IMMERSED LAGRANGIAN QUANTUM COHOMOLOGY

<u>Setup:</u>

* $(M^{2n}, \omega = d\lambda)$ exact symplectic mnfld s.t. $c_1(M) = 0$;
* L^n closed connected manifold, $i : L \to M$ exact Lagrangian immersion with only transverse double points;
* J almost complex str. on M satisfying some "convexity" properties;
* Ω nowhere vanishing top holomorphic form on M.

We assume that i is graded, i.e. $\exists \Theta_L : L \to \mathbb{R}$:

$$e^{2\pi i \Theta_L} = \det^2_\Omega \circ s_L$$

where $s_L(x) := Di(T_xL)$, $x \in L$.

Let $R := \{(p,q) \in L \times L : i(p) = i(q),\ p \neq q\}$ and define

$$|p,q| := n + \Theta_L(q) - \Theta_L(p) - 2 \cdot ∡\left(Di[T_pL], Di[T_qL]\right)$$

Fix a primitive f_L and define $A(p,q) := f_L(q) - f_L(p)$.

KEY ASSUMPTION: If $(p,q) \in R$ s.t. $A(p,q) > 0$,

then $|p,q| \geq 3$.

Ex. $\Pi_N := \{ F_N = 0 \} \subseteq \mathbb{C}^3$, $F_N = xy - \prod_{i=1}^{N} (z - i)$

with induced ω, J from $\mathbb{C}^3$ and

$\Omega := \mathrm{Res}\left(\frac{dx \wedge dy \wedge dz}{F_N} \right)$. $\left(N = 1:\ \Pi_N \cong \mathbb{C}^2,\quad N = 2:\ \Pi_N \cong T^* S^2 \right)$

For $r \in \{1, \dots, N\}$: $\quad L_{N,r} := \{ |x|^2 = |y|^2,\ |z|^2 = r \} \subseteq \Pi_N$

is a Log immersed sphere with one $\pitchfork$ self-intersection. In cylindrical coordinates on S^2:

$$i_{N,r} : (a, e^{ib}) \longmapsto \left(e^{ib} \xi(a),\ e^{-ib} \xi(a),\ -re^{ia} \right)$$

where $\xi(a) := \prod_{i=1}^{N} \sqrt{-re^{ia} - i}$.

Let $p_{S^2} := (\pi, 0)$, $q_{S^2} := (-\pi, 0)$, $i(p_{S^2}) = i(q_{S^2})$

$$|p_{S^2}, q_{S^2}| = -1 \quad \& \quad |q_{S^2}, p_{S^2}| = 3$$

Recall: Floer cohomology with Hamiltonian pert.

counts "Floer strips" between Hamiltonian orbits connecting L to $\varphi(L)$.

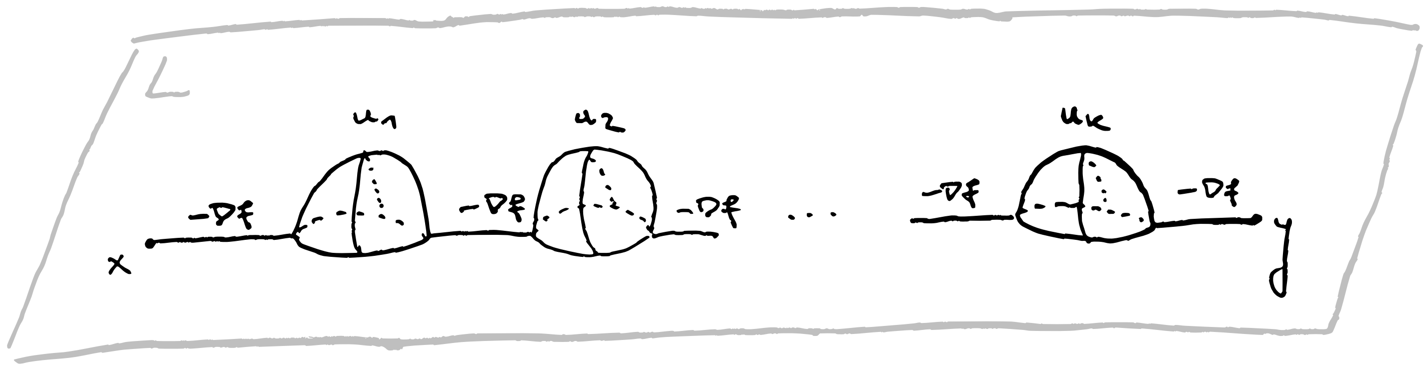
Immersed case:

trickier, as teardrops a priori obstruct $d^2=0$, but it still works under our assumption.

