Lagrangian correspondences

M, N "related" symplectic manifolds.

Transfer knowledge from M to N.

1. The symplectic category Symp

Def Objects: (M, ω)

Morphisms: $(M_1, \omega_1) \xrightarrow{\text{symplecto}} (M_2, \omega_2)$

Super restrictive!

Note: $\varphi : M_1 \to M_2$ symplecto

$\Rightarrow gr(\varphi) \subset M_1^- \times M_2$ Lagrangian

Def' Morphisms: $L \subset M_0^- \times M_1$ Lagrangian

Such L is called a <u>Lagrangian correspondence</u> from M_0 to M_1.

Composition: $M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2$

$$L_{01} \circ L_{12} = \left\{ (x_0, x_2) \in M_0 \times M_2 \;\middle|\; \exists m_1 \in M_1 \text{ with } \begin{array}{l} (m_0, m_1) \in L_{01} \\ (m_1, m_2) \in L_{12} \end{array} \right\}$$

Problems: · $L_{01} \circ L_{12}$ is generally not a manifold

· If $L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2$

then $L_{01} \times L_{12}$ is at least immersed

Two ways to go on

Fukaya '17 — immersed Lagrangians

Woodward-Wehrheim '10 — generalized Lag. corr.

Def'' Morphisms $M_0 \to M_1$ are <u>gneneralized Lagrangian correspondences</u>, i.e. a sequence

$$M_0 = \underbrace{N_0 \xrightarrow{L_{01}} N_1 \to \dots \xrightarrow{L_{r-1,r}} N_r}_{\mathcal{L}} = M_1$$

and $L_{i,i+1}$ are Lag correspondences

Composition: $M_0 = N_0 \to \dots \to N_r = M_1 = N_0' \to \dots \to N'_{r'} = M_2$

Def $\mathcal{L}$ is called cyclic if $M_0 = M_1$.

Ex A Lagrangian $L \subset M$ can be viewed as $pt \to M$ or $M \to pt$.

Hence for $L, L' \subset M$

$$pt \xrightarrow{L} M \xrightarrow{L'} pt$$

is a cyclic Lagrangian

2. Floer homology of cyclic Lagrangians

Generalize $HF(L, L')$ for $L, L' \subset M$

to $HF(\mathcal{L})$ for cyclic correspondences $\mathcal{L}$.

s.t. $HF(pt \xrightarrow{L} M \xrightarrow{L'} pt) = HF(L, L')$

Def $HF(M_0 \xrightarrow{L_{01}} \cdots \xrightarrow{L_{r0}} M_0)$

$:= HF(L_{01} \times L_{23} \times \cdots \times L_{r-1r}, (L_{12} \times \cdots \times L_{r0})^T)$

in $M_0^- \times M_1 \times \cdots \times M_{r-1}^- \times M_r$

r odd otherwise compose with Δ_{M_0}

at the end.

Alternatively: Direct definition by quilts

Want $HF(M_0 \xrightarrow{L_{01}} \cdots \xrightarrow{L_{i-1i}} M_i \xrightarrow{L_{ii+1}} \cdots \xrightarrow{L_{r0}} M_0)$

$\cong$

$HF(M_0 \xrightarrow{L_{01}} \cdots \to M_{i-1} \xrightarrow{L_{i-1i} \circ L_{ii+1}} M_{i+1} \to \cdots \xrightarrow{L_{r0}} M_0)$

if $L_{i-1i} \circ L_{ii+1}$ is embedded.

Ex $HF(pt \xrightarrow{L_0} M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_1} pt)$

$$\cong \begin{cases} HF(L_0, L_{01} \circ L_1) & \text{if } L_{01} \circ L_1 \text{ embedded} \\ HF(L_0 \circ L_{01}, L_1) & \text{if } L_0 \circ L_{01}, L_1 \text{ embedded} \end{cases}$$

3. Functorial properties

Def $\mathrm{Don}^{\#}(M_0, M_1)$ generalized Donaldson-Fukaya category has

objects:

generalized Lagrangian correspondences

morphisms:

Floer cohomology $HF(\underline{L}_1, \underline{L}_2)$

(both $M_0 \to \dots \to M_1$)

Ex $\mathrm{Don}^{\#}(M) := \mathrm{Don}^{\#}(pt, M)$

Thm [WW '10] $\mathrm{Symp} \longrightarrow \mathrm{Cat}$ is a 2-functor:

$$M \longmapsto \mathrm{Don}^{\#}(M)$$

$$\mathrm{Don}^{\#}(M_0, M_1) \overset{\text{Functor}}{\longmapsto} \mathrm{Fun}(\mathrm{Don}^{\#}(M_0), \mathrm{Don}^{\#}(M_1))$$

respecting composition, i.e. if

$$\left(M_0 \xrightarrow{L_{01}} M_1\right) \longmapsto \left(\mathrm{Don}^{\#}(M_0) \xrightarrow{\Phi(L_{01})} \mathrm{Don}^{\#}(M_1)\right)$$

then

by natural transformation

$$\Phi(L_{01}) \circ \Phi(L_{12}) \simeq \Phi(L_{01} \circ L_{12})$$

Work in progress (Abouzaid, Bottman)

Upgrade to $(A_\infty, 2)$-functor

replacing $\mathrm{Don}^{\#}$ by $\mathrm{Fuk}^{\#}$

and functors by A_∞-functors.

Is already partially done by WW

4. Example (Gao '18)

Let $U \subset M$ be a Liouville subdomain, e.g.

Then for $i : U \hookrightarrow M$,

get $gr(i)^T \subset M^- \times U$ is Lagrangian.

$\overset{WW}{\Rightarrow}$ Get $WFuk^\#(M) \to WFuk^\#(U)$

$\overset{Gao}{\Rightarrow}$ Actually get

$$WFuk(M) \to WFuk^{imm}(U)$$

5. Further applications (further talks)

- Transfer of non-displaceability results
- Floer theory of family Dehn twists
- Floer Gysin sequence
- Floer theory of symplectic reduction
- HMS : Products of sheaves $\leftrightarrow$ operation with Lag corresp.

  AG: B-side

  SG: A-side