Symplectic Reduction: Ideas, Examples and Related Constructions

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Recall. Let $\psi : G \to \mathrm{Diff}\,(M)$ be a smooth action of a Lie group G on a symplectic manifold (M, ω). ψ is called a *Hamiltonian action* if there exists a map

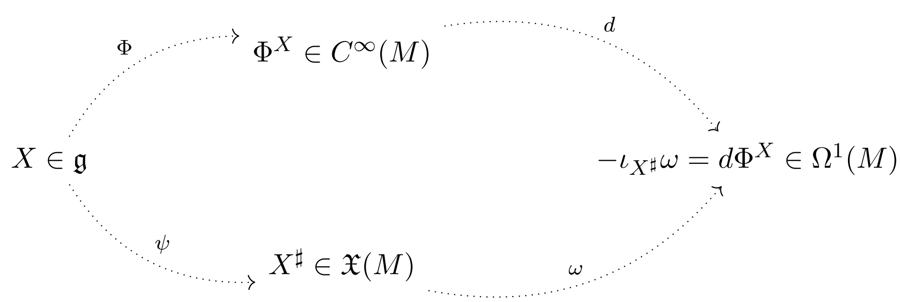
$$\Phi : M \to \mathfrak{g}^*$$

satisfying two conditions:

- Note the following standard constructions:

  1. Φ gives a function on M by considering the component Φ^X of Φ along $X \in \mathfrak{g}$: $\Phi^X(p) = \langle \Phi(p), X \rangle$.
  2. The action gives a vector field for each $X \in \mathfrak{g}$ via the infinitesimal action: $X^\sharp \in \mathfrak{X}(M)$ is defined pointwise as $X^\sharp_p = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \psi_{\exp(tX)}(p)$.
  3. The symplectic form gives a way of defining a 1-form for each vector field: For $Y \in \mathfrak{X}(M)$, the 1-form is given by $-\iota_Y \omega \in \Omega^1(M)$.
  4. Exterior differentiation applied to a function $f \in C^\infty(M)$ is a 1-form $\mathrm{d}f$.

  The first condition is now that these operations are compatible with each other i.e. that the following diagram commutes:



  In other words, for each $X \in \mathfrak{g}$, the component of Φ along X is a Hamiltonian function for the fundamental vector field $X^\sharp$.

- The second condition is that Φ is (ψ, Ad^*)-equivariant i.e. the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & \mathfrak{g}^* \\ \downarrow{\scriptstyle \psi_g} & & \downarrow{\scriptstyle Ad_g^*} \\ M & \xrightarrow{\Phi} & \mathfrak{g}^* \end{array}$$

(M, ω, G, Φ) is called a *Hamiltonian G-space* and Φ is called a *moment map*.

Example. Equip $\mathbb{C}^d$ with the standard symplectic form $\omega = \frac{i}{2} \sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i = \sum r_i \mathrm{d}r_i \wedge \mathrm{d}\theta_i$ and act with the d-torus as

$$(e^{iX_1}, ..., e^{iX_d}) \cdot (z_1, ..., z_d) = (e^{iX_1} z_1, ..., e^{iX_d} z_d)$$

or in polar coordinates:

$$(e^{iX_1}, ..., e^{iX_d}) \cdot (r_1, \theta_1, ..., r_d, \theta_d) = (r_1, \theta_1 + X_1, ..., r_d, \theta_d + X_d).$$

Using these polar coordinates, the vector field generated by this action can then immediately be identified as

$$X^\sharp = X_1 \frac{\partial}{\partial \theta_1} + X_2 \frac{\partial}{\partial \theta_2} + ... + X_d \frac{\partial}{\partial \theta_d}.$$

Consider then the map

$$\begin{aligned} \Phi : \mathbb{C}^d &\to (\mathbb{R}^d)^* \cong \mathbb{R}^d \\ (z_1, ..., z_d) &\mapsto \frac{1}{2} \sum |z_i|^2 + \text{const} = \frac{1}{2} \sum r_i^2 + \text{const}. \end{aligned}$$

Then we can easily compute that

$$\iota_{X^\sharp} \omega = -\sum X_i r_i \mathrm{d}r_i = -\frac{1}{2} \sum X_i \mathrm{d}r_i^2 = \mathrm{d}\left(-\frac{1}{2} \sum X_i r_i^2\right) = -\mathrm{d}\langle \Phi(r_i, \theta_i), X \rangle$$

and so this is indeed a moment map.

Remark. We will come back to this example and make use of the liberty to choose the constant.

