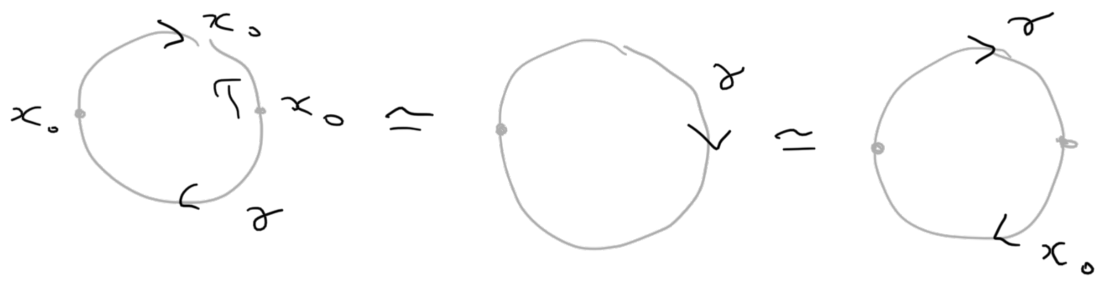
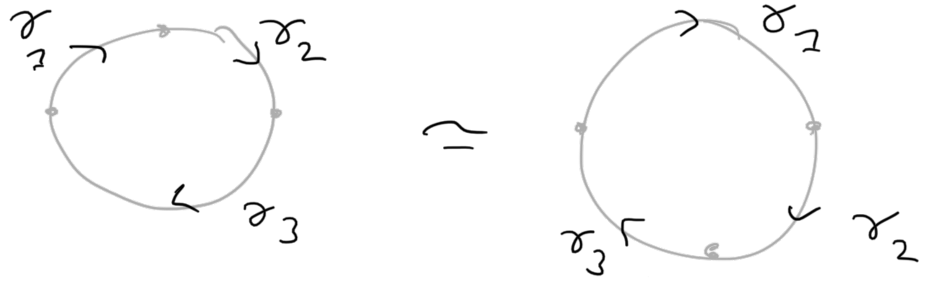
<u>Goal</u>: see how loop spaces have the homology type of A_∞-space

Let (X, x_0) pointed CW-complex and denote $x_0 \in \Omega(X, x_0)$ the constant loop at x_0

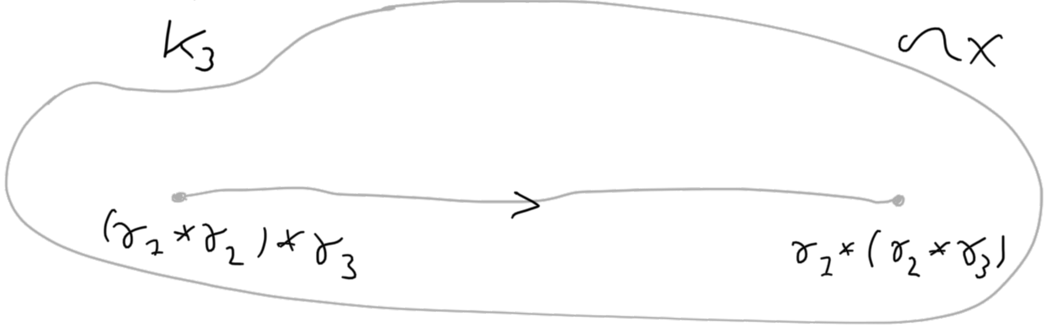
$$\gamma * x_0 \simeq \gamma \simeq x_0 * \gamma$$



