Morse-Bott Functions from Moment Maps

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1 Morse-Bott Functions

We follow the definitions and presentation of [2].

Let M be a smooth manifold, $f : M \to \mathbb{R}$ a smooth function and $\mathrm{Crit}\,(f) = \{p \in M \mid \mathrm{d}f_p = 0\}$ its set of critical points.

Definition. The *Hessian* $\mathrm{H}_p(f)$ of f at a critical point $p \in \mathrm{Crit}\,(f)$ is the symmetric bilinear form given by

$$[\mathrm{H}_p(f)](X_p, Y_p) = X_p(Y(f))$$

where $X_p, Y_p \in T_pM$ are tangent vectors and $Y \in \mathfrak{X}(M)$ is any vector field on M such that $Y(p) = Y_p$.

If $C \subset \mathrm{Crit}\,(f)$ is a submanifold of positive dimension, for any $Y \in \mathfrak{X}(M)$ we have $Y(f)|_C = \mathrm{d}f(Y)|_C = 0$. Hence if $X_p \in T_pC$ it follows that

$$[\mathrm{H}_p(f)](X_p, Y_p) = X_p(Y(f)) = 0$$

and thus $T_pC \subset \ker(\mathrm{H}_p(f))$.

Definition. $f : M \to \mathbb{R}$ is called a *Morse-Bott function* if

1. $\mathrm{Crit}\,(f)$ is a disjoint union of connected submanifolds and

2. for each connected submanifold $C \subset \mathrm{Crit}\,(f)$ and each $p \in C$ we have $\ker(\mathrm{H}_p) = T_pC$.

Remark. The non-degeneracy condition can be equivalently phrased as follows: Denote νC the normal bundle of C with fiber $\nu_p C = T_pM/T_pC$ at $p \in C$. Note that from a(ny) Riemannian metric on M we get a splitting

$$TM|_C = TC \oplus \nu C.$$

As $T_pC \subset \ker(\mathrm{H}_p(f))$, $\mathrm{H}_p(f)$ induces a symmetric bilinear form $\mathrm{H}^\nu_p(f)$ on $\nu_p C$. The non-degeneracy condition is then equivalent to $\mathrm{H}^\nu_p(f)$ being non-degenerate for all $p \in C$.

Lemma (Morse-Bott Lemma). *Let $f : M \to \mathbb{R}$ be a Morse-Bott function and $C \subset Crit(f)$ be a connected component. For any $p \in C$ there is a local chart of M around p and a local splitting $\nu C = \nu^- C \oplus \nu^+ C$ identifying a point $q \in M$ with (u, v, w) where $u \in C$, $v \in \nu^- C$ and $w \in \nu^+ C$ such that within this chart f assumes the form*

$$f(q) = f(u, v, w) = f(C) - |v|^2 + |w|^2.$$

Definition. The *Morse-Bott index* of a connected component C is the dimension of $\nu^- C$.

Example. Consider the height function $f : T \to \mathbb{R}$ on the laying doughnut. This is a Morse-Bott function with two critical submanifolds, both isomorphic to S^1. The circle on top has Morse-Bott index 1 and the circle on the bottom has Morse-index 0.

Example. Consider the square $f = h^2 : S^n \to \mathbb{R}$ of height function $h : S^n \to \mathbb{R}$. This is a Morse-Bott function with 3 critical submanifolds: the north and southpole have dimension 0 but the equator is sphere of dimension $n-1$ i.e. S^{n-1}. The north pole and the southpole have index n while the equator has index 0.

2 Morse-Bott Functions from Moment Maps

Theorem. *Let G be a compact Lie group and let (M, ω, G, μ) be a hamiltonian G-space. For any $X \in \mathfrak{g}$, the component of μ along X given by*

$$\begin{aligned} \mu^X &: M \to \mathbb{R} \\ & p \mapsto \mu^X(p) = \langle \mu(p), X \rangle \end{aligned}$$

is a Morse-Bott function and the critical manifolds are symplectic submanifolds and the indices are all even.

