Spectral Invariants and Irie's Theorem

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Overview

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- Classical Lusternik-Schnirelmann theory requires a nice action functional $\mathcal{E}$.

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- We cannot apply the classical LS theory.
- However, there is still a well-defined notion of min-max values, called **spectral invariants**.

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For $\beta \in \mathrm{HF}_\bullet(H) \setminus \{0\}$ we define the **spectral invariant** $c_\beta(H)$ as

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- Moreover

$$c_\beta(H) = \inf\{a \mid \beta \in \mathrm{im}(i_a)\} = \inf\{a \mid j_a(\beta) = 0\},$$

where $i_a \colon \mathrm{HF}_\bullet^{<a}(H) \to \mathrm{HF}_\bullet(H)$ and $j_a \colon \mathrm{HF}_\bullet(H) \to \mathrm{HF}_\bullet^{\geq a}(H)$.

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  where $i_a \colon \mathrm{HF}_\bullet^{<a}(H) \to \mathrm{HF}_\bullet(H)$ and $j_a \colon \mathrm{HF}_\bullet(H) \to \mathrm{HF}_\bullet^{\geq a}(H)$.
- If $d < c_\beta(H)$ then $\beta \notin \mathrm{im}(i_d)$ and $j_d(\beta) \neq 0$.

Properties

- If H is autonomous and C^2-small, then $c_\beta(H) = c_\beta^{\mathrm{LS}}(-H)$ where $\beta \in \mathrm{HF}_\bullet(H) \cong \mathrm{HM}_\bullet(-H) \cong H_\bullet(M)$.

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- For degenerate Hamiltonians K:

$$c_\beta(K) = \lim_{n\to\infty} c_\beta(H_n), \quad H_n \xrightarrow{C^2} K.$$

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- Phrased "differently",

$$c_\beta(H) \in \mathcal{P}(H) := \{\mathcal{A}_H(x) \mid x \in \mathcal{L}M,\, \dot{x} = X_H(x)\}.$$

Hofer-Zehnder admissible functions

- Now let (W, ω) be a symplectic manifold, possibly with boundary $\partial W = \Sigma \neq \emptyset$. Roughly speaking, $C_{\mathrm{HZ}}(W, \omega) \in [0, +\infty]$ is the biggest possible oscillation of $F \colon W \to \mathbb{R}$.

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- Picture:

Hofer-Zehnder admissible functions

- If additionally the *contractible* periodic orbits $x\colon [0,T] \to W$ of X_F are either trivial (i.e. $T = 0$) or of period strictly greater than 1 (i.e. $T > 1$), we shall call F **Hofer-Zehnder admissible**.

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Definition

We define the **Hofer-Zehnder capacity** as

$$C_{\mathrm{HZ}}(W,\omega) = \sup\left\{-\min F \;\middle|\; F\colon W \to \mathbb{R} \text{ is Hofer-Zehnder admissible}\right\}.$$

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- The finiteness of Hofer-Zehnder Capacity has striking dynamical implications.

Finiteness of Hofer-Zehnder Capacity

Theorem (Hofer-Zehnder)

Let $H\colon W \to \mathbb{R}$ be a Hamiltonian with $S_0 := H^{-1}(0)$ a compact hypersurface. Assume that there exists an open neighbourhood U of S_0 such that

$$C_{HZ}(U, \omega) < +\infty.$$

Then there exists a dense set of nearby hypersurfaces S_λ that carry a non-trivial periodic Hamiltonian orbit x of X_H.

Proof:

- Define an auxiliary function $f\colon \mathbb{R} \to \mathbb{R}$, $V = H^{-1}\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, $K = H^{-1}[-\varepsilon, \varepsilon]$ and $F := f \circ H\colon W \to \mathbb{R}$ as follows:

- Definition of F and HZ capacitiy: $y\colon [0, T] \to W$, $T \in (0, 1)$, $\dot{y} = X_F(y)$.
- Chain rule: $(f' \circ H) \cdot X_H = X_F$. Consequently $\frac{\partial}{\partial t} H(y(t)) = 0$, hence $H(y) \equiv \lambda$.
- Nontriviality of $y \implies f'(\lambda) = f'(H(y)) \neq 0$, thus $\tau := f'(\lambda) \in \left(-\varepsilon, -\frac{\varepsilon}{2}\right) \cup \left(\frac{\varepsilon}{2}, \varepsilon\right) \implies |\lambda| < \varepsilon$.
- Then $x\colon [0, T\tau] \to W$, $x(t) = y(T\tau^{-1})$ satisfies $\dot{x} = X_H(x)$ and $H(x) = \lambda$.

