Junior Symp. Sem.

ETH. 20.10.22.

Superpotentials for the applied symp. geometer.

0. Intro.

(X, ω) : symp mfd.

Study of Lag submfds is central in symp geom.

Def.

A submfd $L^n \subset X^{2n}$ is Lag.

if $\omega|_L = 0$.

e.g. $X = T^*M$. $L = 0_M$.

Two major research topics

$X = \Sigma_g$, $L =$ emb. circle.

$X = S^2$ $L = S^1_{eq}$.

1: rigidity

Most of the "topologically interesting" Lags show strong. rigidity:

e.g. $L \cap \varphi(L) \neq \emptyset$

$\forall \varphi \in Ham(X)$.

2. classification

How many Lag's are there (that are not Ham. isotopic)?

$\rightsquigarrow$ Arnold conj.

$\#(L \cap \varphi(L)) \geq$ ______

$\longrightarrow$ Floer homology. was developped to tackle these sort of problems.

1. Rigidity of Lag.s

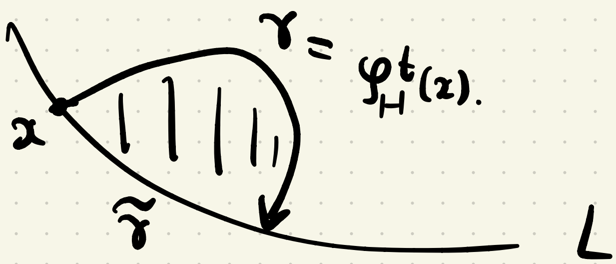
1.1. Lag. Floer homology.

$H : [0,1] \times X \longrightarrow \mathbb{R}$

- $CF(L, H)$

$$:= \bigoplus \mathbb{C} \cdot \tilde{\gamma}$$

$\tilde{\gamma}$: capped Ham. chord of H from L to L.



- $\partial_{Fl} : CF_k(L, H) \longrightarrow CF_{k-1}(L, H)$

$$\tilde{\gamma} \longmapsto \sum_{\substack{\mu(\tilde{\gamma}) \\ -\mu(\tilde{\gamma}') = 1.}} \# \mathcal{M}(\tilde{\gamma}, \tilde{\gamma}') \, \tilde{\gamma}'$$

$\rightsquigarrow$ $(CF(L,H), \partial_{Fl})$ is a chain cplx

$\rightsquigarrow$ $HF(L,H)$.

0.2. Pearl homology. (Biran-Cornea).

"Morse theoretic" approach to HF.

- $\mathcal{D} := (f, g_L, J)$

$f: L \to \mathbb{R}$ Morse.

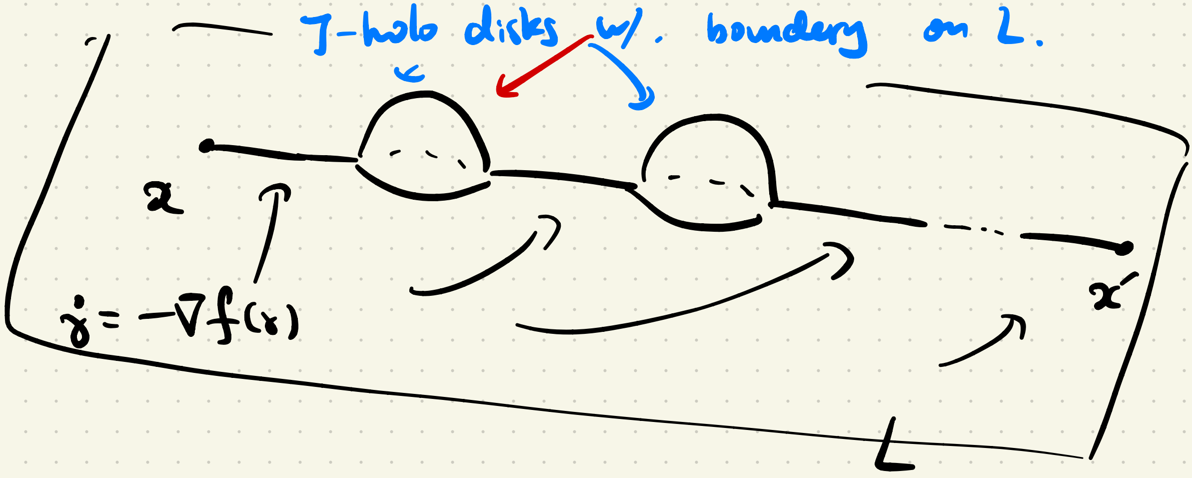
Riem. metric on L

alm. cplx str on X.

