A fundamental (topological) question:

Consider an n-dimensional manifold M^n.

Q: Can M^n be realized as a **boundary** of some $(n+1)$-dim'l manifold W^{n+1}?

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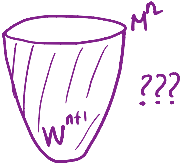
A: Yes!

Simply consider $M \times [1, 2)$

Same question with adjectives:

Consider an n-dimensional **closed** manifold M.

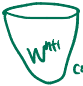


Q: Can M be realized as the **boundary** of a **compact** $(n+1)$-dim'l manifold W?

as

orientable | nonorientable

A: NO!



$\Rightarrow \chi(M) = \text{even}$

Ex: $\chi(\mathbb{CP}^{2n}) = 2n+1$

Ex: $\{pt\}$ is not the ∂ of any compact W^1

When $n = 1,2,3$ A: YES!

A: NO!

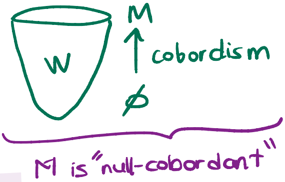
Ex: $\mathbb{RP}^{2n}$ since $\chi(\mathbb{RP}^{2n}) = 1$

(Recall, $\mathbb{RP}^n$ is orientable when n = odd)

What's a filling?

Simply put,

Filling of M^n :=

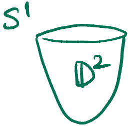


= (n+1)-dim'l manifold W whose boundary is M.

W = filling of M

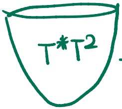
M = fillable manifold

Examples:



,

$T^3 = S^1 \times S^1 \times S^1$



$= T^2 \times \mathbb{D}^2$

Today's fundamental question:

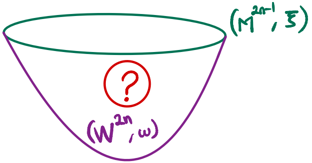
Question

Are all (compact) contact manifolds (symplectically) fillable?

(Somewhat) equivalently;

Question

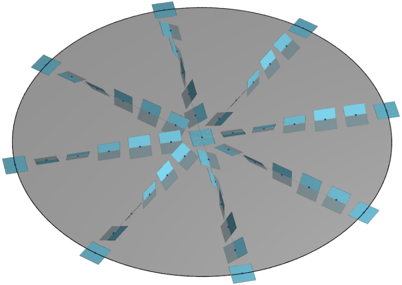
Can all (compact) contact manifolds be realized as the "boundary" of some symplectic manifold?



Answer within the dichotomy: Overtwisted vs. Tight

Overtwisted (OT)

(M,ξ) is OT if $\exists D \underset{\text{emb.}}{\subset} M$ with $\xi_p = T_pD \quad \forall p \in \partial D$



[1] Overtwisted disk in $\mathbb{R}^3$

Tight (not OT)

[1] Source: Patrick Massot

Answer within the dichotomy: Overtwisted vs. Tight

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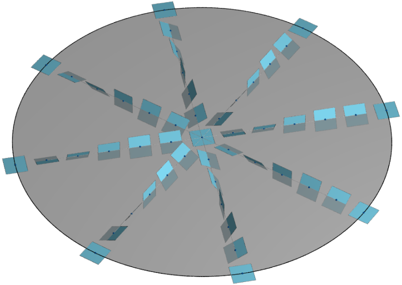


[1]Overtwisted disk in $\mathbb{R}^3$

Theorem (Gromov '85, Eliashberg '89)

If M^3 is OT then it's not (weakly) fillable

(Later shown to be true in all dims.)

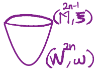
Tight (not OT)

Symplectically Fillable $\subset$ Tight

$\neq$

(Etnyre-Honda)

[1]Source: Patrick Massot



, $\partial W = M$ as oriented manifolds

- Weak filling: $\omega|_\xi > 0$ in dimension three*.

