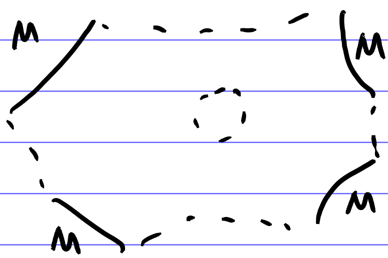
Conj (Kontsevich '09) The Fukaya category of a Stein manifold X can be locally computed on a Lagrangian core.

Thm (HKK '14) The topological Fukaya category (2) of a marked surface Σ (1) can be locally computed (4) on a ribbon graph (3).

1) **Def** A *marked surface* Σ is a smooth surface with corners $\partial_0 \Sigma$ and $M \subset \partial\Sigma$ compact 1-dim with $\partial M = \partial_0 \Sigma$.

Moreover, $S^1 \not\subset M$.

Ex



A_∞ - category:

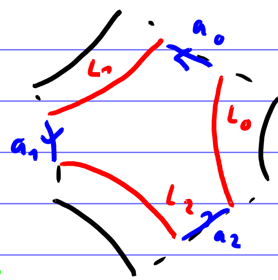
- Objects $\{X_i\}$
- Morphisms $CW(X_1, X_2)$
- $\mu^k : CW(X_0, X_1) \otimes \cdots \otimes CW(X_{k-1}, X_k) \to CW(X_0, X_k)$

$(k \geq 1)$ satisfy relations

(i) μ^1 makes $CW(X_0, X_1)$ a chain complex

(ii) μ^2 composition

<u>Ex</u> $\mathcal{A}_2$



- $Ob(\mathcal{A}_2) = \{L_0, L_1, L_2\}$
- $CW(L_i, L_{i+1}) = \mathbb{Z}_2 a_i$

$CW(L_i, L_i) = \mathbb{Z}_2 \, id_{L_i}$

- $\mu^1 = 0$, $\mu^2(id_{L_i}, a_i) = a_i = \mu^2(a_i, id_{L_{i+1}})$

$\mu^3(a_1, a_2, a_3) = id_{L_i}$

We call $\mathcal{A}_2$ the "universal exact triangle"

Def. The diagram
$$\begin{tikzcd} X_0 \arrow[rr, "f_0"] & & X_1 \arrow[dl, "f_1"] \\ & X_2 \arrow[ul, "f_2"] & \end{tikzcd}$$
is an <u>exact triangle</u> in an A_∞-category $\mathcal{A}$ if $\exists$ an A_∞-functor

$$\mathcal{F} : \mathcal{A}_2 \to \mathcal{A}$$

such that $\mathcal{F}(L_i) = X_i$

$$[\mathcal{F}(a_i)] = [f_i]$$

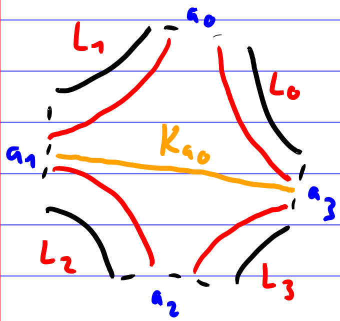
- An A_∞-category is called <u>triangulated</u> if any morphism $X_0 \xrightarrow{f_0} X_1$ with $\mu^1(f_0) = 0$ can be completed into an exact triangle

- By formally adding triangles and triangles of triangles, ... get an inclusion into a triangulated closure (defined up to equivalence)

$$\mathcal{A} \hookrightarrow \underbrace{Tw\,\mathcal{A}}_{\text{triangulated}}$$

Ex $\tilde{\mathcal{A}}_n$ defined as $\mathcal{A}_2$ but for $n \geq 3$

is not yet triangulated:



