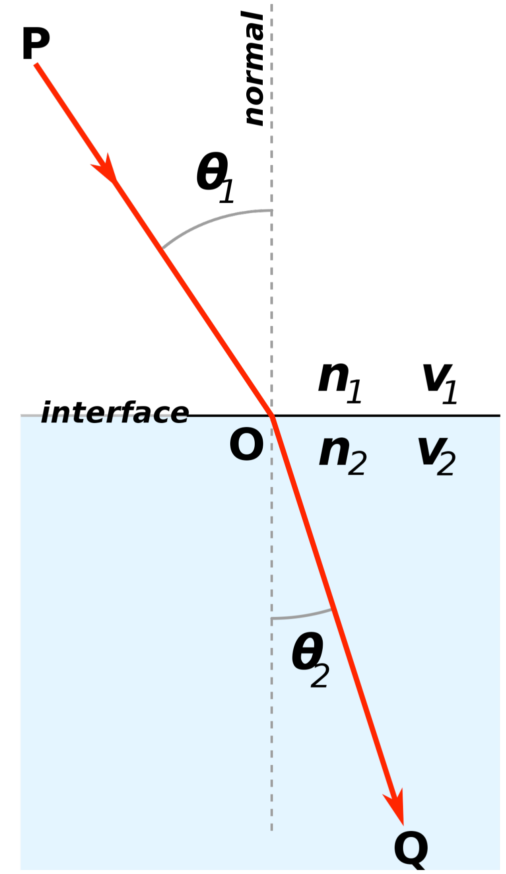
On a proof of the Tree conjecture for triangle tiling billiards

Olga Paris-Romaskevich
University of Rennes 1

Swiss Knots
Zürich, 16 July 2019

Tiling billiards and Snell's law of refraction

1. Water



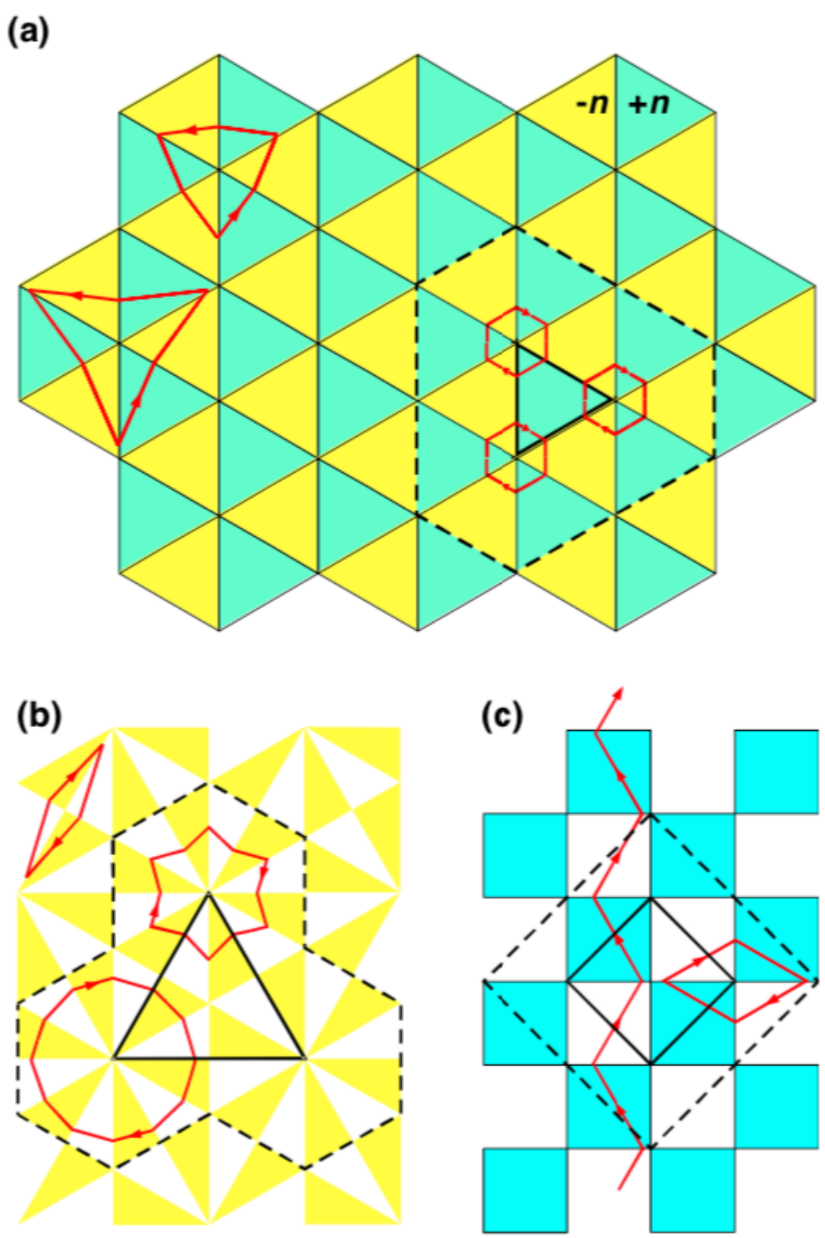
Snell's Law $$\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

Example. The refraction coefficient for water : k=1/1.333

Tiling billiards and Snell's law of refraction

2. A material with refraction coefficient k=-1

Tiling billiards is a dynamical system of movement in a tiling of a plane, in which the coefficient of refraction between two neighbouring tiles is equal to k=-1.



Two basic examples :
square and equilateral triangle tiling

- any trajectory passes by any tile at most once
- all bounded trajectories are periodic (periods: 4 and 6)
- all periodic trajectories are stable under perturbation

Goal : understand the dynamical properties of tiling billiards in different tilings (dynamics strongly depends on a form of the tiling)

© a non-published preprint by A. Mascarenhas, B. Fluegel

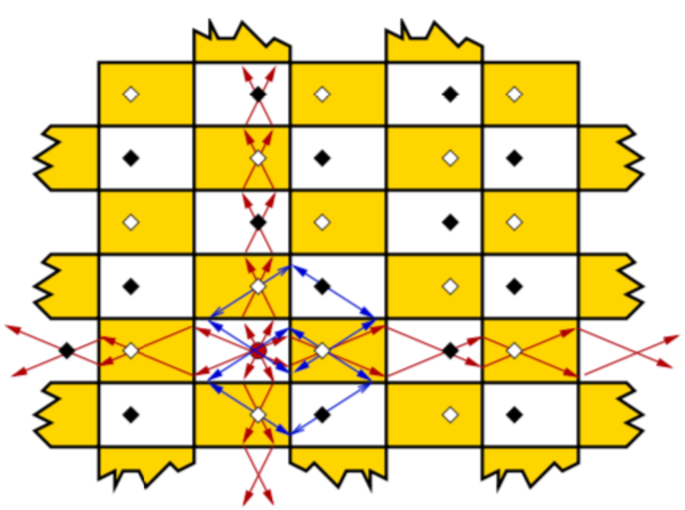
Antisymmetry and the breakdown of Bloch's theorem for light (2015)

Tiling billiards and Snell's law of refraction

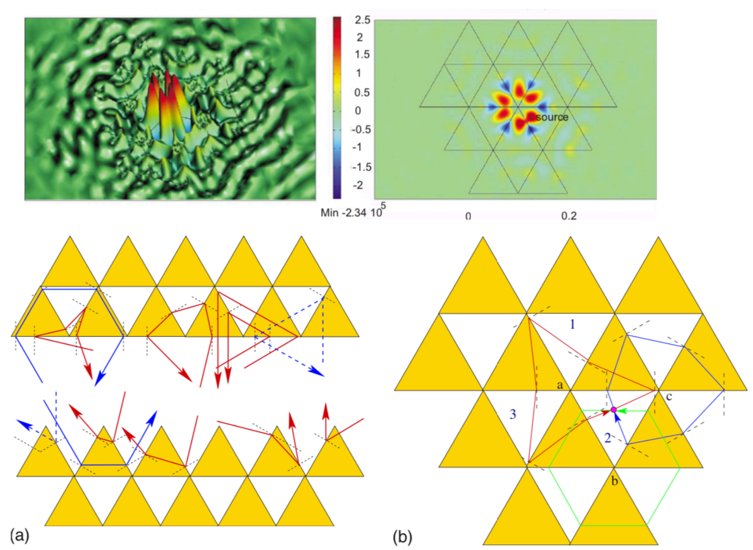
3. How physically relevant is k=-1 ?