- $C(\mathcal{D}) := \bigoplus_{x \in \mathrm{Crit}(f)} \mathbb{C} \cdot x$

- $\partial : C(\mathcal{D}) \longrightarrow C(\mathcal{D})$

$$x \longmapsto \sum \# \mathcal{P}(T) \cdot x'$$



$\to$ $(C(\mathcal{D}), d)$ is a chain cplx

$\rightsquigarrow$ $\underline{QH_*(L)}$: quantum homology of L.

$$HF(L) \xrightarrow[\sim]{PSS_L} QH(L)$$

<u>Rmk.</u>

1. $HF(L) \neq 0$ implies

$$L \cap \varphi(L) \neq \emptyset, \quad \forall \varphi \in Ham(X)$$

2. Unlike $QH(X)$, it isn't always

$$QH(L) \simeq H^*(L) \otimes \Lambda.$$

e.g. if L is displaceable, then. $QH(L) = 0$.

- Nowadays, the most common way to say that L is non-displaceable is by proving $HF(L) \neq 0$.

→ Issue 1: computing HF is extremely difficult!. (unlike $HF(H)$!)

Rmk. Oh's spec. seq. for N_L large

Issue 2: Some Lags have $HF(L) = 0$ even though it's known to be non-displ'ble.

e.g. $T^2_{cn} \subset \mathbb{C}P^2$. has

$$HF(T^2_{cn}) = 0. \;!$$

Is Floer thy not good enough ??

$\rightsquigarrow$ Superpotentials can take care of these issues!

2. Classification.

We want to classify Lag's in X.

i.e, given two Lags L, L' in X, we want to study if

$$L = \varphi(L') \text{ for some } \varphi \in Ham(X).$$

e.g. $\mathbb{C}P^2$.

T^2_{Ch}, T^2_{Clif} are they Ham. isotopic?

$S^2 \times S^2$.

T^2_{Ch}, T^2_{Clif} are they isotopic?

$\rightsquigarrow$ Superpotentials can take care of this!

2. Superpotential

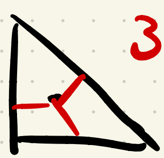
- About the terminology:
  - disk potential
  - potential function
- Idea comes from
  - Chekanov (?)
  - physics (string thy)

Def. 1. (weak)

W_L = # Maslov 2-disks w/. bd on L. (passing through a generic pt)

- This is easier to compute than HF. etc.

e.g. $\mathbb{C}P^2$. T^2_{clf}, T^2_{ch}.



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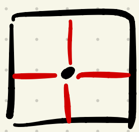
$S^2 \times S^2$ T^2_{clf} T^2_{ch}.



4. 5

Cor.

T^2_{Ch} and $T^2_{Cl.f}$ are not Ham. isotopic in $\mathbb{C}P^2$, $S^2 \times S^2$

Def. (real one!)

$$W_L : \mathrm{Hom}(H_1(L);\mathbb{C}) \longrightarrow \mathbb{C}$$
$$\rho \longmapsto W_L(\rho)$$

local system.

$$:= \sum_{A \in \pi_2(X,L)} \rho(\partial A)\,\#(\text{Maslov 2-disk in class } A)$$

Usually one takes a basis of $H_1(L)$ and thus of $\hom(H_1(L);\mathbb{C})$ (denote it by $z_1 \dots z_k$) and express.