Really: "Take limit of $CF(L,\varphi(L))$ as $\varphi \to id$"

$\Rightarrow$ Morse theory with disk perturbations.

Ex In embedded (+ more ass.) case we count



$$\begin{cases} x, y \in \mathrm{Crit}\, f \\ u_i \text{ pseudoholomorphic disk with boundary on } L \end{cases}$$

$\partial_s u + J_u \partial_t u = 0$

$\Rightarrow$ More efficient for computations.

Towards the definition of the pearl complex.

For $\gamma_-, \gamma_+ \in \mathbb{R}$, $k \in \mathbb{N}$, $\alpha : \{1, \dots, k\} \to \mathbb{R}$ let

"moduli space of k-marked strips"

$\mathcal{M}_J(\gamma_-, \gamma_+, \alpha) :=$

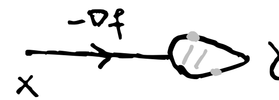
$$\left\{ \begin{array}{l} (\Delta, u, e) \quad \text{s.t.} \\ * \ \Delta = \{z_1, \dots, z_k\} \subseteq \partial(\mathbb{R} \times [0,1]) \\ * \ u : \mathbb{R} \times [0,1] \to M \quad C^0 \text{ on } \mathbb{R} \times [0,1], \\ \quad C^\infty \text{ and pseudoholomorphic on } \mathbb{R} \times [0,1] - \Delta \\ * \ e : \partial(\mathbb{R} \times [0,1]) - \Delta \to L \quad C^0 \text{ s.t.} \\ \qquad i \circ e = u|_{\partial(\mathbb{R} \times [0,1]) - \Delta} \\ \quad \text{with branch jump of type } \alpha(i) \\ \quad \text{at } z_i \\ * \ \lim\limits_{s \to \pm\infty} e(s, i) = \gamma_\pm^i, \quad i \in \{0,1\} \end{array} \right\}$$

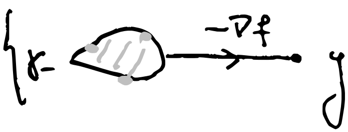
(following counterclockwise orientation on ∂)

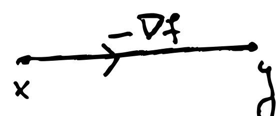
We define $\mathcal{M}_J(\gamma_-,\phi,\alpha)$, $\mathcal{M}_J(\phi,\gamma_+,\alpha)$ by requiring u,ℓ to have removable sing at $\pm\infty$ resp.

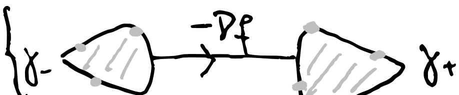
"teardrops" : 

Pick $f: L \to \mathbb{R}$ Morse, g Riem. metric s.t. (f,g) Morse-Smale and let $x,y \in \mathrm{Crit} f := \{ z \in L : Df(z) = 0 \}$.

$\mathcal{M}_J(x,\gamma_+,\alpha) := \{$  $\}$

$\mathcal{M}_J(\gamma_-,y,\alpha) := \{$  $\}$

$\mathcal{M}(x,y) := \{$  $\}$

$\mathcal{M}_J^{\mathrm{Morse}}(\gamma_-,\gamma_+,\alpha_1,\alpha_2) := \{$  $\}$

In all cases, we allow constant teardrops iff $|\Delta| > 0$.

What about <u>regularity</u> of those moduli spaces?

Under nice conditions: M_j's smooth mnflds of dim.

$$|\mathrm{start}| - |\mathrm{target}| - \sum |\alpha(j)| + |\Delta| - 1$$

according to standard theory (see McDuff-Salamon).

These "nice conditions" are likely to be generic (proof in progress).

The pearl complex.