It happens that materials with k=-1 can be (quiet simply) constructed, and their properties are intensively studied by physicists at this very moment. For example, slabs of « photonic crystals » can have this property. It is necessary that any negative refractive index material is strongly dispersive with frequency. The periodic trajectories in the tiling billiards model correspond to the resonances in the full wave picture, which are important for super resolution. Negative refraction materials have many interesting properties, not yet completely understood (one of the applications: invisibility cloaks !).

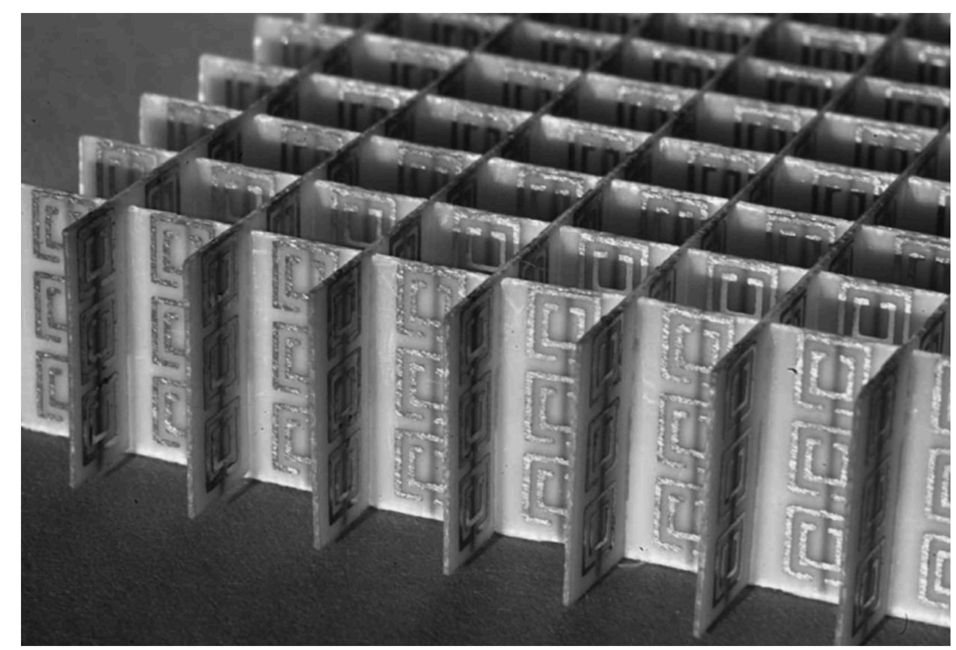
Mathematical community learned about the idea of tiling billiards from the work by A. Mascarenhas, B. Fluegel *Antisymmetry and the breakdown of Bloch's theorem for light* (2015?) but it is certainly not the first work on the subject…



© S. Guenneau, S. Anantha Ramakrishna, Amar C. Vutha, J.B. Pendry *Negative refraction in 2-D checkerboards related by mirror anti-symmetry and 3-D corner lenses (2007)*



© S. Anantha Ramakrishna, S. Guenneau, S. Enoch, G. Tayeb, B. Gralak *Confining light with negative refraction in checkerboard metamaterials and photonic crystals (2007)*



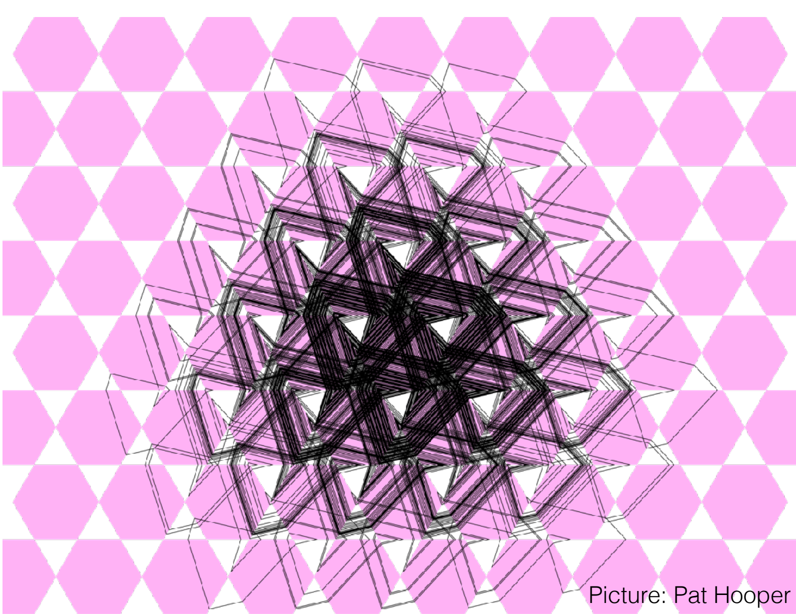
© A split ring structure etched into copper circuit board plus copper wires giving negative refraction material, from J. Pendry *Negative refraction (2009)*

Tiling billiards: mathematical bibliography

Tiling billiard trajectory
— in Matisse's collage

Henri Matisse The Snail (1953)

Tiling billiard trajectory
— in trihexagonal tailing



The following « mathematical bibliography» is,
as far as I know, complete.

1. D. Davis, K. DiPietro, J. Rustad, A. St Laurent *Negative refraction and tiling billiards (2016)*
2. P. Glendinning *Geometry of refractions and reflections through a biperiodic medium* (2016)
3. D. Davis, P. Hopper *Periodicity and ergodicity in the trihexagonal tiling (2016)*
4. P. Baird-Smith, D. Davis, E. Fromm, S. Iyer *Tiling billiards on triangle tilings, and interval exchange transformations* (2019)
5. P. Hubert, O. Paris-Romaskevich *Triangle tiling billiards draw fractals only if aimed at the circumcenter* (2019)
6. O. Paris-Romaskevich *On a proof of the Tree conjecture for triangle tiling billiards* (2019+), preprint

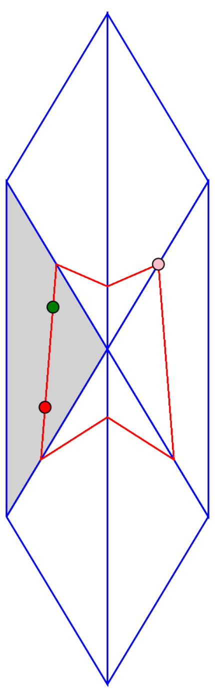
I learned about tiling billiards in a talk by Diana Davis
on the 14th February 2017 in CIRM, Marseille.

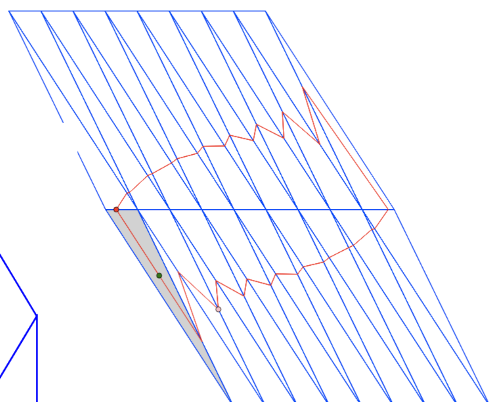
A dynamical system: triangle tiling billiard.

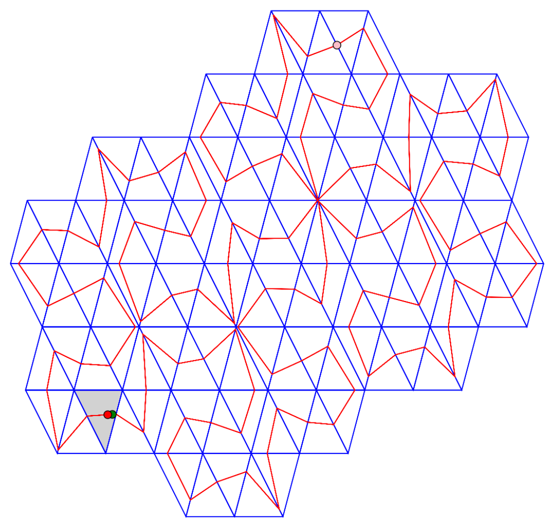
periodic triangle tiling obtained from the equilateral tiling by a linear transformation +
dynamics of light in the heterogeneous medium with k=-1, for each tile-to-tile transition

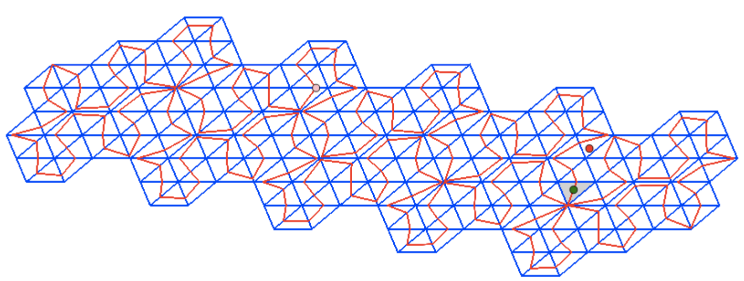
Examples of trajectories and qualitative behavior.