W_L in terms of $z_1 \cdots z_k$:

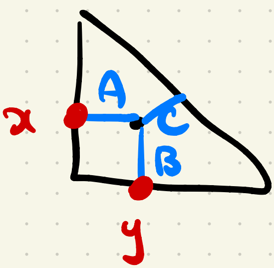
$$W_L(z) = \sum z^{\partial A} \#(\text{Maslov 2-disks in class } A)$$

$$= \sum z_1^{(\partial A)_1} \cdots z_k^{(\partial A)_k} \#(\text{Maslov 2-disks in class } A)$$

(**Rmk** if we take a trivial local system $\rho = 1$, then W_L is just the count of Maslov 2-disks.)

e.g.

2. $\mathbb{CP}^2$. T^2_{Clif}.



$$H_1(T^2_{\text{Clif}}) = \underset{\langle x \rangle}{\mathbb{Z}} \oplus \underset{\langle y \rangle}{\mathbb{Z}}.$$

$(e^{i\theta}:1:1)$ $(1:e^{i\theta}:1)$

Maslov 2-disks:

$\partial A = x$

$\partial B = y$

$\partial C = x^{-1}y^{-1}$ $\left(= (1:1:e^{i\theta}) \;\text{''}\; (e^{-i\theta}:e^{-i\theta}:1)\right)$

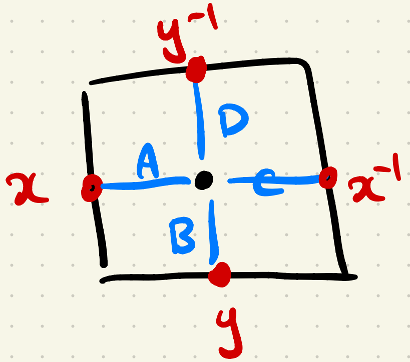
$\rightsquigarrow$

$$W_{T^2_{\mathrm{Cl'f}}} = x + y + \frac{1}{xy}.$$

Rmk.

$$W_{T^2_{\mathrm{Ch}}} = z_1 + z_2 + \frac{1}{z_1 z_2^4} + \frac{2}{z_2^2}.$$

3. e.g. $S^2 \times S^2$, T^2_{Clif}



$H_1(T^2_{\mathrm{Clif}}; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$

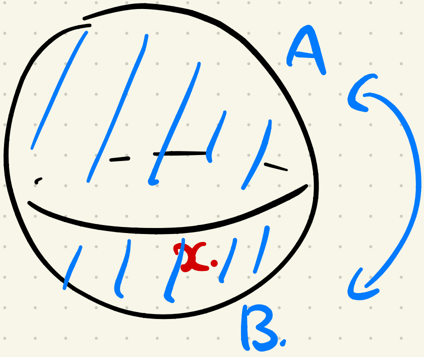
x, y

$$W_{T^2_{\mathrm{Clif}}} = x + y + \frac{1}{x} + \frac{1}{y}.$$

Cmt.

Vianna found ∞-ly many monotone Lag tori in $\mathbb{CP}^2$. they are not Ham isotopic. He uses SP to

1\. e.g. S^2. S^1_{eq}. distinguish them.



$H_1(S^1_{eq}) = \mathbb{Z}$

x

Maslov 2-disks.

$\partial A = x$

$\partial B = x^{-1}$.

$$W_{S^1_{eq}} = x + x^{-1} \quad \lrcorner$$

3. HF and superpotentials

<u>Thm.</u> (Hori–Vafa, Fukaya–Oh–Ohta–Ono (FOOO))

If. $\mathrm{Crit}(W_L) = \emptyset$

$\Rightarrow HF(L) = 0$.

If. $L \simeq T^n$. then. $\mathrm{Crit}(W_L) \neq \emptyset \Rightarrow HF \neq 0$.

e.g. (S^2, S^1_{eq})

$$W_L = x + x^{-1}$$

$$\frac{dW_L}{dx} = 1 - \frac{1}{x^2}, \quad \mathrm{Crit}(W_L) = \{x = \pm 1\}$$

$\therefore HF(L) \neq 0.$

· $(\mathbb{C}P^2, T^2_{cl f})$

$$W_L = x + y + \frac{1}{xy}.$$

$$dW_L = \left(1 - \frac{1}{x^2 y}\right) dx + \left(1 - \frac{1}{xy^2}\right) dy$$

$\longrightarrow$ $(x, y) = \left(e^{\frac{2\pi i}{3} j}, e^{\frac{2\pi i}{3} j}\right)$,

$j = 1, 2, 3$

are crit pts of W_L.