homotopy associative



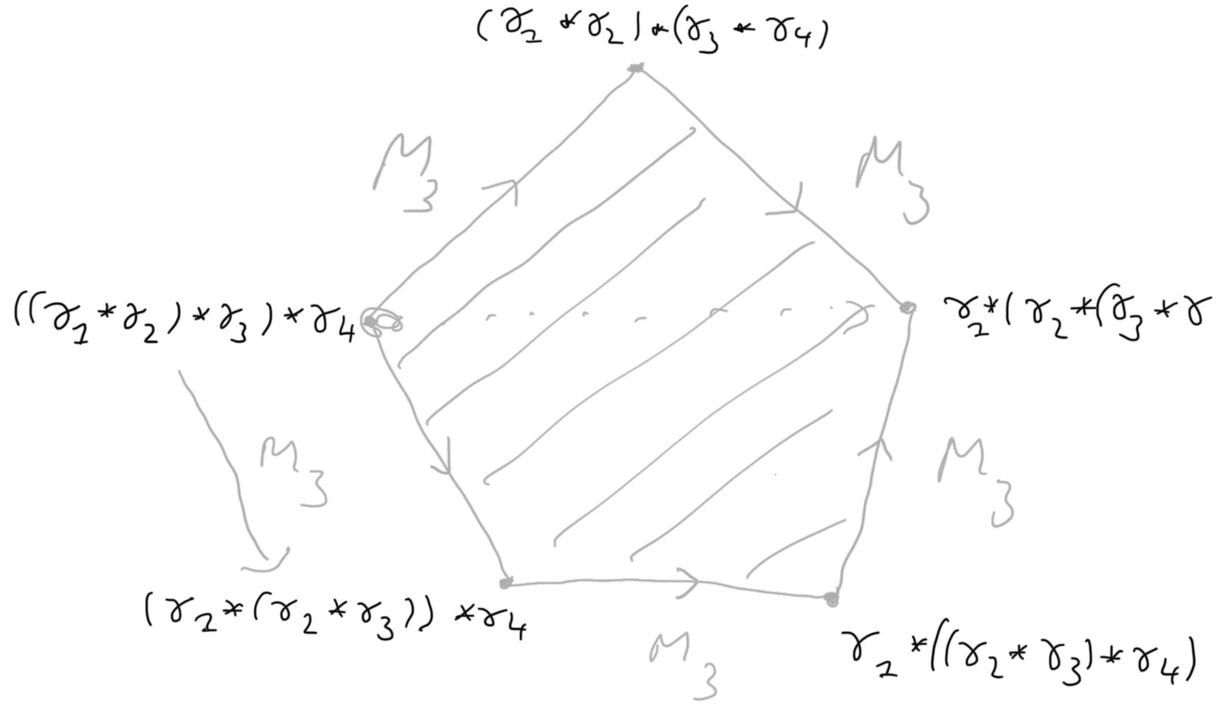
$$M_3 : \underset{\substack{\| \\ K_3}}{I} \times \Omega X^3 \longrightarrow \Omega X$$



M_3 satisfies coherence condition $\partial K_3 = \{0, 1\}$

Next step is taking 4 loops

$\rightsquigarrow$ 5 ways of concatinating



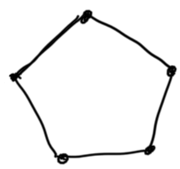
$$M_4 : \underset{\substack{\| \\ \text{convex hull}}}{K_4} \times \Omega\Omega^4 \longrightarrow \Omega X$$

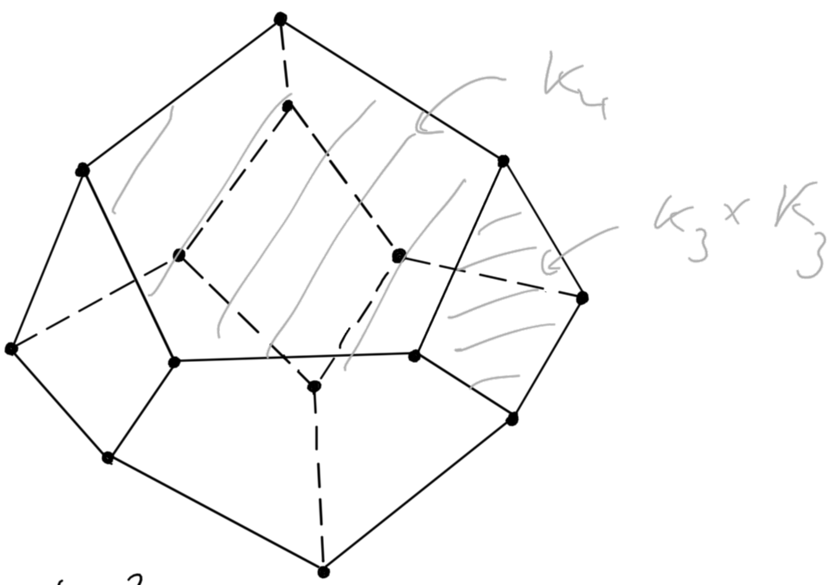
convex hull()

M_4 satifies coherence condition on ∂K_4

$\vdots$

$$M_n : \underset{\substack{\cong \\ I^{n-2}}}{K_n} \times \Omega X^n \longrightarrow \Omega X$$

$K_2 = \cdot$ $\quad$ $K_3 = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ $\quad$ $K_4 =$ 

$K_5 =$ 

$K_n \cong I^{n-2}$ and ∂K_n build up from $K_2, \dots, K_{n-1}$.