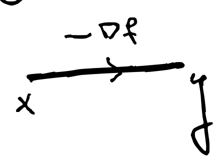
$$QC(i;\, f, g, J) := \mathbb{Z}_2 \operatorname{Crit} f \oplus \mathbb{Z}_2 R$$

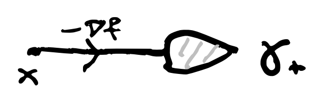
graded by Morse index & self-int. index.

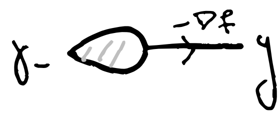
Define the differential matrixwise

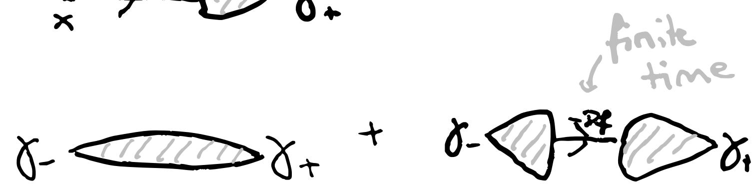
$$d := \begin{pmatrix} d_{CC} & d_{CR} \\ d_{RC} & d_{RR} + d^{\text{Morse}}_{RR} \end{pmatrix}$$

by counting (mod 2), for $y \in \operatorname{Crit} f$, $\gamma_+ \in R$:









respectively, in dim. 0.

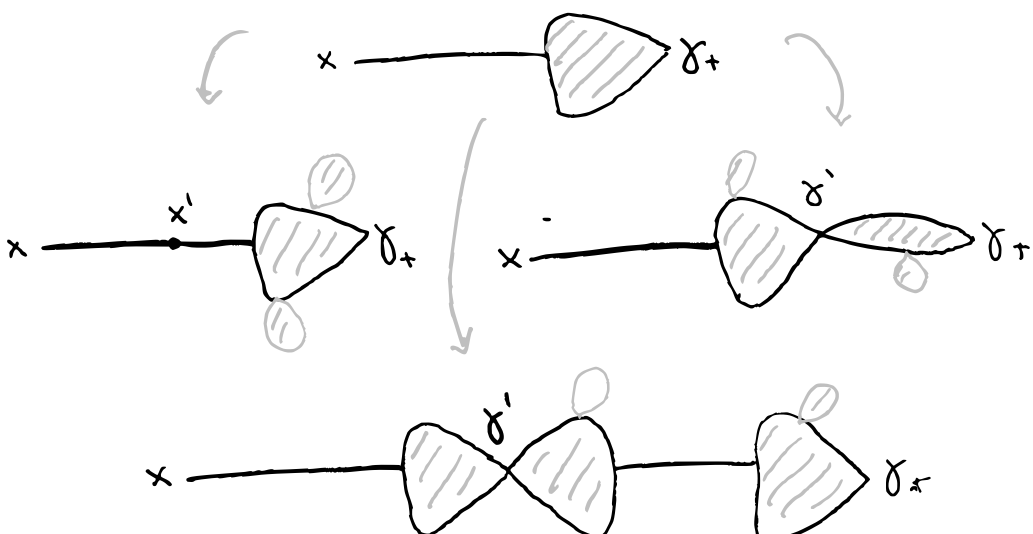
⚠ Strips are not allowed to jump branches.

Prop. Under our assumptions, $(QC(i), d)$ is a well-def. cochain complex.

Idea Study compactness / compactifications of Π's in dimension $0/1$. (includes standard gluing results)

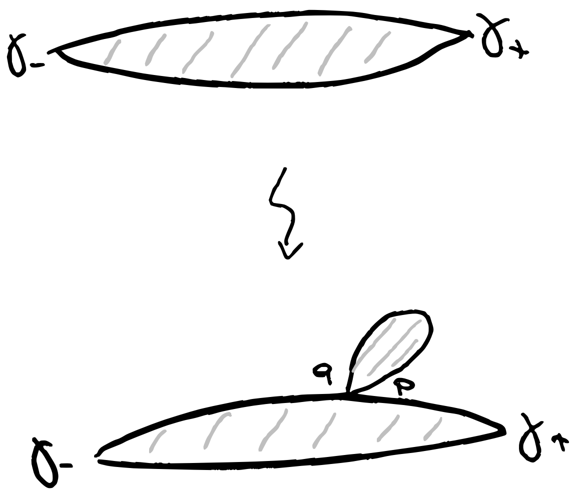
+ classification of compact 1-mnflds with bdry

E.g.