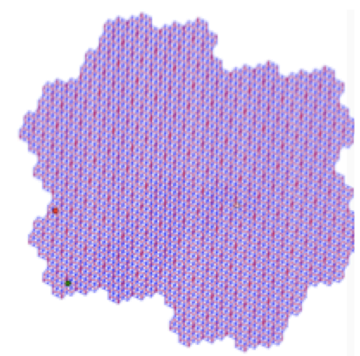
Periodic trajectories





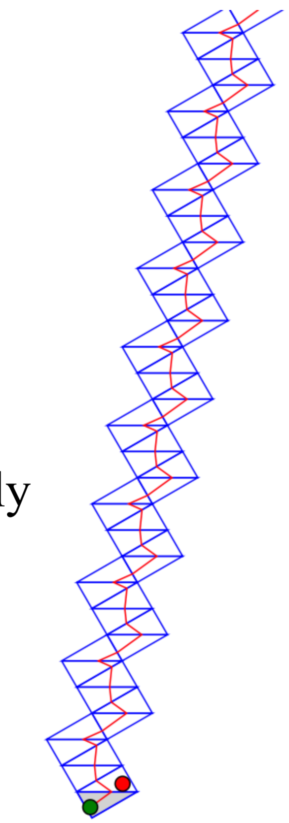




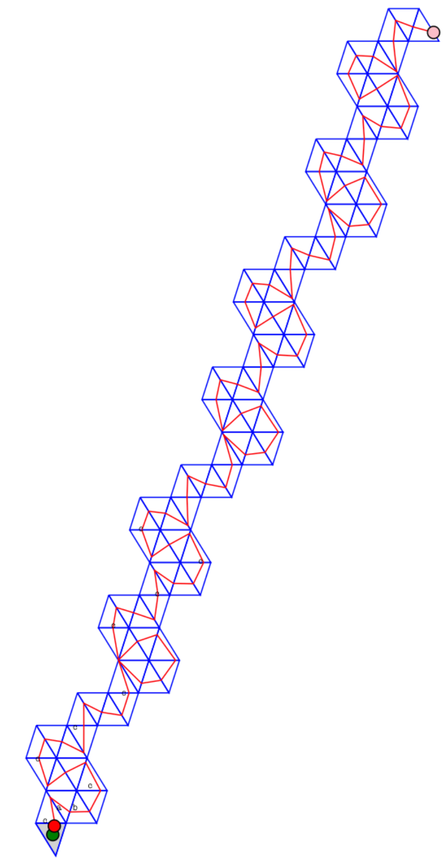


approaching non-linearly escaping trajectory

Linearly escaping trajectories



periodic drift



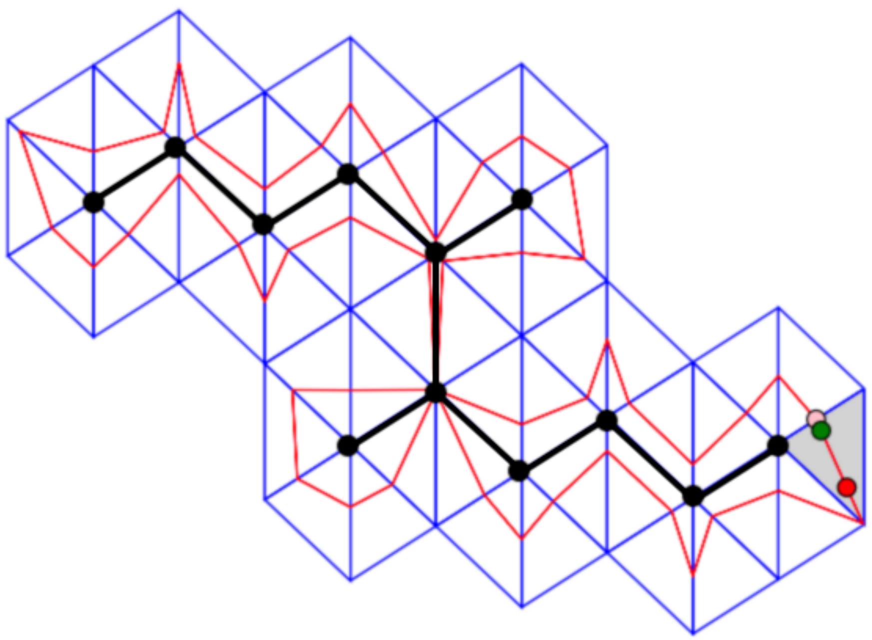
non-periodic drift

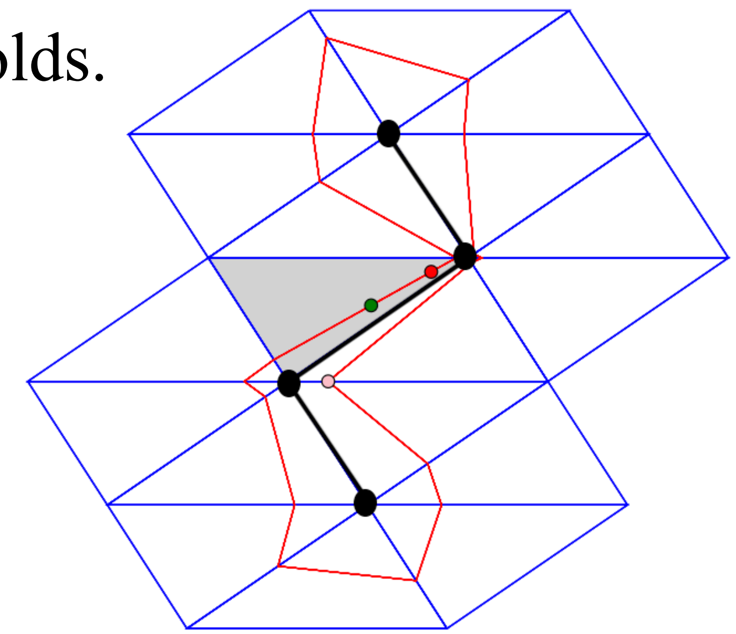
Theorem. For a triangle tiling billiard,
1.(P. Baird-Smith, D. Davis, E. Fromm, S. Iyer) Any trajectory is either **periodic**, or **linearly escaping**, or **non-linearly escaping**.
(moreover, the three basic properties stated for quadrilateral and equilateral triangle tiling billiards still hold)
2. (P. Hubert, O. P.-R.) **Almost any** trajectory is either periodic or linearly escaping.

A result : Tree conjecture holds.

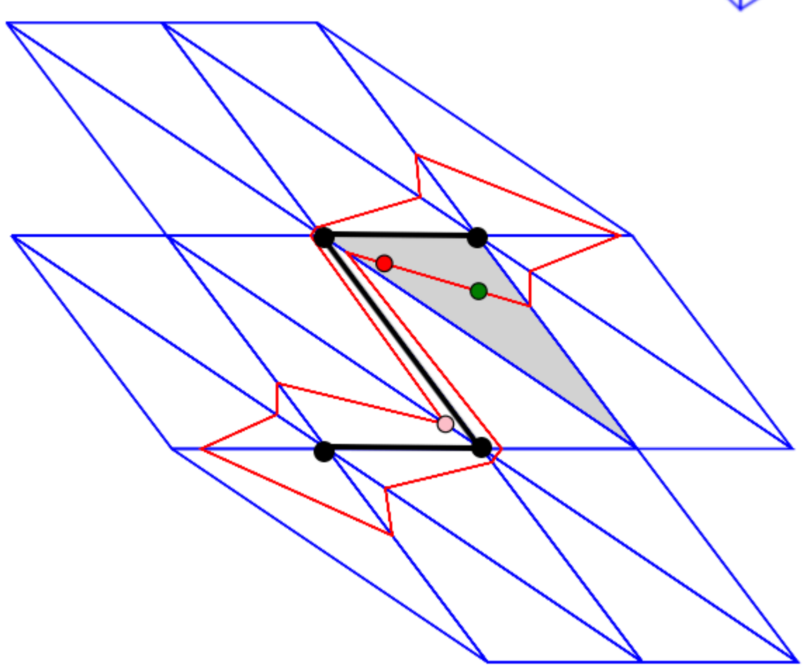
Conjecture. (P. Baird-Smith, D. Davis, E. Fromm, S. Iyer)
Any periodic trajectory in a triangle tiling billiard doesn't contour triangles.