Def. Y is called an A_∞-space if there exist maps

$$M_n : K_n \times Y^n \longrightarrow Y \quad \forall n \geq 2$$

such that

(i) $M_2(*, y, e) = y = M_2(*, e, y)$

(ii) M_n can be written in terms of $M_2, \dots, M_{n-1}$ on ∂K_n

Milnor 59' X a CW-complex then ΩX with the compact open topology has the homotopy type of a CW-complex

Example from "Morse theory"

$X = S^n$ then ΩS^n has the homotopy type of a CW-compl with one cell each in dimension

$0, n-1, 2(n-1), 3(n-1), \dots$

Milnor $\rightsquigarrow$ $Y \underset{g}{\overset{f}{\rightleftarrows}} \Omega X$ homotopy equivalence

then set

$$\underbrace{K_n \times Y^n \xrightarrow{f^n} K_n \times \Omega X^n \xrightarrow{M_n} \Omega X \xrightarrow{g} Y}_{=: N_n}$$

$\rightsquigarrow$ N_n satisfy the A_∞-space definition up to homotopy.

$\rightsquigarrow$ we can deform $N_n \simeq \tilde{N}_n$ such that $(Y, \tilde{N}_n)$ is an A_∞-space.

Let's find $\tilde{N}_2 \simeq N_2$.

For assume that

$y_0 = g(x_0)$ is a 0-cell

we know that $\forall y$

$$N_2(y,y_0) \simeq y \simeq N_2(y_0,y)$$

$\Leftrightarrow$

$$\begin{tikzcd}
 & Y\times Y \arrow[dr, "N_2"] \arrow[d, dashed, "h"] & \\
Y\vee Y \arrow[ur] \arrow[rr, bend right, "\nabla"'] & {} & Y
\end{tikzcd}$$

$$\nabla(y,y_0) = y = \nabla(y_0,y)$$

The inclusion $\{y_0\} \longrightarrow Y$ is a cofibration, i.e.

$$\begin{tikzcd}
\{y_0\} \arrow[rr, "i_0"] \arrow[d] & & \{y_0\}\times I \arrow[dl, "h"'] \arrow[d] \\
Y \arrow[r, "f"] \arrow[rr, bend right] & Z & Y\times I \arrow[l, dashed, "\tilde{h}"']
\end{tikzcd} \quad (HEP)$$

<u>Theorem</u> Let X be CW-complex and $A \subseteq X$ a closed union of cells then $i: A \longrightarrow X$ is a cofibration

$$\begin{tikzcd}
Y\vee Y \arrow[rr, "i_0"] \arrow[d] & & Y\vee Y\times I \arrow[dl, "h"'] \arrow[d] \\
Y\times Y \arrow[r, "N_2"] \arrow[rr, bend right] & Y & Y\times Y\times I \arrow[l, dashed, "\tilde{h}"']
\end{tikzcd}$$

Set $\widetilde{N}_2 := \tilde{h}_1$ then $N_2 \simeq \widetilde{N}_2$

and $\widetilde{N}_2(y, y_0) = \tilde{h}_1(y, y_0)$

$= h_1(y, y_0)$

$= \nabla(y, y_0)$

$= y$

Similarly for $N_n \simeq \widetilde{N}_n \quad \forall n \geq 2$

$\boxed{\Rightarrow \Omega X \text{ has homotopy type of an } A_\infty\text{-space}}$

Let $(X, \{M_n\})$ be an A_∞-space

$m_1 := \partial^{cell} : C_*^{cell}(X) \circlearrowleft$

$m_n := M_{n\,*} : C_*^{cell}(X)^{\otimes n} \longrightarrow C_*^{cell}(X)$

$\Rightarrow \left(C_*^{cell}(X), \{m_n\} \right)$ is an

$\underline{A_\infty\text{- algebra}}$

this means that

$$\sum_{r+s+t=n} \pm\, m_{r+t+1}\left(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}\right) = 0$$

for all $n \geq 1$.

$n=1$: $m_1 m_1 = 0$

$n=2$: $m_2(id \otimes m_1 + m_1 \otimes id) - m_1 m_2 = 0$

$n=3$: $m_2(m_2 \otimes id + id \otimes m_2)$

$= m_3(id \otimes id \otimes m_1 + m_1 \otimes id \otimes id + id \otimes m_1 \otimes id)$

$+ m_1 m_3$

$\vdots$

<u>Sanity</u> <u>check</u>: On ΩS^2

prove that

$$M_{2*}(id \otimes \partial^{cell} + \partial^{cell} \otimes id) = \partial^{cell} M_{2*}$$

(Excercise)

Thank you for listening!