The assumption rules out teardrops.

E.g.



$$2 = |\gamma_-| - |\gamma_+|$$

$$|\gamma_-| - |\gamma_+| - |\alpha(1)| + |\Delta|$$

$$= 2 - |\alpha(1)| + 1 \leq 0$$

as $0 < \text{Area}(\text{bubble}) = A(p,q)$

Hence $\mathcal{M}_J(\gamma_-, \gamma_+, \alpha) = \emptyset$.

Prop. The ~~pearly~~ cohomology $H^*(QC(i; f, g, J), d)$ does not depend on f, g, J and is denoted $QH^*(i)$.

Back to our example:

Prop. If $r \geq 1$, $QH^*(i_{N,r})$ is isomorphic to $\mathbb{Z}_2$ in degrees $-1, 0, 2, 3$ and trivial elsew.

If $r = 0$, $QH^*(i_{N,r})$ is trivial.

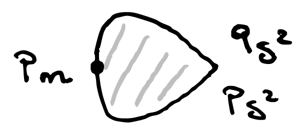
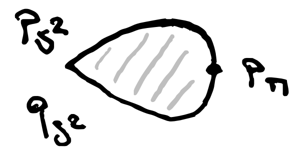
Sketch of proof

* Pick $f: S^2 \longrightarrow \mathbb{R}$ with max at $p_M := (0,1)$ & min at $p_m := (0,-1)$ & no other crit. point.

$\Rightarrow$ $QC^{-1}(i) = \mathbb{Z}_2 (p_{S^2}, q_{S^2})$ $\qquad QC^0(i) = \mathbb{Z}_2 p_m$

$QC^2(i) = \mathbb{Z}_2 p_M$ $\qquad QC^3(i) = \mathbb{Z}_2 (q_{S^2}, p_{S^2})$

* Interesting differentials: $d(p_{S^2}, q_{S^2})$ & $d(p_M)$

counting  , 

Claim $\left| \Pi_J(p_m, (p_{S^2}, q_{S^2})) \right| = \left| \Pi_J((q_{S^2}, p_{S^2}), p_M) \right| = 2^{r-1}$

We prove it for $\Pi_J(p_m, (p_{S^2}, q_{S^2}))$:

Consider $\Pi_J(\phi, (p_{S^2}, q_{S^2}))$ and view strips in there as α-disks with $\Delta = \{1\}$

* Let $u = (f, g, r \cdot h) : D^2 \to \Pi_N$ such an α-disk.

Notice that $|h^{-1}(1)| = 1$ & $h|_{\mathring{D}^2} : \mathring{D}^2 \to \mathbb{C}$ holomorphic

$\overset{\text{max}}{\underset{\text{pr.}}{\Longrightarrow}}$ $h : D^2 \to D^2$ holomorphic

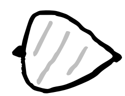
$\overset{\text{std}}{\underset{\text{CA}}{\Longrightarrow}}$ $h(z) = \lambda \dfrac{z - \beta}{\bar{\beta} z - 1}$, $|\lambda| = 1$, $|\beta| < 1$.

$$\Longrightarrow \begin{cases} f = e^{i\Theta} \prod_{i=1}^{r-1} \xi_i \cdot (\text{terms with } h) \\ g = e^{-i\Theta} \prod_{i=1}^{r-1} \eta_i \cdot (\text{terms with } h) \end{cases}$$

ξ_i, η_i Blaschke products s.t. $\overbrace{\xi_j \cdot \eta_j}^{\text{determines connected component}} = \dfrac{rh - j}{jh - r}$.

Up to reparam. $h = \mathrm{id}$.

see them as discs!

