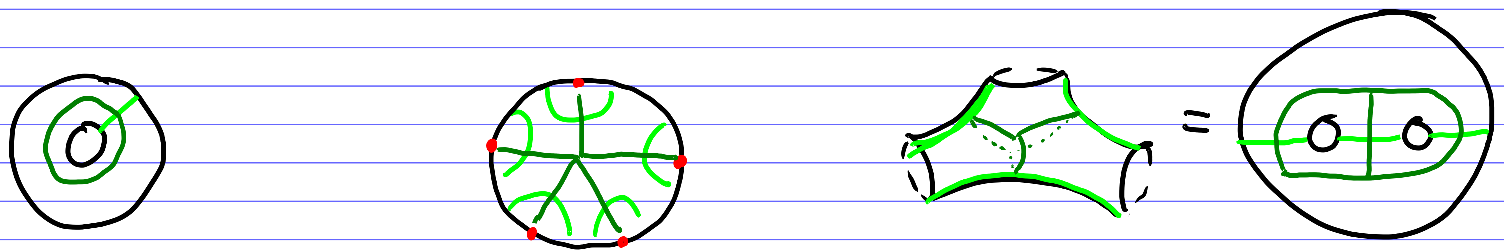
Def A <u>full arc system</u> in an open surface (Σ, S) is a set $\{\gamma_i\}$ of disjoint arcs such that $\Sigma \setminus \bigcup \gamma_i$ consists of topological disks with at most one stop.

Ex



Rem $\{$ Ribbon graphs $\} \longleftrightarrow \{$ arc systems $\}$

Ex



The cobordism groups of open surfaces

Thm Suppose (Σ, S) is an open surface Σ with stops S on $\partial\Sigma$

Then $\mathrm{Cob}^{\text{embedded, exact}}(M, S)$ is generated by any full arc system $\{\gamma_i\}$.

Moreover,

$$\mathrm{Cob}^{\text{embedded, exact}}(\Sigma, S) = H_1(\Sigma, \partial\Sigma \setminus S)$$

Conj (Higher dimensions) For a Weinstein domain $(M, d\lambda)$

$$\mathrm{Cob}^{\text{embedded, exact}}(M)$$

is generated by cocores of $Sk(M)$.