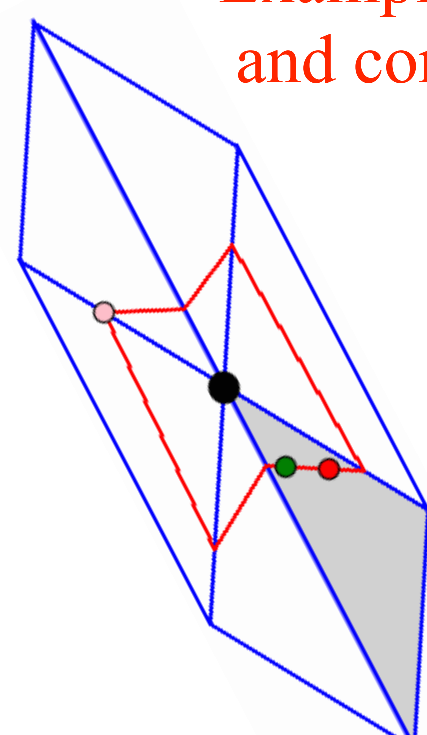
Theorem. (O. P.-R.)
The Tree Conjecture for triangle tiling billiards holds.

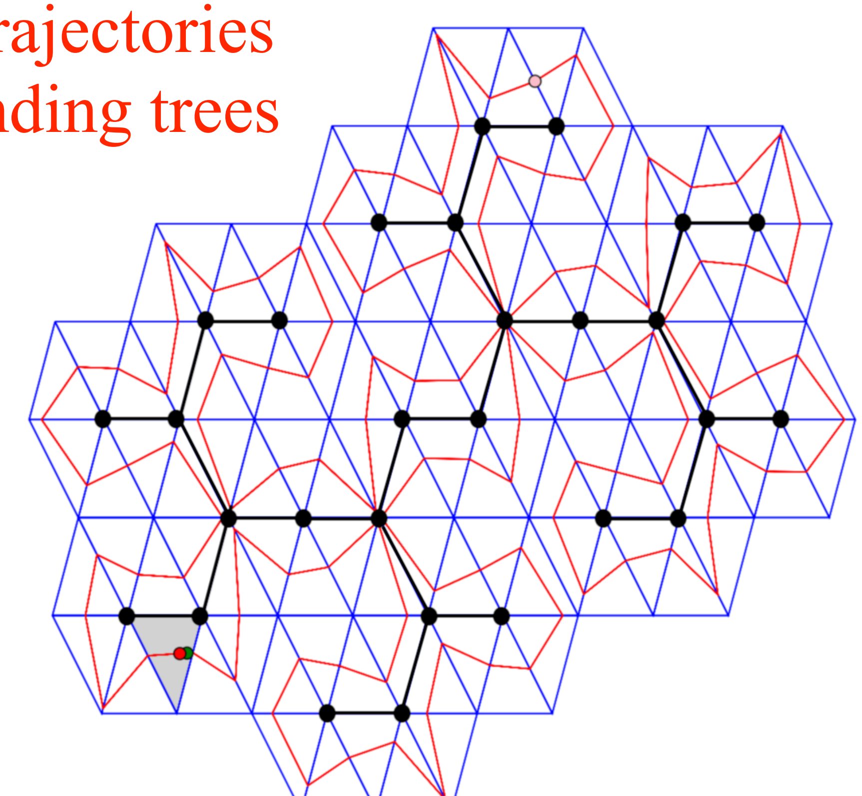




Examples of trajectories
and corresponding trees







Why care about the Tree conjecture ?...

Triangle tiling billiards ⟷ (D., B.-S., F., I.) ⟷ Fully flipped interval exchange transformations on the circle

Parametrized families of interval exchange transformations & families of translation surfaces

Non-linearly escaping trajectories of triangle tiling billiards ⟷ (D., B.-S., F., I. H., P.-R.) ⟷ Arnoux-Rauzy family of interval exchange transformations, Arnoux-Yoccoz surface, Rauzy gasket, Rauzy fractal, etc.

Triangle tiling billiards happen to be related to rich objects that have been studied for the last thirty years (Arnoux, Avila, Hooper, Hubert, McMullen, Rauzy, Skripchenko, Yoccoz, Weiss,…)

Tree conjecture is related to the symbolic behaviour of periodic leaves on Arnoux-Rauzy surfaces. It gives a more precise understanding of arithmetic orbits of Arnoux-Yoccoz map.

Corollary. (O. P.-R.) There exists a triangle tiling billiard trajectories passing by all tiles in a tiling (and converging, up to rescaling, to the Rauzy fractal).

A proof of the Tree conjecture.

Step 1. **Folding** (D., B.-S., F., I.): all the tiled plane can be **folded** into a circle, and a trajectory is folded into a chord in this circle. Such a folding map F is well defined.

Consequences of folding, « integrability » (D., B.-S., F., I.):
Any trajectory passes by any tile at most once; all bounded trajectories are periodic; all periodic trajectories are stable under perturbation

Step 2. **Tiling billiard foliations**

Any trajectory of a triangle tiling billiard can be considered as a leaf of a foliation of the tiled plane - **parallel foliation**, which is constructed as a lift of a parallel foliation on a folded circle. This is an oriented foliation.

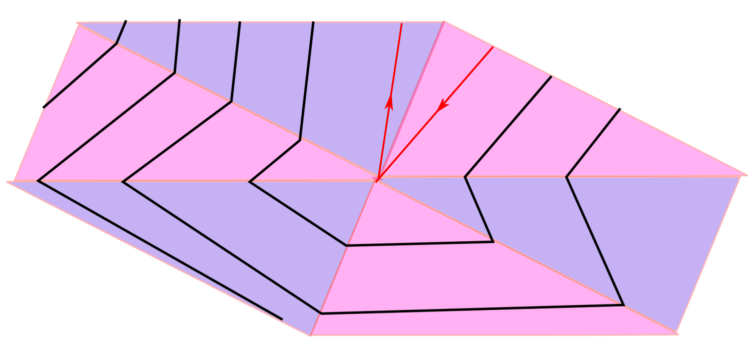
Step 3. **Separatrices**

A separatrix is a leaf of the parallel foliation that passes through some vertex V and such that it follows the billiard law even for two triangles sharing V. We call these leaves **singular trajectories**.
These trajectories are very important in the parallel foliation since they define the symbolic behaviour of all other trajectories.

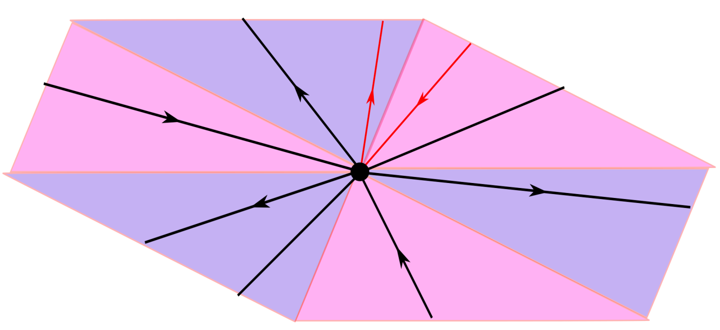
1. Separatrices fold into chords on a folded circle C that pass by a point F(V) on it.
2. Unfolding a chord gives separatrices that follow the billiard law in neighbouring triangles.
3. Every singular trajectory can be included in a unique parallel foliation, and a unique ray foliation.

Step 4. **Ray foliation**

Each oriented separatrix defines a so-called **ray foliation** of the plane which is defined as a lift of a foliation of chords passing by F(V).