Example: Consider $T^3 = T^2 \times S^1$, $(x,y,\theta) \in T^2 \times S^1$

$(T^3, \xi_n = \ker(\alpha_n))$ where $\alpha_n = \cos(n\theta)dx + \sin(n\theta)dy$ and $n \in \mathbb{Z}^+$

Consider $\beta_t = d\theta + t\alpha_n$ and verify $\beta_t \wedge d\beta_t > 0$ i.e. β_t is contact for $t>0$

Corresponding hyperplanes ζ_t are contact $\forall t>0$ & converges to ξ_n as $t \to \infty \Rightarrow$ ξ_n and ζ_t are contact isotopic

Key idea

(T^3, ξ_n) is weakly fillable by $(T^2 \times \mathbb{D}^2, dx \wedge dy + \omega_{\mathbb{D}^2})$ since $\omega(\partial_x, \partial_y) = 1$

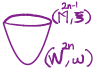
i.e. $\omega|_{\zeta_0} > 0 \Leftrightarrow \omega|_{\zeta_t} > 0$ for t small

! First classification result due to Gromov '85

$(\mathbb{D}^4, \omega_{std})$ is the unique weak symplectic filling of (S^3, ξ_{std})/blow-up

*since 3D is when ω can be positive on the 2D contact planes.

Symplectic fillabilty: Flavors



, $\partial W = M$ as oriented manifolds

- Weak filling: $\omega|_\xi > 0$ in dimension three.
- Strong filling: $\exists\ \lambda \in \Omega^1(M)$ s. t. $\omega = d\lambda$ on a **nhd** of ∂W and $\xi = ker(\lambda|_{\partial W})$

Equivalently, $\exists$ Liouville v.f. Z ($\mathcal{L}_Z \omega = \omega$) defined near ∂W
$\pitchfork \partial W$ and $\xi = \ker(\iota_Z \omega|_{\partial W})$

Symplectic fillabilty: Flavors

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Example: Consider $\omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on $\mathbb{R}^4$ in coords (x_1, y_1, x_2, y_2)

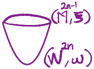
Let $\lambda := \frac{1}{2}[x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2]$ be the primitive of ω_{std}, i.e. $\omega_{std} = d\lambda$

Then ξ_{std} on $S^3 \subset \mathbb{R}^4$ is $\xi_{std} = \ker\alpha$ where $\alpha = \lambda|_{S^3}$

The vector field $Z = x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}$ is a Liouville v.f. $\pitchfork S^3$ for ω_{std}

$\Rightarrow (\mathbb{D}^4, \omega_{std})$ is a strong symp. filling of (S^3, ξ_{std})

Symplectic fillabilty: Flavors



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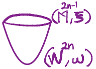
- Exact filling: $\exists\ \lambda \in \Omega^1(M)$ s. t. $\omega = d\lambda$ on **all** of ∂W and $\xi = ker(\lambda|_{\partial W})$

(also known as Liouville domain)

Symplectic fillabilty: Flavors



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- Exact filling: $\exists\ \lambda \in \Omega^1(M)$ s. t. $\omega = d\lambda$ on **all** of ∂W and $\xi = ker(\lambda|_{\partial W})$
- Stein filling: $(\underbrace{W, J}_{\text{complex manifold w/ boundary}}, \phi)$

ϕ: exhausting (proper, bdd from below) J-convex ($\omega_\phi > 0$ on complex lines in TW) $\phi : W \to \mathbb{R}$ s.t. M is a regular evel set.

$W :=$ Stein domain

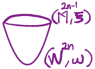
Recall the prev. ex:

Ex: Consider J_{std} on $\mathbb{R}^4$; $J_{std}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$ & $J_{std}\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$ for $j = 1,2$

Let $\phi : \mathbb{R}^4 \to \mathbb{R}$
$(x_1, y_1, x_2, y_2) \mapsto x_1^2 + y_1^2 + x_2^2 + y_2^2$ } Observe that ϕ is an exhausting J-convex fnc on $\mathbb{R}^4$ where S^3 is a regular level set.