$\longrightarrow$ $HF(L) \neq 0.$

- $S^2 \times S^2$. T^2_{Ch}. (FOOO)

$$W_L = z_1 + z_1 z_2^{-1} + z_1^{-1} z_2 + z_1^{-1}!$$

$\mathrm{Crit}(W_L) =$

- What does the crit pts do?

→ a slight modification of the Floer bd. map: (local systems) in the pearl formulation.

without local. systems.

$$\partial(x) = \sum_{T} \# \mathcal{P}(T) \cdot y.$$

with local systems.

$$\rho : H_1(L) \longrightarrow \mathbb{C} \quad (\in \mathrm{Hom}(H_1(L), \mathbb{C}))$$

$$\partial(x) = \sum_{T} \underline{\underline{\rho(\partial T)}} \cdot \# \mathcal{P}(T) \cdot y.$$

modification of the boundary op.

Why? ("Prf").

Biran–Cornea formulation.

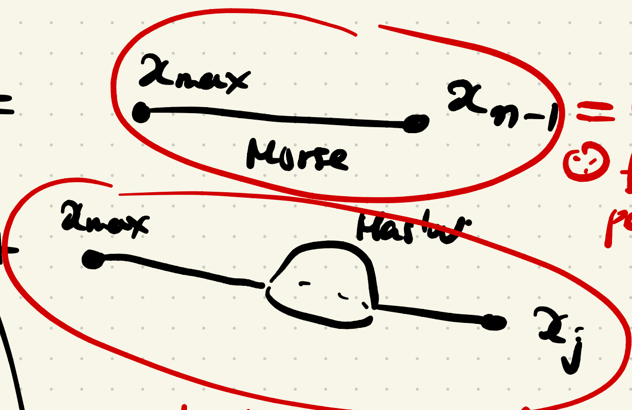
- $L \simeq T^n$

$\longrightarrow$ L admits a perfect Morse fct. $f : L \to \mathbb{R}$

- $QH(L) \neq 0 \iff [L] \neq 0$ in $QH(L)$.

So we focus on the crit pt. x_{max} (degree n).

- $\partial x_{max} =$



$= 0$ ☺ f is perfect.

$+$

$|x| - |y| + \mu(T) - 1$

$n - (n-1) + 0 - 1$

$n - \boxed{n+1} + 2 - 1$

× !

don't appear for degree reason. ($\deg(x_j) \geq n$)

$\rightsquigarrow$ x_{max} represents a class.

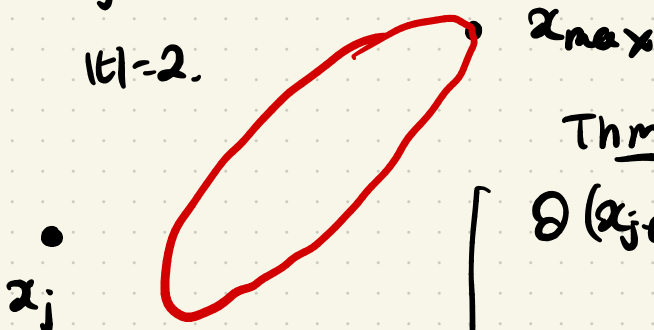
Is it a boundary,

i.e. $x_{max} \in \mathrm{Im}(\partial)$??

- $C_{n+1} \longrightarrow C_n$

x_j t.

$|t|=2$.



Thm (BC., FOOO)

$$\left[\begin{array}{l} \partial(x_{j+1}, t) \\ \\ = z_j \dfrac{\partial W_L}{\partial z_j} \cdot x_{max} \end{array}\right.$$

$$|x| - |y| + \beta(T) - 1 = 0$$

$n-1 \quad n \quad + \quad 2 \quad - 1$

$\cancel{n+1} \quad - \quad \cancel{n} \quad 0 \quad - 1$

$n-3 \quad -n \quad + \quad 4 \quad - 1$