b-Contact Structures on Symplectic Hyperboloids

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Symplectic hyperboloids

Consider $(T^*\mathbb{R}^n, \omega_0 = dq \wedge dp)$ and a Hamiltonian

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A representative of an equivalence class of the action of the $\mathrm{Sp}(n)$ on the set $\left\{ H^{-1}(0) \mid H(x) = \frac{1}{2}x^T Ax - 1, \quad A \in \mathrm{Sym}(2n) \cap \mathrm{Gl}(2n) \right\}$ is called:

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- symplectic ellipsoid when A is positive definite, $H^{-1}(0) \simeq S^{2n-1}$;
- symplectic hyperboloid when $\mathrm{sgn}(A) = (k, 2n-k)$ with $1 \leq k \leq 2n-1$, $H^{-1}(0) \simeq S^{k-1} \times \mathbb{R}^{2n-k}$.

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Where $J_0 := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$

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Note: If α is an eigenvalue of $J_0 A$, then $-\alpha$ and $\overline{\alpha}$ are too.

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a) Real eigenvalues of $J_0 A$, $\lambda, -\lambda$ with $\lambda > 0$.

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$$\begin{pmatrix} & & & \lambda & & \\ & & & 1 & \ddots & \\ & & & & 1 & \lambda \\ \lambda & 1 & & & & \\ & \ddots & 1 & & & \\ & & \lambda & & & \end{pmatrix}$$

Signature (m, m).

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b) Eigenvalues of $J_0 A$ with non-zero both real and imaginary parts, $\pm\lambda_1 \pm i\lambda_2$ with $\lambda_1, \lambda_2 > 0$.

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$$\begin{pmatrix} & & & & & \lambda_1 & -\lambda_2 & 1 & & \\ & & & & & \lambda_2 & \lambda_1 & & \ddots & \\ & & & & & & & \ddots & & 1 \\ & & & & & & & & \lambda_1 & -\lambda_2 \\ & & & & & & & & \lambda_2 & \lambda_1 \\ \lambda_1 & \lambda_2 & & & & & & & & \\ -\lambda_2 & \lambda_1 & & & & & & & & \\ 1 & & \ddots & & & & & & & \\ & \ddots & & \lambda_1 & \lambda_2 & & & & & \\ & & 1 & -\lambda_2 & \lambda_1 & & & & & \end{pmatrix}$$

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 & & \mu & & & \\
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Signature (m, m) if m is even, $(m \pm 1, m \mp 1)$ if m odd.

Tentacular hyperboloids

Theorem: Pasquotto - van der Vorst - W. 2017

Consider a Hamiltonian

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where A_0 is positive definite and A_1 is such that $J_0 A_1$ is hyperbolic and $A_1 = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ with $B + B^T$ positive definite. Then the Rabinowitz Floer homology of $\Sigma := H^{-1}(0)$ is well defined.

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In local coordinates $(x_1, \dots x_n)$ we have $Z = x_1^{-1}(0)$. The vector fields spanned by $(x_1^m \partial_{x_1}, \partial_{x_2}, \dots \partial_{x_n})$ are called b^m-vector fields.

Definition

A *b-symplectic manifold* is a triple (M, Z, ω), where (M, Z) is a b-manifold and $\omega \in {}^b\Omega^2(M)$ is closed and non-degenerate i.e. $\omega^n \neq 0 \in {}^b\Omega^{2n}(M)$.

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A *Liouville b-vector field* is a b-vector field Y, such that $d(\iota_Y \omega) = \omega$.
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A *b-contact manifold* is a triple $(N, Z, \xi = \ker \alpha)$, where (N, Z) is a b-manifold and $\alpha \in {}^b\Omega^1(N)$ such that $\alpha \wedge (d\alpha)^n \neq 0 \in {}^b\Omega^{2n+1}(N)$.

McGeehe transform

Diffeomorphism $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+ \times S^{n-1}$ can be lifted to a symplectomorphism

$$T^*\mathbb{R}^n \setminus \{0\} \ni (q, p) \xrightarrow{\Phi} (r, P_r, \phi, \eta) \in T^*\mathbb{R}_+ \times T^*S^{n-1},$$
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$$X := \mathbb{R}^2 \times T^*S^{n-1} \ni (x, P_r, \phi, \eta), Z := \{x = 0\}$$
$$\tau_{\mathrm{McG}} : X \setminus Z \to T^*\mathbb{R}_+ \times T^*S^{n-1}$$
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$$(\tau_{\mathrm{McG}})^*(dr \wedge dP_r + \phi \wedge d\eta) = -\frac{4}{x^3} dx \wedge dP_r + d\phi \wedge d\eta$$

is a b^3-symplectic form on (X, Z).