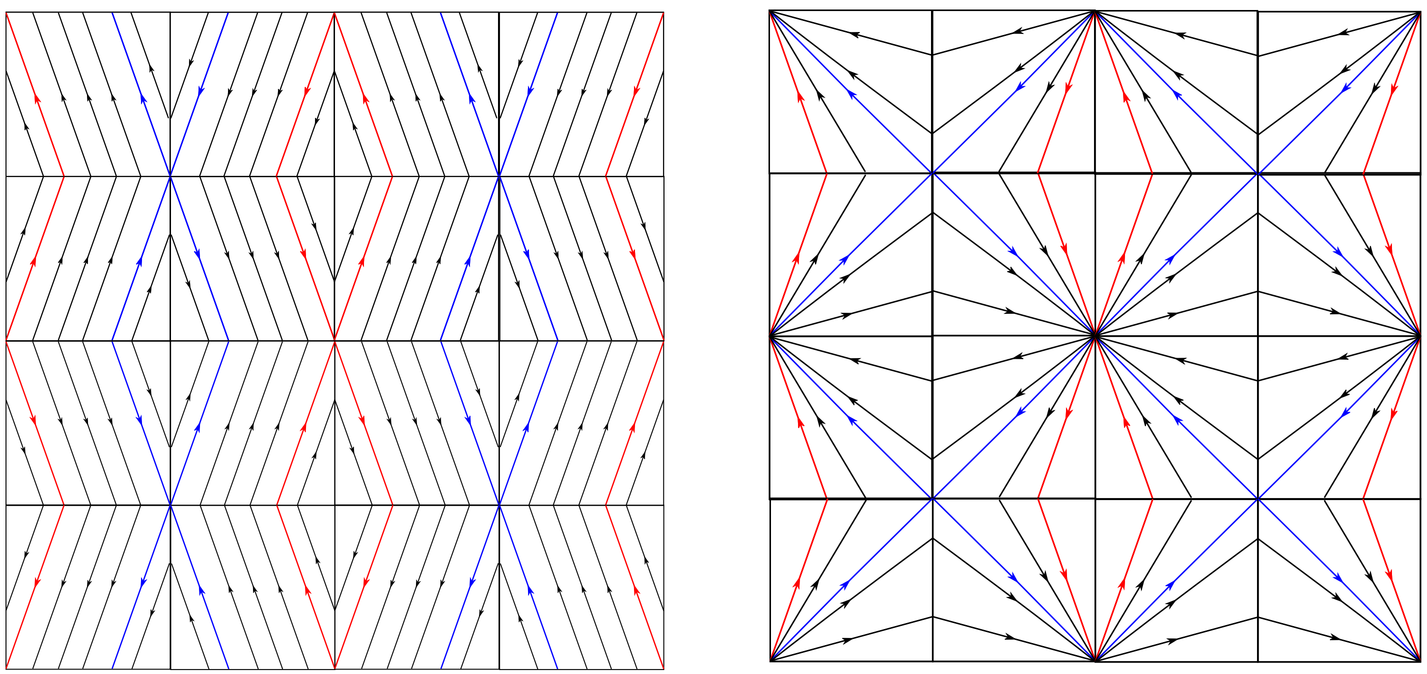
parallel foliation



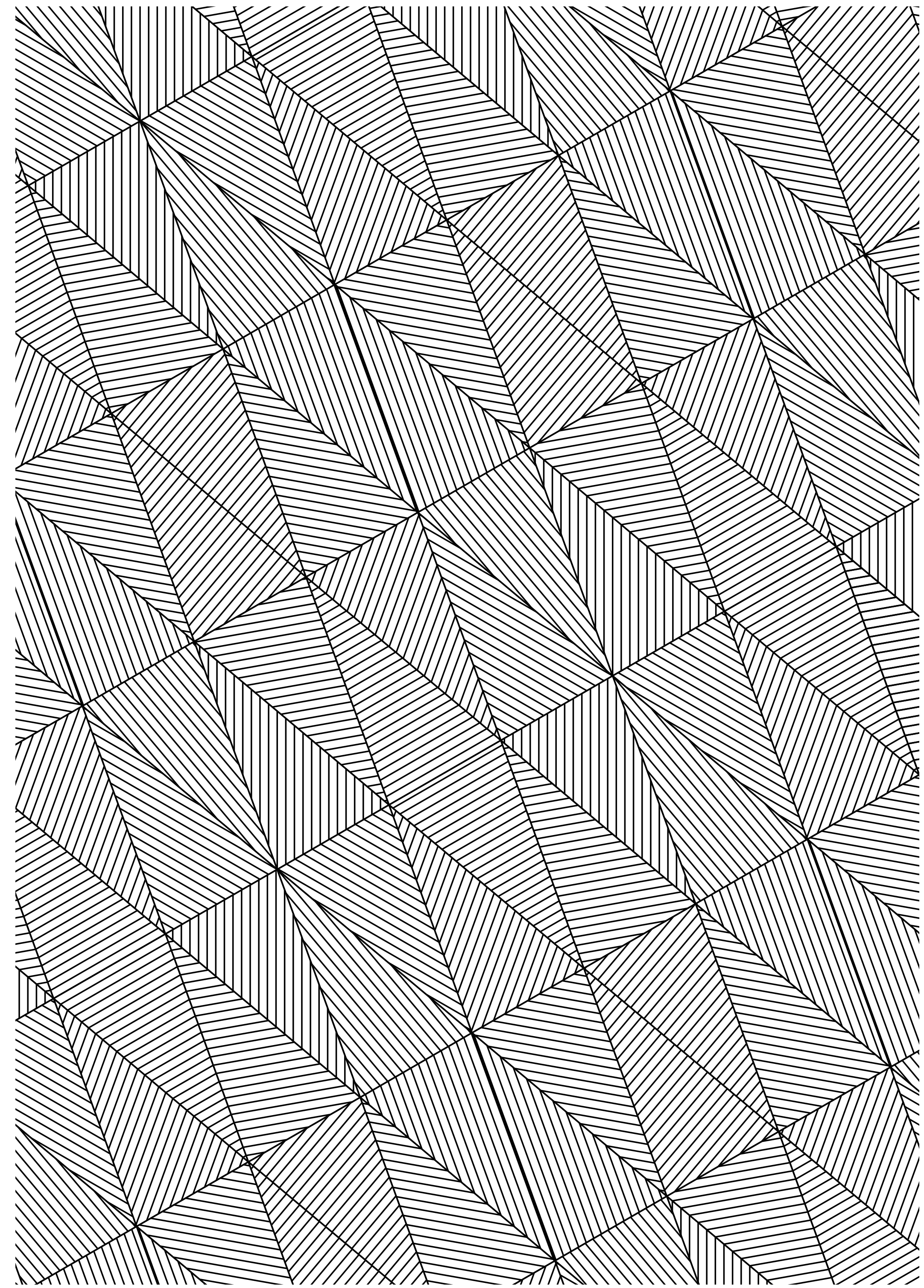
ray foliation

The main idea of the proof of the Tree Conjecture is first, to reduce the study of symbolic behaviour of periodic trajectories to that of singular trajectories. And second, to use both parallel and ray foliation in order to study singular trajectories.

A (very simple) example: parallel and ray foliations on the square tiling are periodic. In red, their common singular leaves.



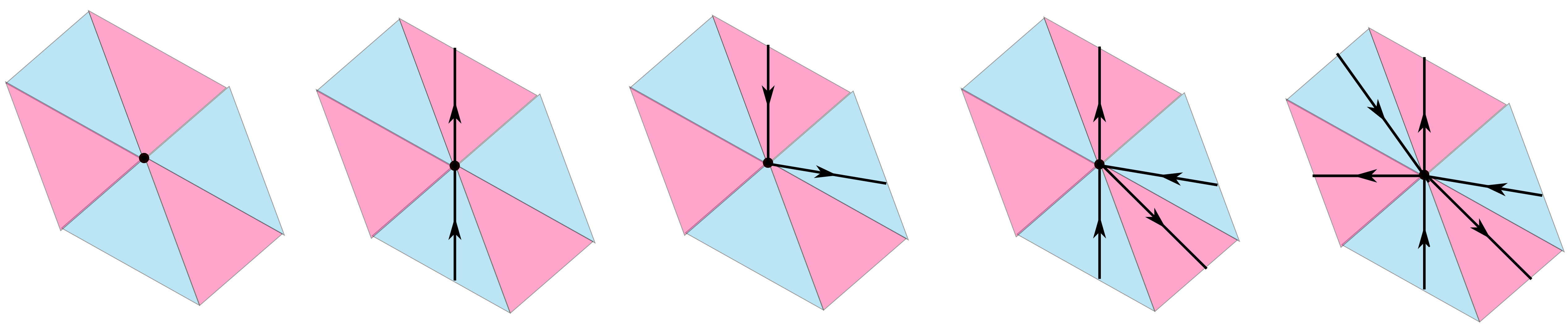
In our case, the parallel foliation for triangle tiling is not periodic but quasi-periodic



Step 5. **Flowers**

A **flower** is a union of all singular trajectories passing by some vertex V that belong to the same parallel foliation. A flower is **bounded** if all of these trajectories are bounded (and hence, periodic). Every bounded singular trajectory (separatrix loop) is called **a petal**.