Symplectic Homology

- Let $(W, -d\lambda)$ be a Liouville domain, $\Sigma = \partial W$, and $(M, -d(r \cdot \lambda))$ its symplectic completion.

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Definition

The symplectic homology $\mathrm{SH}_\bullet(W)$ is defined as the direct limit (for $\tau \to \infty$) of the groups $\mathrm{HF}_\bullet(H^\tau)$, where $H^\tau \colon M \to \mathbb{R}$ are Hamiltonians that are linear at infinity with slope τ.

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Some properties

- For $\tau_0 > 0$ small enough and H^{τ_0} C^2-small on W, one has $\mathrm{HF}(H^{\tau_0}) = \mathrm{HM}\left(-H^{\tau_0}|_W\right) = H(W, \partial W)$.

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- A Theorem of Viterbo asserts $\mathrm{SH}(D_1^* Q) \cong H(\mathcal{L}Q)$ (also see Salamon-Weber, Abbondandolo-Schwarz).

Irie's Theorem

- Denote by β_τ the image of the fundamental class $[W]$ under $\mathrm{HF}_\bullet(H^{\tau_0}) \to \mathrm{HF}_\bullet(H^\tau)$. As before, we define $c_\beta(H^\tau)$ for $\beta \in \mathrm{HF}_\bullet(H^\tau)$.

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Theorem (Irie 2012)

Let (W, λ) be as above. If there exists a slope $\tau > 0$ such that $\beta_\tau = 0$, then

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- Combining this with Hofer and Zehnder's results shows that $\mathrm{SH}(W) = 0$ has strong dynamical implications for compact hypersurfaces of $H\colon M \to \mathbb{R}$ inside W^0.

Proof

- For contradiction, assume that there is a HZ admissible $H\colon W^0 \to \mathbb{R}$ with $-\min H > \tau$ (goal: show that $c_{\beta_\tau}(H^\tau) = -\min H$).
- Extend H to $H_\varepsilon^{\delta\tau}\colon M \to \mathbb{R}$, where $H_\varepsilon^{\delta\tau}|_{W^0} = \varepsilon \cdot H$.
- Then $c_{\beta_{\delta\tau}}(H_\varepsilon^{\delta\tau}) = c_{[W]}(-\varepsilon \cdot H) = -\varepsilon \cdot \min H$, for $\varepsilon, \delta > 0$ small.
- Let $\varepsilon \to 1$: using spectrality one can show $c_{\beta_{\delta\tau}}(H^{\delta\tau}) = -\min H$, for $\delta > 0$ small.
- Since $\tau < -\min H$: $\mathcal{P}(H^\tau) \subseteq (-\infty, -\min H]$, thus $c_{\beta_\tau}(H^\tau) \leq -\min H$.
- Now we show $c_{\beta_\tau}(H^\tau) \geq -\min H$:

Proof

- For small $e > 0$ and $d := -\min H - e$ we have

$$\begin{array}{ccc} \beta_{\delta\tau} \in \mathrm{HF}_{2n}(H^{\delta\tau}) & \xrightarrow{j_d} & \mathrm{HF}_{2n}^{\geq d}(H^{\delta\tau}) = \langle [x] \rangle \\ \downarrow & & \downarrow \cong \\ \beta_{\tau} \in \mathrm{HF}_{2n}(H^{\tau}) & \xrightarrow{j_d} & \mathrm{HF}_{2n}^{\geq d}(H^{\tau}) = \langle [x] \rangle \end{array}$$

- But $j_d(\beta_{\delta\tau}) \neq 0$ (since $d < c_{\beta_\tau}(H^{\delta\tau})$), thus $j_d(\beta_\tau) \neq 0$, hence $d \leq c_{\beta_\tau}(H^\tau)$.
- Therefore we have shown $c_{\beta_\tau}(H^\tau) = -\min H$, but $\beta_\tau = 0$ by assumption, hence $c_{\beta_\tau}(H^\tau) = -\infty$. Contradiction.

Thanks for listening!