Proof. (Inspired by [4], Theorem 2.2) $X \in \mathfrak{g}$ generates a one-parameter subgroup of G. Its closure is connected, compact since G is and Abelian. Thus it is a torus $\mathcal{T} < G$ of some dimension. By definition of the moment map, $\mathrm{d}\mu^X = -\iota_{X^\sharp}\omega$ so that for any $p \in M$

$$\begin{aligned} p \in \operatorname{Crit}\left(\mu^X\right) \quad &\iff \quad \mathrm{d}\left(\mu^X\right)_p = 0 \\ &\iff \quad -\iota_{X^\sharp_p}\omega_p = 0 \\ &\iff \quad X^\sharp_p = 0 \\ &\iff \quad p \in M^{\mathcal{T}}. \end{aligned}$$

or in other words,

$$\operatorname{Crit}\left(\mu^X\right) = M^{\mathcal{T}}.$$

We thus conclude that the connected components of Crit $\left(\mu^X\right)$ are symplectic submanifolds.

To see the non-degeneracy, we look at the moment map μ in a neighbourhood of a point $p \in M^{\mathcal{T}}$ in its local form of the Toric Darboux Theorem

$$\mu = \mu(p) + \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 \lambda^{(i)},$$

where the $\lambda^{(i)}$ are the weights of the isotropy representation. Evaluating at X gives the candidate for the Morse-Bott function in these local coordinates:

$$\mu^X = \langle \mu(p), X\rangle + \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 \left\langle \lambda^{(i)}, X \right\rangle.$$

If $\left\langle \lambda^{(i)}, X \right\rangle = 0$ then it follows by continuity of the action and the fact that $\exp(\mathbb{R}X)$ is dense in $\mathcal{T}$ that the i-th summand V_i in T_pM is fixed by $\mathcal{T}$. It follows that

$$(T_pM)^{\mathcal{T}} = \bigoplus_{i\colon \lambda^{(i)}=0} V_i = \bigoplus_{i\colon \left\langle \lambda^{(i)}, X\right\rangle = 0} V_i = \ker(\mu^X).$$

But since we can identify $T_p(M^{\mathcal{T}}) = (T_pM)^{\mathcal{T}}$, this is exactly the non-degeneracy condition. Since $|z_i|^2 = x_i^2 + y_i^2$ we also conclude that all the indices are even. ■

3 Application: Preimage of a Facet of a Moment Polytope

Theorem (Atiyah, Guillemin-Sternberg)**.** *Let $(M, \omega, \mathcal{T}, \mu)$ be a compact connected Hamiltonian $\mathcal{T}$-space for a torus $\mathcal{T}$. Then:*

1. *the levels of μ are connected;*

2. *the image of μ is convex;*

3. *the image of μ is the convex hull of a finite number of points, that are images of the fixed points of the action.*

Proof. See e.g. the paper by Atiyah [1] or the one by Guillemin Sternberg [3]. ■

Let Δ be the moment polytope of a Hamiltonian $\mathcal{T}$-space M:

$$\mu(M) = \Delta = \bigcap_{i \in \mathcal{I}} \mathbb{H}_{(v_i, c_i)} = \{\varphi \in \mathfrak{t}^* \mid \langle \varphi, v_i \rangle \leq c_i \qquad \text{for all} \qquad i \in \mathcal{I}\}$$

Let F be a facet of Δ, that is, there is a $j \in \mathcal{I}$ such that

$$F = \{\varphi \in \Delta \mid \langle \varphi, v_j \rangle = c_j\}$$

and consider its preimage under the moment map $N := \mu^{-1}(F)$. Since μ is continuous and F is closed, N is closed.

v_j determines an S^1-subgroup of $\mathcal{T}$ which also acts Hamiltonian way on M with moment map $i^* \circ \mu$, where $i : \mathfrak{s} \hookrightarrow \mathfrak{t}$ is the inclusion. We get

$$\langle i^* \circ \mu(p), v \rangle = \langle \mu(p), i(v) \rangle = \langle \mu(p), v \rangle .$$

But $\mu(p) \in \Delta$ so that

$$\langle \mu(p), v_j \rangle = c_j \quad \iff \quad \mu(p) \in F \quad \iff \quad p \in N$$

which shows that N is a level set of $\mathrm{d}i_e^* \circ \mu$. Therefore, by the convexity Theorem N is connected.

We know that $(i^* \circ \mu)^{v_j}$ is a Morse-Bott function and the connected components of $\mathrm{Crit}\,((i^* \circ \mu)^{v_j})$ are thus symplectic submanifolds. N is by construction such a critical submanifold of $(i^* \circ \mu)^{v_j}$. This shows that it is a symplectic submanifold of M.

References

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