Then, $\Pi_J(\phi, (p_{S^2}, q_{S^2})) = \Big\{$  $\begin{matrix} q_{S^2} \\ p_{S^2} \end{matrix} \Big\}$ has 2^{r-1}

components, each diffeomorphic to $\mathbb{R} \times S^1$.

position of $-\infty$

$e^{i\theta}$ in fig

We have then that:

$$ev_{-\infty} : \Pi_J \longrightarrow S^2$$

$$(\Delta, u, e) \longmapsto \lim_{s \to -\infty} e(s)$$

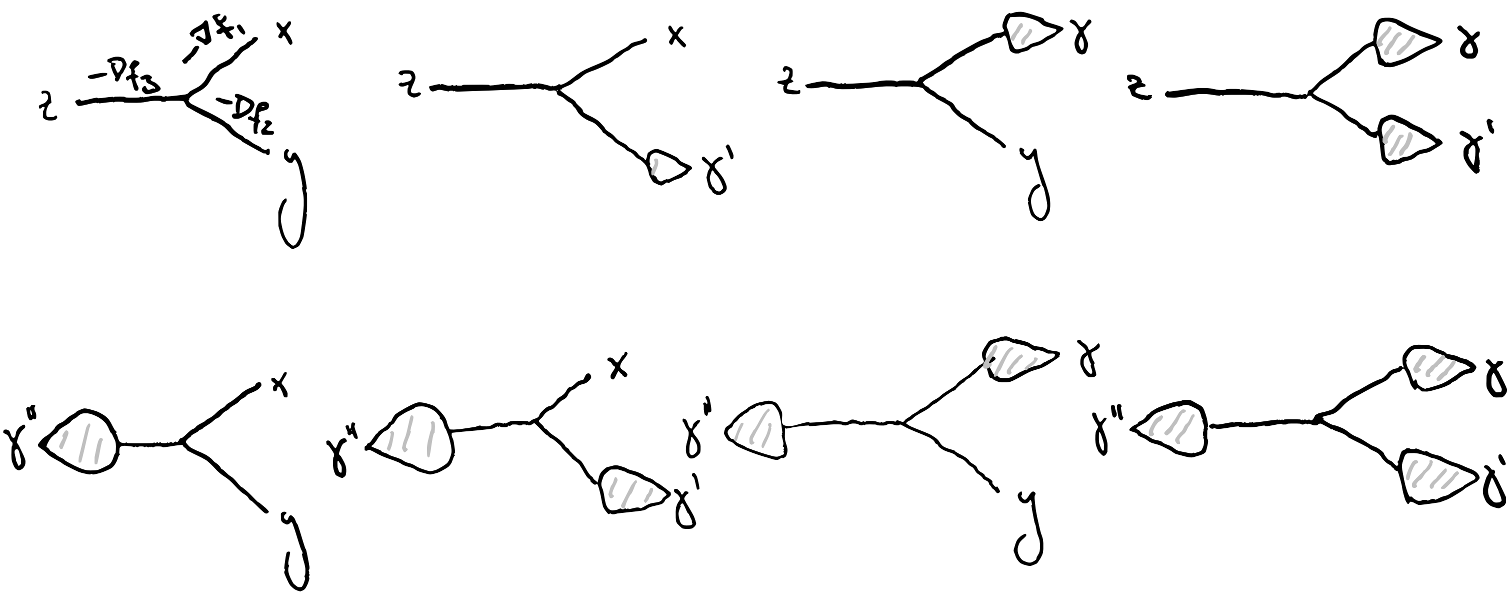
restricts to a diffeo. from each component of Π_J to the cylindrical patch of S^2.

$(\phi, (p, q))$

$\blacksquare$

Product & higher operations.

Pick tree generic Morse functions f_1, f_2, f_3 on L and $(x,\gamma) \in QC(i, f_1)$, $(y,\gamma') \in QC(i, f_2)$ define $(x,\gamma) * (y,\gamma')$ by counting (mod 2)



This way we can define

$$*: QC(i,f_1) \otimes QC(i,f_2) \longrightarrow QC(i,f_3)$$

Prop $*$ is a well-defined cochain map that does not depend on choices.

Hence it induces a product in cohomology.

Prop $*$ is associative in cohomology

Pick $f_2 = f_3$ and f_1 with unique minimum m, then

Prop $(m,0) \in QC(i,f_1)$ is a unit for $*$.

Moreover, it is canonical.

Critf R

Similarly, by considering couples of trees with no edges with two vertices, one can define A_∞-operations on $QC(i)$

$$\sum_{i+j=n} \mu_i(\dots, \mu_j(\dots), \dots) = 0$$

with $\mu_1 = d$, $\mu_2 = *$

THANK

YOU!