Lemma. *Let (M, ω, G, Φ) be a hamiltonian G-space. Then for every $p \in M$ we have*

$$\ker(\mathrm{d}\Phi_p) = (T_p(\mathcal{O}_p))^\omega \quad \textit{and} \quad \mathrm{im}(\mathrm{d}\Phi_p) = (\mathfrak{g}_p)^0.$$

Proof. By the first condition in the definition of a moment map we have

$$\langle \mathrm{d}\Phi_p[v], X \rangle = \omega_p(v, X_p^\sharp) \quad \text{for all} \quad X \in \mathfrak{g},\ v \in T_p M.$$

Claim. It holds that

$$\mathrm{d}\Phi_p[v] = 0 \quad \iff \quad \omega_p(v, X_p^\sharp) = 0,\ \forall X \in \mathfrak{g}.$$

Proof of Claim. If $\mathrm{d}\Phi_p[v] = 0$, then for all $X \in \mathfrak{g}$

$$\omega_p(v, X_p^\sharp) = \langle \mathrm{d}\Phi_p[v], X\rangle = 0.$$

Reversely, assume that $\omega_p(v, X_p^\sharp) = 0$ for all $X \in \mathfrak{g}$. Then this holds in particular for the elements of any basis for $\mathfrak{g}$. If $\mathrm{d}\Phi_p[v]$ vanishes on all basis elements, it is identically zero.

Using this claim and that $T_p(\mathcal{O}_p)$ is spanned by the fundamental vector fields we conclude

$$\ker(\mathrm{d}\Phi_p) = \left\{ v \in T_pM \mid \omega_p(v, X_p^\sharp) = 0 \ \forall X \in \mathfrak{g} \right\} = (T_p(\mathcal{O}_p))^{\omega_p}.$$

Next, we observe that

$$\begin{aligned}
\dim(\ker(\mathrm{d}\Phi_p)) &= \dim\left((T_p\mathcal{O})^{\omega_p}\right) \\
&= \dim(T_pM) - \dim(T_p\mathcal{O}) \\
&= \dim(M) - \dim(\mathcal{O}) \\
&= \dim(M) - (\dim(G) - \dim(G_p))
\end{aligned}$$

so that

$$\begin{aligned}
\dim(\mathrm{im}(\mathrm{d}\Phi_p)) &= \dim(T_pM) - \dim(\ker(\mathrm{d}\Phi_p)) \\
&= \dim(G) - \dim(G_p) \\
&= \dim(\mathfrak{g}) - \dim(\mathfrak{g}_p) \\
&= \dim\left((\mathfrak{g}_p)^0\right).
\end{aligned}$$

Hence to prove the second result it suffices to show one inclusion. But it is clear that $\mathrm{im}(\mathrm{d}\Phi_p) \subset (\mathfrak{g}_p)^0$ since for $v \in T_pM$ and any $X \in \mathfrak{g}_p$ we observe

$$\langle \mathrm{d}\Phi_p[v], X\rangle = \omega_p(v, X_p^\sharp) = \omega_p(v, 0) = 0$$

as $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid X_p^\sharp = 0\}$. ■

Corollary. *Let (M, ω, G, Φ) be a hamiltonian G-space and let $p \in M$ be a point. Then p is a regular point of Φ if and only if G acts locally freely at p.*

Proof. Using the lemma we get

$$\begin{aligned}
p \text{ regular point of } \Phi \quad &\iff \quad \mathrm{d}\Phi_p \text{ surjective.} \\
&\iff \quad (\mathfrak{g}_p)^0 = \mathfrak{g}^* \\
&\iff \quad \mathfrak{g}_p = 0
\end{aligned}$$

■

Hence, $\varphi \in \mathfrak{g}^*$ is a regular value of the moment map Φ if and only if $\mathfrak{g}_p = 0$ for all $p \in \Phi^{-1}(\varphi)$. If this is the case, then the implicit function theorem gives that $\Phi^{-1}(\varphi)$ is an embedded submanifold of dimension $\dim(M) - \dim(\mathfrak{g}^*) = \dim(M) - \dim(G)$. Moreover, the tangent space to a point $p \in \Phi^{-1}(\varphi)$ is given by $T_p(\Phi^{-1}(\varphi)) = \ker(\mathrm{d}\Phi_p) = (T_p(\mathcal{O}_p))^\omega$. However, since we want to study symplectic actions, we now have two major issues:

1. $\Phi^{-1}(\varphi)$ is in general not invariant under the group action. Indeed, by the second condition of the moment map: for $p \in \Phi^{-1}(\varphi)$ and $g \in G$

$$\Phi \circ \psi_g(p) = \mathrm{Ad}_g^* \circ \Phi(p) = \mathrm{Ad}_g^*(\varphi)$$

and so $\psi_g(p)$ lies in $\Phi^{-1}(\varphi)$ if and only if φ is fixed by the coadjoint action. There are two main cases where this holds and which are of interest to us:

   - If $\varphi = 0$, then $\mathrm{Ad}_g^*(\varphi) = \varphi$ regardless of of G.
   - If G is abelian, the coadjoint representation is trivial and so $\mathrm{Ad}_g^*(\varphi) = \varphi$ for any $\varphi \in \mathfrak{g}^*$.

