Junior symplectic seminar: The C^1 flux conjecture

Thm: [flux conjecture; Ono '06]
Let M be a closed symplectic manifold. Then, $\mathrm{Ham}(M)$ is closed in $\mathrm{Symp}_0(M)$ in the C^1-topology.

We will assume $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$.

1. The flux group

The flux homomorphism is

$$\mathrm{Flux} : \widetilde{\mathrm{Symp}_0}(M) \longrightarrow H^1_{dR}(M)$$
$$[\{\psi_t\}] \longmapsto \left[\int_0^1 i_{X_t}\omega \, dt\right],$$

where $\frac{d}{dt}\psi_t = X_t \circ \psi_t \;\forall t$

i) is surjective;
ii) has kernel $\{[\{\psi_t\}] \mid \psi_1 \in \mathrm{Ham}(M)\}$.

Defn: The flux group of M is
$$\Gamma_\omega := \mathrm{Flux}(\pi_1(\mathrm{Symp}_0(M))).$$

Rem: Since $H^1_{dR}(M) = \mathrm{Hom}_{\mathbb{R}}(\pi_1(M), \mathbb{R})$, we can consider for $\gamma : S^1 \xrightarrow{C^\infty} M$

$$\begin{aligned}
\mathrm{Flux}([\{\psi_t\}])([\gamma]) &= \left\langle \int_0^1 i(X_t)\omega\, dt, \gamma \right\rangle \\
&= \int_0^1\int_0^1 \omega_{\gamma(t)}(X_t(\gamma(s)), \dot\gamma(s))\, dt\, ds \\
&= \langle \omega, f_*[\mathbb{T}^2]\rangle \quad \text{for } f : \mathbb{T}^2 \longrightarrow M,\ (t,s) \longmapsto \psi_t(\gamma(s)) \\
&\in \langle \omega, H_2(M;\mathbb{Z})\rangle.
\end{aligned}$$

In particular, Γ_ω is at most countable.

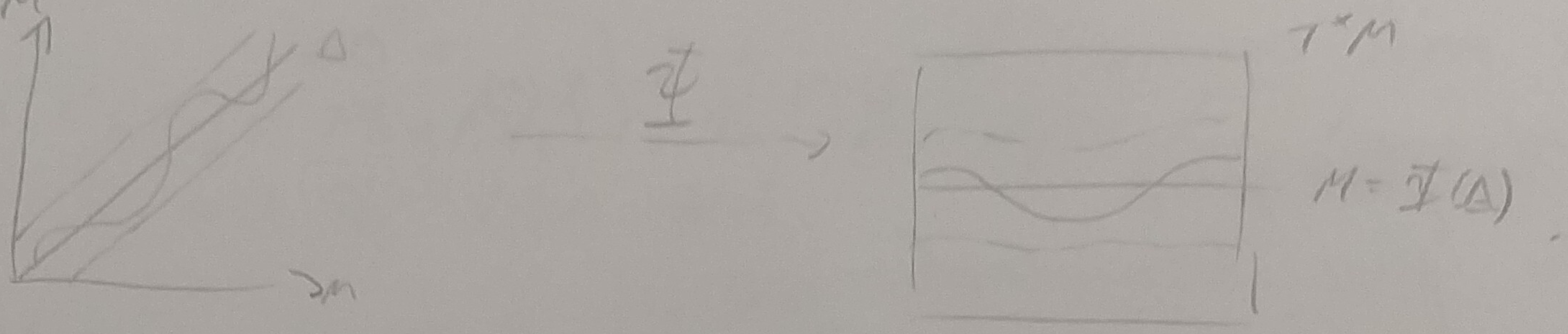
Prop: [Banyaga, '78]
$\mathrm{Ham}(M)$ is C^1-closed $\iff$ Γ_ω is discrete.