$\Rightarrow (D^4, J, \phi)$ is a Stein filling of (S^3, ξ_{std})

Symplectic fillabilty: Flavors



, $\partial W = M$ as oriented manifolds

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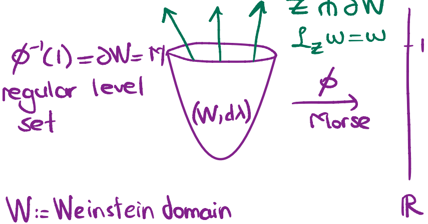
- Stein filling: (W, J, ϕ)
- Weinstein filling $(W, d\lambda, Z, \phi)$

Liouville v. f.

gen. Morse function



W := Weinstein domain

Ex: Take any (smoothed) link of singularity $\{f=0\} \cap S^{2n+1}$

Then $W = \{f=0\} \cap B^{2n+2}$

Symplectic fillabilty: Flavors

$$\overbrace{\text{Stein} = \text{Weinstein}}^{\text{Cieliebak-Eliashberg}} \subset \text{Exact} \subset \text{Strong} \subset \text{Weak} \subset \text{Tight}$$

$\cap \not\subset$ dim ≥ 5 Strong

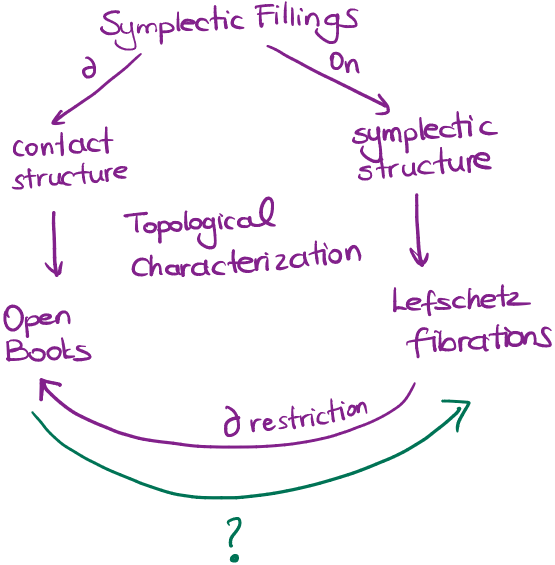
$\neq$ in dim 3

$\neq$ in dim 3

$\neq$ up to dim 5

$\neq$ all dims

A tree with a loop?

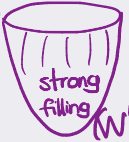


From open books to Lefschetz fibrations via fillability

Theorem (Wendl)

(M^3, ξ) supp. $OB(F^2, B^1)$

planar surface (genus 0 surf.)



(W^4, ω)

$\Longrightarrow$

(W^4, ω)

$\downarrow$ LF

D^2

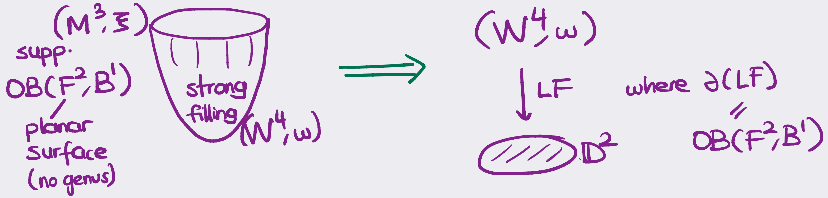
where $\partial(LF) = OB(F^2, B^1)$

$(S^3, \xi_{std}) = OB(D^2, id)$

page planar surf.

From open books to Lefschetz fibrations via fillability

Theorem



An important consequence of this theorem:

Strongly fillable $\equiv$ Stein fillable

when (M^3, ξ) is planar

From Lefschetz-Bott fibrations to open books

Lefschetz-Bott fibrations = "complexified Morse-Bott functions"

<u>Example:</u> Let $f_1 : W_1 \longrightarrow \mathbb{D}^2$ be any fibration

and $f_2 : W_2 \rightarrow \mathbb{D}^2$ be a LF.

Then the fiber product $f_1 \times_{\mathbb{D}^2} f_2 : W_1 \underset{\mathbb{D}^2}{\times} W_2 \longrightarrow \mathbb{D}^2$ is a LB-fibration

From Lefschetz-Bott fibrations to open books

(Oba) • Total space of a symplectic LB fibration serves as a strong symplectic filling of a contact mfld.

• Also constructs strong symp. fillings using links of A_k-singularity $= \{ z_0^2 + \ldots + z_n^2 + z_{n+1}^{k+1} = 0 \}$ and LB fibrations