2. The tangent space $T_p(\Phi^{-1}(\varphi)) = (T_p(\mathcal{O}_p))^\omega$ is in general not a symplectic subspace of T_pM so a priori we do not have a symplectic form on it. However, in the two cases described above, equivariance reduces to invariance meaning that Φ is constant on the orbits. In particular its differential thus maps the tangent space to the orbits to zero:

$$T_p(\mathcal{O}_p) \subset \ker(\mathrm{d}\Phi_p) = (T_p(\mathcal{O}_p))^\omega$$

so that $T_p(\mathcal{O}_p) \subset T_pM$ may not be a symplectic subspace, but an isotropic subspace. Hence we can apply the following:

Lemma. *Let (V, ω) be a symplectic vector space and suppose that $U \subset V$ is an isotropic subspace i.e. $U \subset U^\omega$ or equivalently $\omega|_{U\times U} = 0$. Then ω induces a canonical symplectic form $\bar{\omega}$ on U^ω/U such that*

$$\pi^*\bar{\omega} = i^*\omega \quad \textit{where} \quad \begin{cases} i : U^\omega \hookrightarrow U \textit{ is the inclusion and} \\ \pi : U^\omega \to U^\omega/U \textit{is the projection.} \end{cases}$$

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Proof. ([1], Lemma 23.3.) Let $v, w \in U^\omega$ and write $[v], [w] \in U^\omega/U$ for their equivalence classes. Define then a 2-form on the quotient as

$$\begin{aligned} \bar{\omega} : U^\omega/U \times U^\omega/U &\to \mathbb{R} \\ ([v], [w]) &\mapsto \omega(v, w) \end{aligned}$$

and check that this is

1. well-defined since for any $u, u' \in U$

$$\omega(v + u, w + u') = \omega(v, w) + \underbrace{\omega(v, u')}_{=0} + \underbrace{\omega(u, w)}_{=0} + \underbrace{\omega(u, u')}_{=0} = \omega(v, w)$$

by definition of the symplectic orthocomplement and

2. non-degenerate: If $v \in U^\omega$ is such that $\omega(v, w) = 0$ for all $w \in U^\omega$, then $v \in (U^\omega)^\omega = U$ so that $[v] = 0$ in the quotient. ■

The idea is thus to get the quotient of $T_p(\Phi^{-1}(\varphi)) = (T_p(\mathcal{O}_p))^{\omega_p}$ by $T_p(\mathcal{O}_p)$ as tangent space. It would therefore be natural to consider the orbit space $\Phi^{-1}(\varphi)/G$. However, for this to be a manifold, we need a stronger hypothesis: If we assume that the action restricted to $\Phi^{-1}(\varphi)$ is free, not only locally free, the quotient $\Phi^{-1}(\varphi)/G$ is actually a manifold. Assembling all those arguments, one can then prove the following theorem:

Theorem (Marsden-Weinstein, Meyer)**.** *Let (M, ω, G, Φ) be a hamiltonian G-space for a compact Lie group G and take $\varphi \in \Phi(M)$. Assume that $\varphi = 0$ or that G is Abelian. Let $i : \Phi^{-1}(\varphi) \hookrightarrow M$ denote the inclusion map and assume further that G acts freely on $\Phi^{-1}(\varphi)$. Then*

1. *the orbit space $M_{red} = \Phi^{-1}(\varphi)/G$ is a manifold,*
2. *$\Pi : \Phi^{-1}(\varphi) \to M_{red}$ is a principal G-bundle and*
3. *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \Pi^*\omega_{red}$.* $\blacksquare$.

Definition. The pair (M_{red}, ω_{red}) is called the *symplectic reduction* or the *symplectic quotient* of (M, ω) by G and Φ.