We first recall a basic construction. Fix a symplectomorphism
$$\Psi : U \longrightarrow V,$$
where U is a neighborhood of Δ in $M \times M$ and V of M in T^*M, s.t. $\Psi(x,x) = (x,0)$. We get a chart

$$\mathcal{C}: \mathcal{U} \longrightarrow \mathcal{V}$$

from a C^1-neighborhood of $\mathbb{1}$ in $\mathrm{Symp}_0(M)$ to a C^1-neighborhood of 0 in $\ker d \subseteq \Omega^1(M)$ by setting

$$\mathrm{graph}\, \mathcal{C}(\psi) = \Psi(\mathrm{graph}\, \psi),$$



WLOG $\mathcal{V}$ is star-shaped and symmetric w.r.t. 0.

Lem.: $\mathcal{C}(\mathcal{U} \cap \mathrm{Ham}(M)) = \mathcal{V}_\Gamma := \{\sigma \in \mathcal{V} \mid [\sigma] \in \Gamma_\omega\}$.

Proof: ($\subseteq$) Take $\psi \in \mathcal{U} \cap \mathrm{Ham}(M)$. Then,

i) $\exists \{\psi_t\}$ symplectic s.t. $\mathcal{C}(\psi_t) = t\sigma = t\,\mathcal{C}(\psi)$ $\forall t \in [0,1]$

ii) $\exists \{\varphi_t\}$ Hamiltonian s.t. $\varphi_1 = \psi$.

Consider

$$\psi_t' := (\psi \# \bar{\varphi})_t = \begin{cases} \psi_{2t} & \text{if } t \le \frac{1}{2} \\ \varphi_{2(1-t)} & \text{if } t \ge \frac{1}{2} \end{cases}.$$

computation!

Then

$$\Gamma_\omega \ni \mathrm{Flux}(\{\psi_t'\}) = \mathrm{Flux}(\{\psi_t\}) - \overbrace{\mathrm{Flux}(\{\varphi_t\})}^{=0\ \checkmark} = [\sigma]$$

($\supseteq$) Take $\sigma \in \mathcal{V}_\Gamma$. Then, $\exists [\{\psi_t''\}] \in \pi_1(\mathrm{Symp}_0(M))$ s.t. $\mathrm{Flux}(\{\psi_t''\}) = -[\sigma]$. Take $\{\psi_t\}$ and $\psi_t' = (\psi'' \# \psi)_t$ as above. Then,

$$\mathrm{Flux}(\{\psi_t'\}) = \mathrm{Flux}(\{\psi_t''\}) + \mathrm{Flux}(\{\psi_t\}) = -[\sigma] + [\sigma] = 0$$

Therefore, $\psi_1' = \psi_1 = \mathcal{C}^{-1}(\sigma) \in \mathrm{Ham}(M)$. ∎

Proof of Banyaga's proposition:

($\Leftarrow$) Suppose that Γ_ω is not discrete. Then, it is countable. In particular, it is not closed, and we can take $\{\alpha_n\} \subseteq \Gamma_\omega$ s.t. $\alpha_n \longrightarrow \alpha \in \bar{\Gamma}_\omega - \Gamma_\omega$. Therefore if σ_n is the harmonic rep. of α_n and σ, of α, then for N large, $\bar{\sigma} := \sigma - \sigma_N \in \mathcal{V} - \mathcal{V}_\Gamma$ and $\bar{\sigma}_n := \sigma_n - \sigma_N \in \mathcal{V}_\Gamma$. Furthermore, $\bar{\sigma}_n \xrightarrow{C^\infty} \bar{\sigma}$ s.t.

$$\psi_n \in \mathcal{C}^{-1}(\bar{\sigma}_n) \xrightarrow{C^1} \psi := \mathcal{C}^{-1}(\bar{\sigma}) \notin \mathrm{Ham}(M).$$

Therefore, $\mathrm{Ham}(M)$ is not C^1-closed.

($\Rightarrow$) Go backwards. ∎

2 Novikov - Floer homology

For a closed 1-form η on M, denote by $\pi: M_\eta \to M$ the covering associated to $\ker \eta \subseteq \pi_1(M)$. Then, we get a covering between the free loop spaces: $\Lambda M_\eta \to \Lambda M$, and contractible free loop spaces: $\mathcal{L} M_\eta \to \mathcal{L} M$.