To have a triangulated A_∞-category

need to add new objects K_{a_0}, K_{a_1}

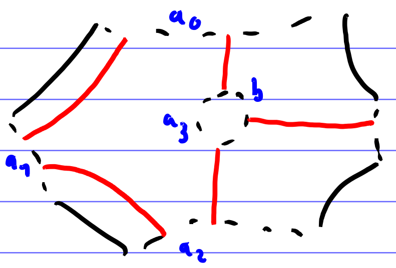
$$\begin{tikzcd} L_0 \arrow[rr, "a_0"] & & L_1 \arrow[dl] \\ & K_{a_0} \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd} L_1 \arrow[rr, "a_1"] & & L_2 \arrow[dl] \\ & K_{a_1} \arrow[ul] & \end{tikzcd}$$

Set $A_n := \mathrm{Tw}\, \tilde{\mathcal{A}}_n$

Def (topological Fukaya category $\mathcal{F}_A(\Sigma)$)

- $Ob(\mathcal{F}_A(\Sigma)) = A$ = set of non-isotopic arcs that divide Σ into polygons with maximal one component of M in its boundary

- $CW(L, L')$ generated by boundary arcs and $CW(L,L)$ has preferred morphism id_L.



- $\mu^1 = 0$, μ^2 unital wrt id_L

If $a_0, \dots, a_n$ build a disk sequence and b can be composed with a_n

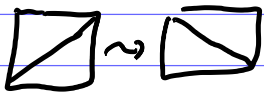
then $\mu^n(a_0, \dots, a_n * b) = b$

(similarly precompose)

Thm $Tw(\mathcal{F}_A(\Sigma))$ is independent of A up to equivalence

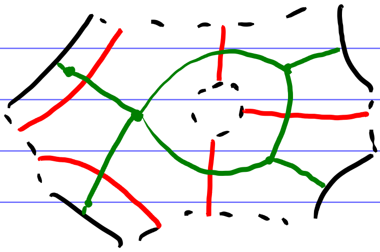
Set: $Fuk(\Sigma) = Tw(\mathcal{F}_A(\Sigma))$

Proof Need to show that starting with an arc system A, any element in a arc system A' can be produced by iterated exact triangles of elements in A.

(Proved by Hatcher §1' using that two triangulations of a surface can be connected by moves )

Thm $Fuk(\Sigma)$ is geometric, i.e. can think of objects as unobstructed immersed curves on Σ. If a curve is closed $(\approx S^1)$ then it is not allowed to be contractible.

Def The dual graph G_A of an arc system A is given by vertices for any face of $\Sigma \setminus A$ and edges for adjacent faces and halfedges for marked boundaries



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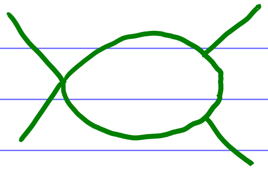

These graphs arising from arc systems are called "ribbon graphs" of Σ

Rem $\Sigma \overset{\swarrow \text{def. retract}}{\simeq} G_A \qquad \forall A$

Def A cosheaf $\mathcal{C}$ of A_∞-categories on a graph G a system of A_∞-categories

- $\mathcal{C}_v$ for vertex v
- $\mathcal{C}_e$ for an edge e
- functors $\mathcal{C}_e \to \mathcal{C}_v$ if v is an endpoint of e

Def (topological Fukaya category as a cosheaf)

Given Σ, A get G_A. Define

- C_e with one object, the arc L dual to e and morphism space $\mathbb{Z}_2 \mathrm{id}_L$
- $C_v = \mathcal{A}_{\deg v - 1}$
- Obvious functors $C_e \to C_v$

Thm $\Gamma(G, \mathcal{C})$ = the set of global sections of this cosheaf is equivalent to $\mathrm{Fuk}(\Sigma)$.

Higher dimensions — dim = 2

stopped Weinstein manifold $\leftrightarrow$ marked surface

Liouville flow is gradient-like for a Morse function

core = union of stable manifolds $\leftrightarrow$ ribbon graph (is a singular Lagrangian)

unstable manifolds of critical points of dim n (are Lagrangian) $\leftrightarrow$ arc system

Wrapped Fukaya category $\leftrightarrow$ topological Fuk category