Example. Let $\omega = \frac{i}{2}\sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i = \sum r_i \mathrm{d}r_i \wedge \mathrm{d}\theta_i$ be the standard symplectic form on $\mathbb{C}^{n+1}$. Consider then the action of S^1 on $(\mathbb{C}^{n+1}, \omega)$ given by multiplication, meaning rotation in each factor by the angle corresponding to $e^{i\theta} \in S^1$. Then we can consider $\Phi^{-1}(\frac{1}{2}) = S^{2n+1}$ and since the action is free everyhere except at the origin, we can apply the reduction theorem:

$$\Phi^{-1}(0)/S^1 = S^{2n+1}/U(1) = \mathbb{CP}^n$$

One checks that the induced form is exactly the Fubini-Study symplectic form.

Example. Consider now $\mathbb{C}^{n\times k}$ (we assume $k \leq n$) and equip it again with the standard symplectic structure. We might think of $\mathbb{C}^{n\times k}$ as the space of complex $n \times k$-matrices and act on it by $U(k)$ with multiplication on the right. Recall then that the Lie algebra $\mathfrak{u}(k)$ is given by the skew-hermitian $k \times k$-matrices and that it can be identified with its dual via the inner product $\langle A, B\rangle = \mathrm{Tr}(A^*B)$. The map

$$\begin{aligned} \Phi : \mathbb{C}^{n\times k} &\to \mathfrak{u}(k) \cong \mathfrak{u}(k)^* \\ A &\mapsto \frac{i}{2}AA^* - \frac{i}{2} \end{aligned}$$

is a moment map for this action. $\Phi^{-1}(0)$ is the set of matrices $A \in \mathbb{C}^{n\times k}$ with $AA^* = 1$, that is the Stiefel manifold $V_k(\mathbb{C}^n)$ corresponding to the set of k-frames in $\mathbb{C}^n$. Since multiplication on the right acts freely, we can apply the symplectic reduction theorem and obtain

$$\Phi^{-1}(0)/U(k) = V_k(\mathbb{C}^n)/U(k) = Gr_{\mathbb{C}}(k, n)$$

the Grassmannian. Further the Reduction-theorem tells us that the projection $\pi : V_k(\mathbb{C}^n) \to Gr_{\mathbb{C}}(k, n)$ is a principal $U(k)$-bundle. Passing to the limit $n \to \infty$, this becomes the universal bundle for $U(k)$.

Related Construction 1: Delzant's Construction

Let

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle \varphi, v_i \rangle \leq c_i, i = 1, ..., d\}$$

be a Delzant polytope i.e. for any vertex η of the polytope, the v_i such that $\langle \eta, v_i \rangle = c_i$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Then define the map

$$\Pi : \mathbb{R}^d \to \mathbb{R}^n$$
$$e_i \mapsto v_i$$

and note that it is surjective and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$. It follows that this induces a map π on the quotients $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Writing $K = \ker(\pi)$, we get a short exact sequence

$$0 \longrightarrow K \xhookrightarrow{i} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \longrightarrow 0.$$

Moreover, it follows from the Delzant condition (more precisely the fact that $\Pi(\mathbb{Z}^d) = \mathbb{Z}^n$) that K is a subtorus of $\mathbb{T}^d$.

We now come back to our example of the moment map from the beginning. Now we choose the constant to be the constants coming from the polytope $(c_1, ..., c_d)$:

$$\Phi : \mathbb{C}^d \to (\mathbb{R}^d)^* \cong \mathbb{R}^d$$
$$(z_1, ..., z_d) \mapsto \frac{1}{2} \sum |z_i|^2 + (c_1, ..., c_d) = \frac{1}{2} \sum r_i^2 + (c_1, ..., c_d).$$

Fact. The subgroup $K \subset \mathbb{T}^d$ inherits a Hamiltonian action on $\mathbb{C}^d$: its moment map is given by $i^* \circ \Phi$, where i^* is the map dual to the inclusion of the Lie algebras.

Fact. K acts freely on $(i^* \circ \Phi)^{-1}(0)$.

Hence we can apply the reduction theorem and consider the quotient $(i^* \circ \Phi)^{-1}(0)/K$. The result is a symplectic manifold $(M_\Delta, \omega_\Delta)$ which can be equipped with a moment map whose image is exactly Δ. This shows, that to any Delzant-polytope, we can find a Hamiltonian $\mathbb{T}$-space whose image under the moment map is precisely this polytope. More generally:

Theorem (Delzant)**.** *Symplectic toric manifolds are classified up to equivalence by unimodular polytopes up to translation. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\{\textit{symplectic toric manifolds}\}\Big/_{\textit{equiv.}} \quad \overset{1-1}{\to} \quad \{\textit{unimodular polytopes}\}\Big/_{\textit{transl.}}$$
$$(M, \omega, \mathcal{T}, \Phi) \quad \mapsto \quad \Phi(M)$$

The main point to keep in mind for the second talk, is that we can construct any symplectic toric manifold as a quotient of $\mathbb{C}^d$ by a given torus K.