Suppose that $[\eta] = \mathrm{Flux}(\{\psi_t\})$. By the deformation lemma, we can find X_t and H_t 1-per. s.t.

$$\iota_{X_t}\omega = \eta + dH_t .$$

Fix also $f: M_\eta \to \mathbb{R}$ s.t. $\pi^*\eta = df$.
We define an action functional on $\mathcal{L} M_\eta$ via

$$\mathcal{A}_{\{\psi_t\}}(\tilde\gamma) := \int_D v^*\omega + \int_0^1 (f + H_t \circ \pi)(\tilde\gamma(t))\, dt, \quad \text{where } v|_{S^1} = \pi\circ\tilde\gamma$$

For a 1-parameter family $J = \{J_t\}_{t\in S^1}$ of ω-compatible a.c.s., then

$$CFN_*(\{\psi_t\}) = \Big\{ \sum_i a_i \tilde\gamma_i \,\Big|\, a_i \in \mathbb{Z}_2,\ \tilde\gamma_i \in \mathrm{Crit}\,\mathcal{A}_{\{\psi_t\}},\ \#\{i \mid a_i \neq 0,\ \mathcal{A}_{\{\psi_t\}}(\tilde\gamma_i) < c\} < \infty\ \forall c \Big\},$$

and

$$\langle \partial \tilde\gamma_-, \tilde\gamma_+\rangle = \#_2 \Big\{ u: \mathbb{R}\times S^1 \to M \,\Big|\, \partial_s u + J_t(u)(\partial_t u - X_t(u)) = 0,\ \lim_{s\to\pm\infty} \tilde u(s,\cdot) = \tilde\gamma_\pm,\ \pi\circ\tilde u = u \Big\}/\mathbb{R}$$

Thm.
1) $\partial^2 = 0$
2) If $\mathrm{Flux}(\{\psi_t'\}) = [\eta] = \mathrm{Flux}(\{\psi_t\})$ and $J' = \{J_t'\}$ is another family, then
$$HFN_*(\{\psi_t\}, J) \cong HFN_*(\{\psi_t'\}, J').$$
3) If $[\{\psi_t\}] \in \mathrm{Ham}(M)$, then
$$HFN_*(\{\psi_t\}) \cong HF_*(\{\psi_t\}; \mathbb{Z}_2)$$
4) If η is Morse and C^1-small, then
$$HFN_*(\{\psi_t\}) \cong HN_{*-n}(M; [\eta])$$

Take

$$\Lambda_\eta := \Big\{ \sum_i a_i g_i \,\Big|\, a_i \in \mathbb{Z}_2,\ g_i \in \pi_1(M)/\ker\eta,\ \#\{i \mid a_i \neq 0,\ \eta(g_i) < c\} < \infty\ \forall c \Big\}$$

Note that $HFN_*(\{\psi_t\})$ is a f.g. Λ_η-module with the action given by $g \cdot \sum a_i \tilde\gamma_i = \sum_i a_i (g\cdot\tilde\gamma_i)$.

3. Ono's proof of the flux conjecture

Let $V \subseteq H^1_{dR}(M)$ a neighborhood of 0 which is star-shaped, symplectic, and rep by C^1-small Morse 1-forms. We want to show that

$$\Gamma_\omega \cap V = \{0\}.$$

Suppose thus that $\exists [\sigma] \in \Gamma_\omega \cap V - \{0\}$, and let $\{\psi_t\} \in \pi_1(\mathrm{Symp}_0(M))$ s.t. $\mathrm{Flux}[\{\psi_t\}] = [\sigma]$.

Idea of the proof:

1) We construct $\Phi: \mathcal{L}M \to \mathcal{L}M$ sending zeros of σ to orbits of a Hamiltonian isotopy $\{\varphi_t\}$.

2) We show that Φ induces compatible isomorphisms

$$HFN_\circ(\{\psi^{-\sigma}_t\}) \xrightarrow{\sim} HFN_{\circ+l}(\{\varphi_t\})$$

$$\Lambda_{-\sigma} \xrightarrow{\sim} \Lambda_0 = \mathbb{Z}_2,$$

where $\{\psi^{-\sigma}_t\}$ is the isotopy gen. by $X^{-\sigma}$ s.t. $\iota_{X^{-\sigma}}\omega = -\sigma$.