Proposition. Any flower in V in restriction to the six tiles surrounding V, has one of the five combinatorial behaviours (up to the change of orientation):



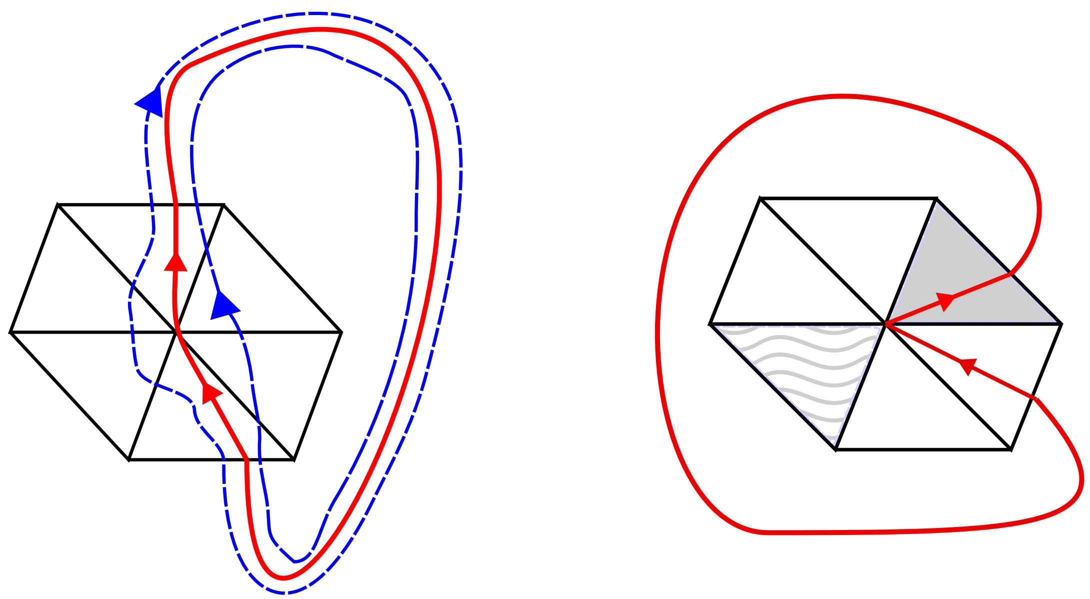
$s = 0$ $s = 2$ $s = 4$ $s = 6$

Proof. The ray foliation in a vertex V has a very simple form in six tiles surrounding the vertex — all the leaves pass through V, and their orientation alternates, from one triangle to its neighbour.

Step 5. **Flower Conjecture**

It happens that the understanding of topological behaviour of bounded flowers suffices to prove the Tree Conjecture.

Flower Conjecture (for triangle tiling billiards).
Consider a (bounded) flower passing by a vertex V. Then, any of its petals passes by two neighbouring tiles. And second, this petal contains an edge between these two tiles inside.



Behaviours excluded by the Flower Conjecture.

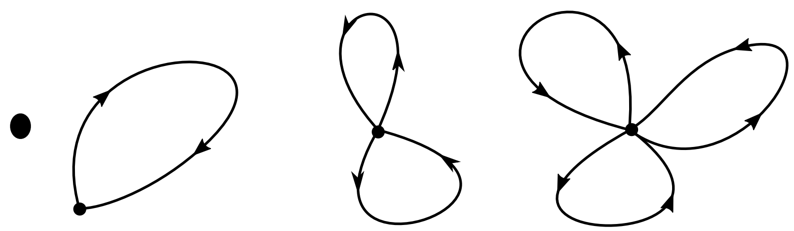
<u>Our plan of the proof of the Tree Conjecture:</u>

Thm 1. Flower Conjecture implies Tree Conjecture.
Thm 2. Flower Conjecture holds.

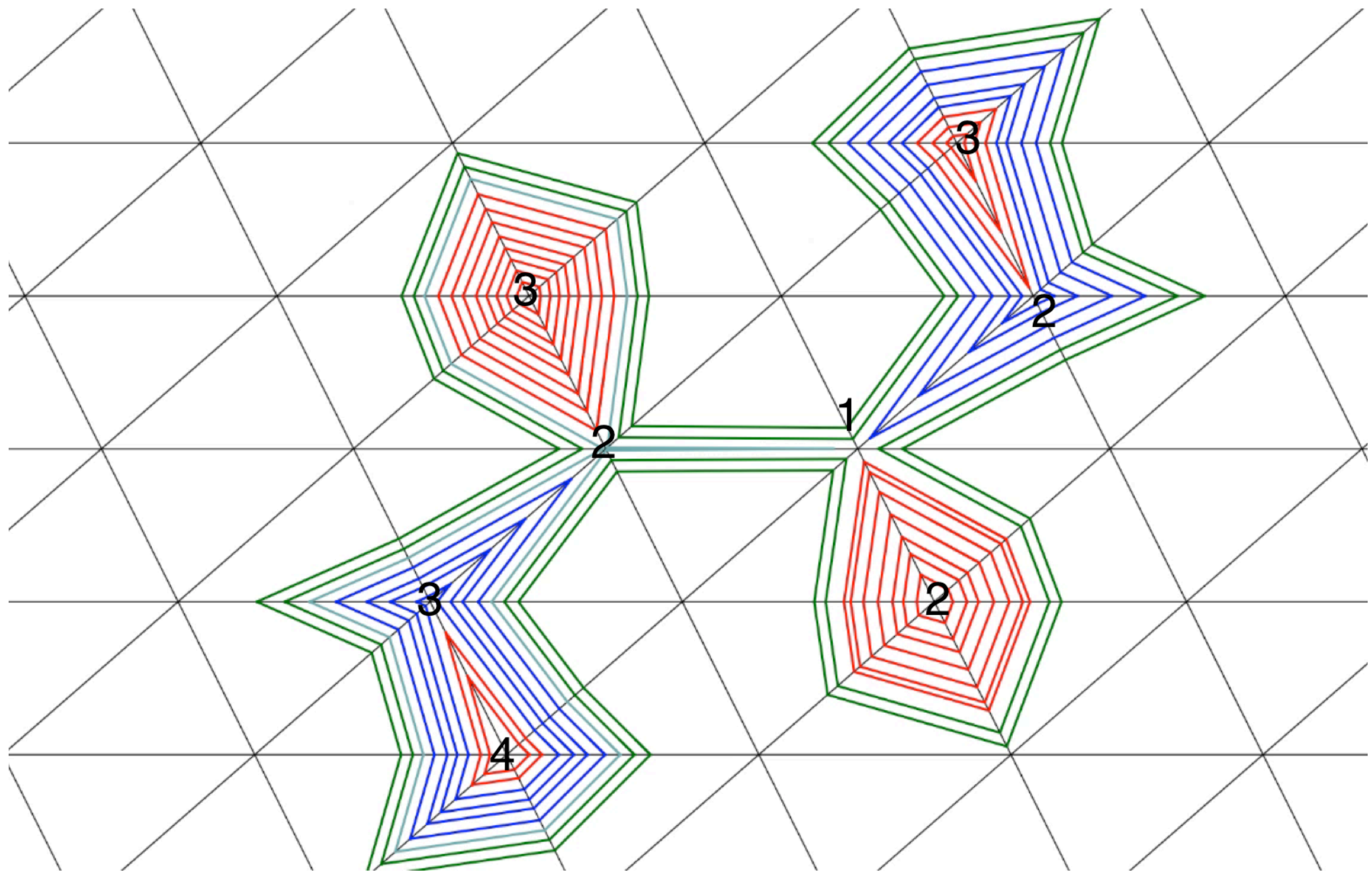
Step 6. **Flower Conjecture implies Tree Conjecture**

Idea of the proof.