Related Construction 2: Symplectic Cutting

This presentation of the basic construction of symplectic cutting follows [2].

Let (M, ω, S^1, Φ_M) be a Hamiltonian S^1-space. Parametrise S^1 by $\theta \in [0, 2\pi[$ and let $\sigma : S^1 \to \mathrm{Diff}(M)$ denote the action. S^1 also acts on the complex plane $\mathbb{C}$ with the standard symplectic structure $\omega_0 = \frac{i}{2} \mathrm{d}z \wedge \mathrm{d}\bar{z}$ by multiplication

$$\begin{aligned} \tau : S^1 &\to \mathrm{Diff}(\mathbb{C}) \\ \theta &\mapsto \tau_\theta : z \mapsto e^{-i\theta} z. \end{aligned}$$

This action is Hamiltonian and the corresponding moment map is

$$\begin{aligned} \Phi_{\mathbb{C}} : \mathbb{C} &\to \mathbb{R} \cong \mathfrak{s}^* \\ z &\mapsto \frac{1}{2}|z|^2. \end{aligned}$$

The product $M \times \mathbb{C}$ is again a Hamiltonian S^1-space where the moment map is

$$\begin{aligned} \Phi : M \times \mathbb{C} &\to \mathbb{R} \\ (p, z) &\mapsto \Phi_M(p) + \frac{1}{2}|z|^2. \end{aligned}$$

For an arbitrary $c \in \mathbb{R}$, the level set is

$$\Phi^{-1}(c) = \left(\Phi_M^{-1}(c) \times \{0\}\right) \sqcup \bigsqcup_{r>0} \left(\Phi_M^{-1}(c-r) \times \left\{\frac{1}{2}|z|^2 = r\right\}\right).$$

In other words, for each $c \in \mathbb{R}$, the preimage $\Phi^{-1}(c)$ is a disjoint union of two S^1-invariant subsets where the first can be identified with

$$\Phi_M^{-1}(c) \times \{0\} \cong \Phi_M^{-1}(c) = \{p \in M \mid \Phi_M(p) = c\}$$

and the second with

$$\bigsqcup_{r>0} \left(\Phi_M^{-1}(c-r) \times \left\{\frac{1}{2}|z|^2 = r\right\}\right) \cong \{p \in M \mid \Phi_M(p) < c\} \times S^1.$$

If S^1 acts freely on $\Phi_M^{-1}(c)$, then it also acts freely on $\Phi^{-1}(c)$. Indeed, it also acts freely on the second part since the S^1 action on $\mathbb{C}$ is free except at the origin. It follows that one can find a symplectic quotient by S^1 and Φ as in the Reduction-Theorem at such a value $c \in \mathbb{R}$.

Definition. The resulting space is denoted by $(M_{\leq c}, \omega_{\leq c})$ and call it the *symplectic cut of M below c* with respect to Φ_M.

Because the decomposition of $\Phi^{-1}(c)$ into two disjoint components above was S^1-invariant, also $M_{\leq c}$ is the union of two disjoint components: The first one can be identified with the symplectic quotient of M at $c \in \mathbb{R}$

$$\Phi_M^{-1}(c) \big/ S^1 := M_c$$

and the second with

$$\{p \in M \mid \Phi_M(p) > c\} \times S^1 \Big/ _{S^1} \cong \{p \in M \mid \Phi_M(p) > c\} := M_{<c}.$$

In conclusion we constructed a new symplectic manifold $M_{\leq c}$ which is made up of the two disjoint components:

$$M_{\leq c} = M_c \sqcup M_{<c}.$$

Remark. A similar construction but with the twisted product symplectic manifold $(M_1 \times \mathbb{C}, pr_M^*\omega - pr_{\mathbb{C}}^*\omega_0)$ and the the corresponding moment map yields the *symplectic cut* $M_{\geq c} = M_c \sqcup M_{>c}$ *of* M *above* c with respect to Φ_M. These two cut spaces can be glued together along the submanifolds M_c to recover the original symplectic manifold (M, ω).

References

[1] Ana Cannas da Silva. *Lectures on Symplectic Geometry.* Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2004.

[2] Shubham Dwivedi, Jonathan Herman, Lisa C Jeffrey, and Theo Van den Hurk. *Hamiltonian Group Actions and Equivariant Cohomology.* Springer, 2019.