3) We compute

$$\dim_{\Lambda_{-\sigma}} HFN_\circ(\{\psi^{-\sigma}_t\})$$

and $\dim_{\mathbb{Z}_2} HFN_\circ(\{\varphi_t\})$

and get a contradiction.

Step 1: Since

$$\mathrm{Flux}(\{(\psi \# \psi^{-\sigma})_t\}) = [\sigma] - [\sigma] = 0,$$

$\exists \{\{\varphi_t\}\} \in \mathrm{Ham}(M)$ s.t. $\varphi_1 = (\psi \# \psi^{-\sigma})_1 = \psi^{-\sigma}_1$. Take $[\Phi_t := \varphi_t \circ (\psi^{-\sigma}_t)^{-1}\}] \in \pi_1(\mathrm{Symp}_0(M))$ and define

$$\Phi: \Lambda M \longrightarrow \Lambda M$$
$$x \longmapsto (t \mapsto \Phi_t(x(t))).$$

Lemma: Φ preserves contractibility, i.e. it defines a map $\mathcal{L}M \to \mathcal{L}M$.

Proof: By continuity, Φ sends $\mathcal{L}M$ to a connected component of ΛM. It also induces a bijection between the 1-periodic orbits of $\{\psi^{-\sigma}_t\}$ and those of $\{\varphi_t\}$.

However, $\mathrm{Per}(\{\psi_t^{-\sigma}\}) = \mathrm{Zeros}(\sigma) \subseteq \mathcal{L}M$ and $\{\psi_t\}$ has at least one 1-per. orbit by Arnold's conjecture. Therefore, this argument must be $\mathcal{L}M$.

∎

Step 3: Assuming step 2, we have

$$\dim_{\Lambda_{-\sigma}} HFN_\circ(\{\psi_t^{-\sigma}\}) = \dim_{\mathbb{Z}_2} HFN_\circ(\{\psi_t\})$$
$$\| \qquad\qquad\qquad \|$$
$$b_{tot}(M;[-\sigma]) \qquad\qquad b_{tot}(M).$$

Prop.: [Le-Ono, '93] If $\ker\eta \subseteq \ker\eta'$, then
$$b_p(M;[\eta]) \leq b_p(M;[\eta']) \quad \forall p \in \mathbb{Z}.$$

Therefore, $b_p(M;[-\sigma]) \leq b_p(M)$. But $HN_p(-\sigma) = 0$ for $p=0, 2n$ since σ is not exact.
$$\Rightarrow b_{tot}(M;[-\sigma]) < b_{tot}(M).$$

Step 2: This follows from a general fact of Novikov-Floer homology.

Prop.: If $[\{\psi_t\}], [\{\psi_t'\}] \in \widetilde{\mathrm{Symp}}_0(M)$ are s.t. $\psi_1 = \psi_1'$ and
$$\Phi(\gamma)(t) = (\psi_t' \circ \psi_t^{-1})(\gamma(t))$$
preserves contractibility, then
$$\Phi^*\eta' = \eta \quad \text{and} \quad \Phi_*(\ker\eta) = \ker\eta'$$
for some C^1-small rep. of the flux.

From this, we get iso.
$$\Phi_* : \Lambda_\eta \longrightarrow \Lambda_{\eta'}$$
$$\widehat{\Phi} : \mathcal{L}M_\eta \longrightarrow \mathcal{L}M_{\eta'}$$
s.t. $\widehat{\Phi}(g\tilde{\gamma}) = \Phi_*(A)\cdot\widehat{\Phi}(\tilde{\gamma})$. We thus get a twisted chain iso:
$$\widehat{\Phi}_\# : CFN_\circ(\{\psi_t\}, J) \longrightarrow CFN_\circ(\{\psi_t'\}, J'),$$
where
$$J_t' := d\Phi_t \circ J_t \circ d\psi_t^{-1}.$$

∎