1. Consider a periodic trajectory and a corresponding parallel foliation.
2. Take a domain bounded by this trajectory - it is foliated by parallel foliation leaves.
3. Contract a periodic trajectory onto a union of separatrices (which has the same symbolic behavior). If the Flower Conjecture holds, four combinatorial cases for this union are possible.

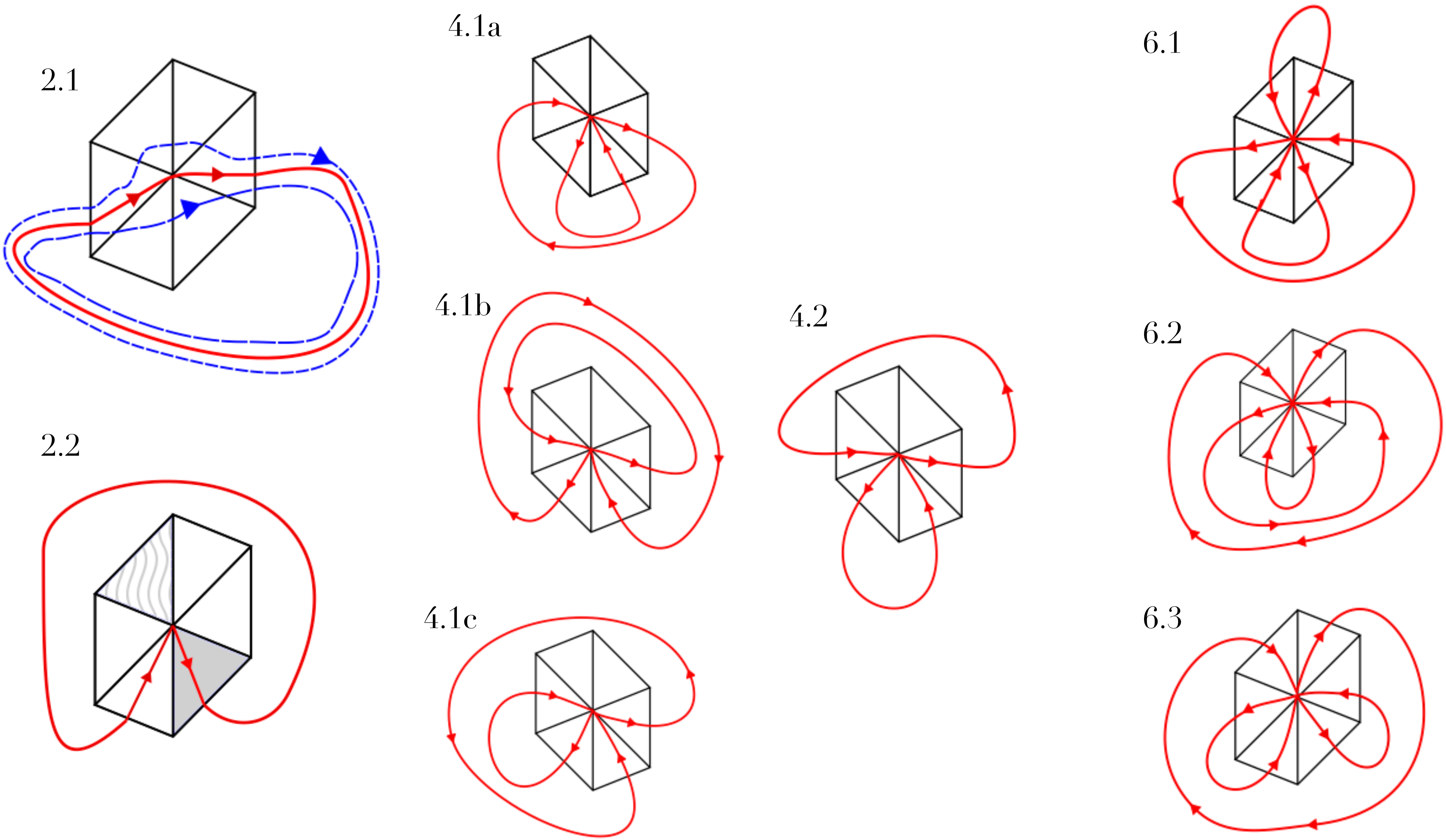


4. Take periodic trajectories inside each of the petals, and proceed by recurrence.



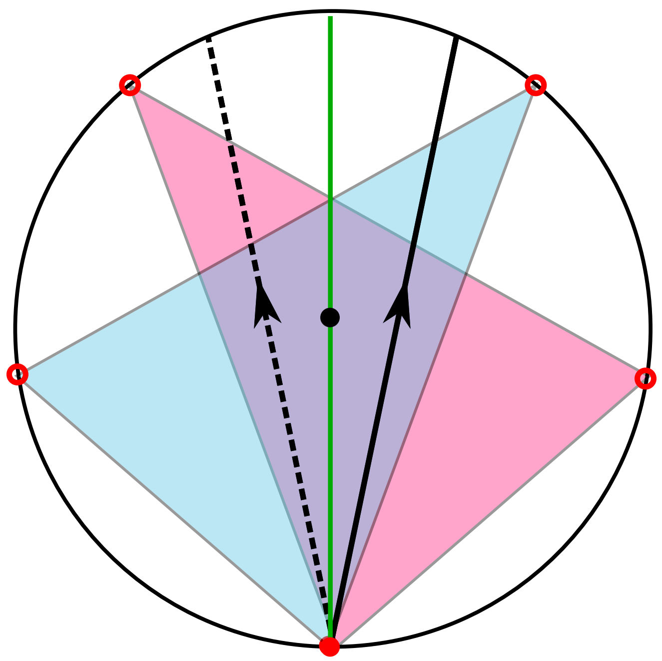
Step 7. **A proof of the Flower Conjecture.**

There exists a finite list of possibilities of topological obstructions to the Flower conjecture. We exclude all of them by using two main tools (proper to triangle tilings) symbolic dynamics and symmetries.

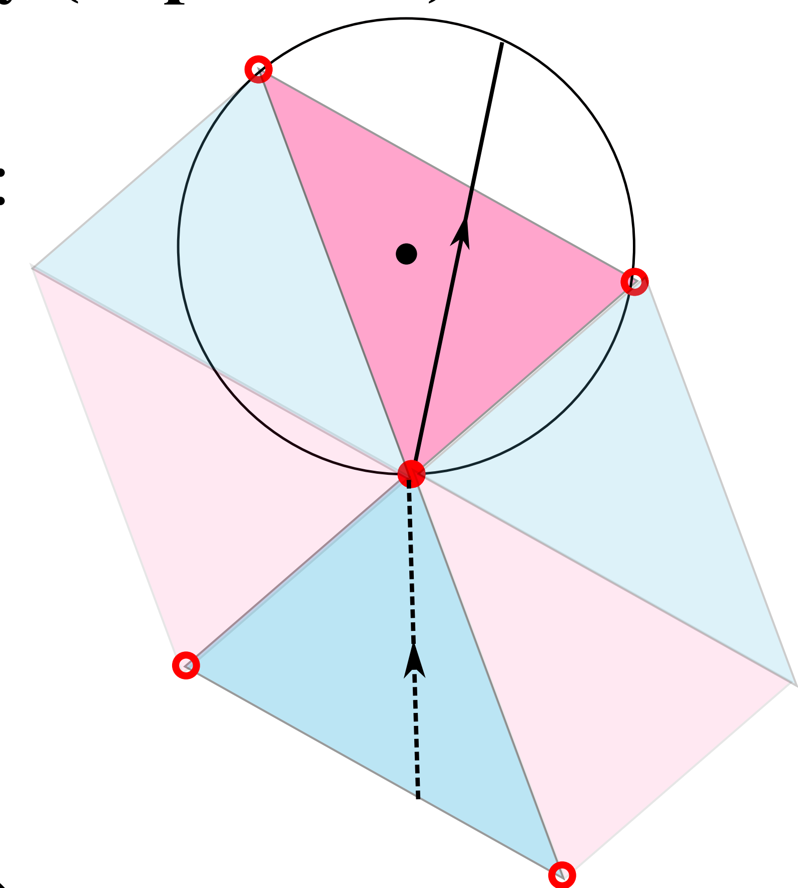


Step 7. **Exclusion of the « Pacman », proof by symmetry (in pictures).**

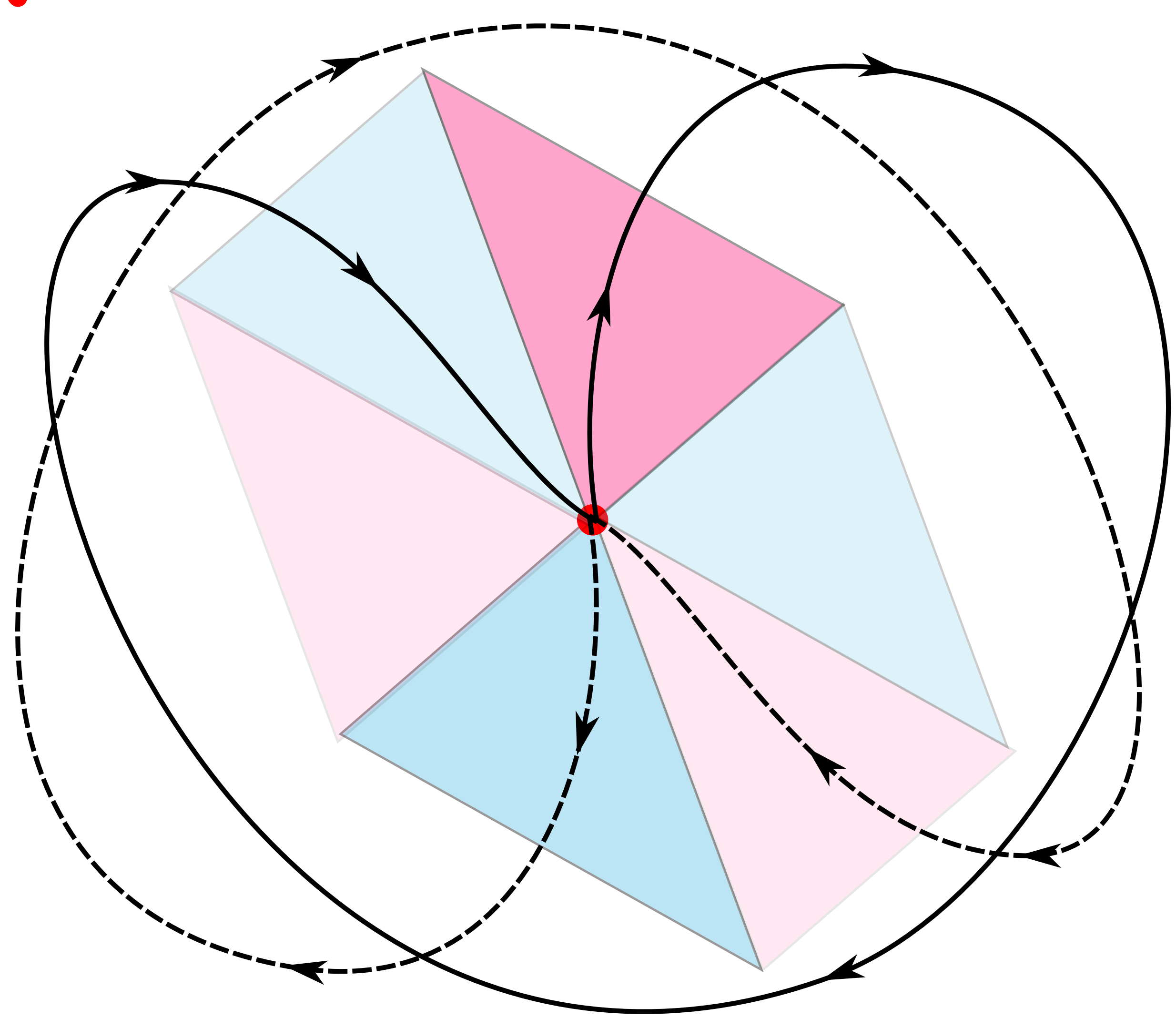
Folding:



Unfold the folding:



Contradiction:



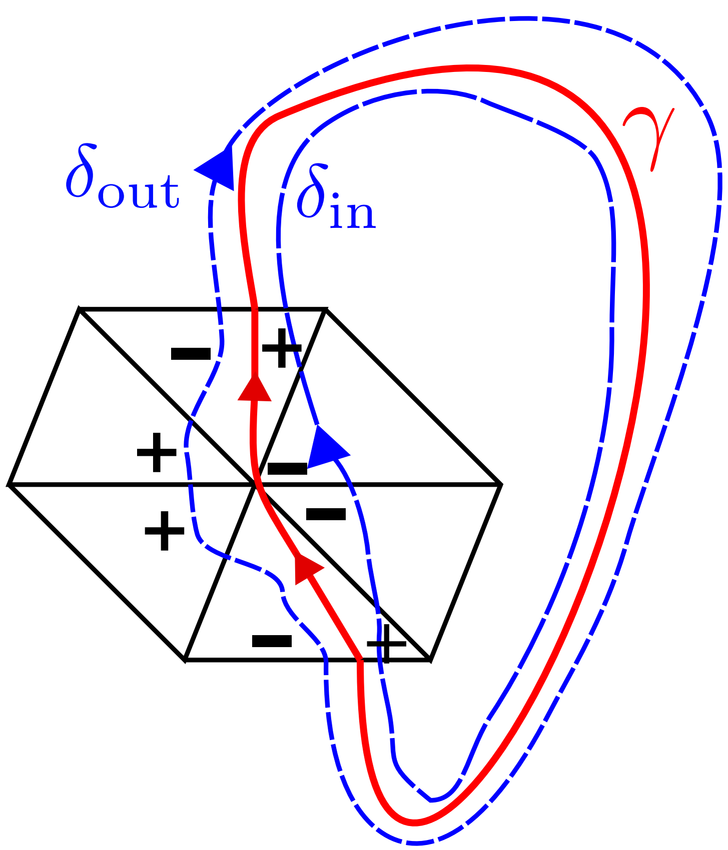
Step 8. **Exclusion of the « Ear », proof by symbolic dynamics.**

Symbolic dynamics of triangle tiling billiards is defined by a coding in the alphabet {a,b,c}. We use the following result.

Thm. (P. H., O. P.-R.) Any periodic trajectory has a symbolic code in the alphabet {a,b,c} which is a **square** of the word of odd length. Hence, any periodic orbit has a period of the form 4n+2.

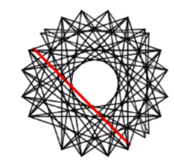
One can consider the symbolic dynamics of triangle tiling billiards as defined by a coding in the alphabet{ab, ba, ac, ca, bc, cb}. Consider a reduced alphabet {+,-} by replacing *ab, bc, ca* by + and *ba, ac, cb* by -.

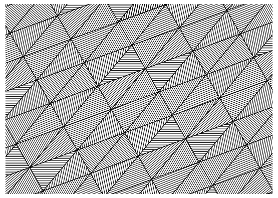
Idea of the proof.
The « Ear » is excluded since periodic trajectories approaching it can't be symbolic squares.

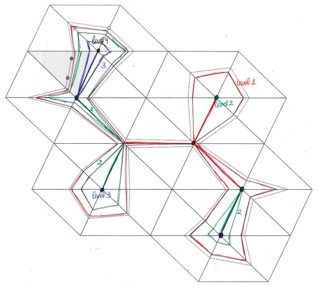


The remaining cases are treated analogously.

Recap of the proof:

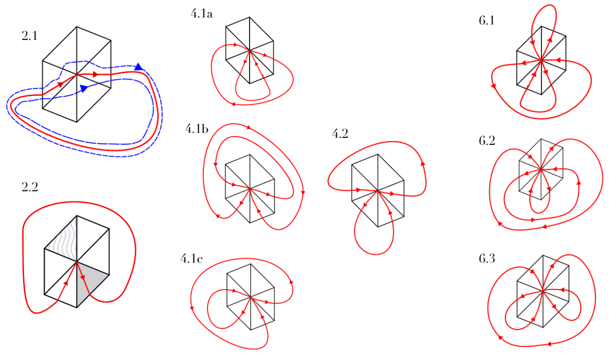




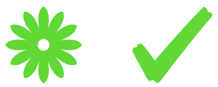


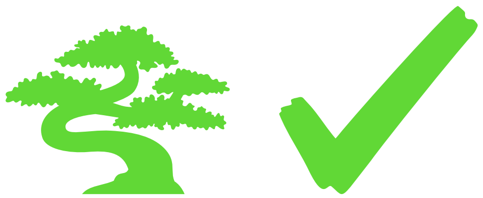
1. Folding is well-defined for triangle tilings
2. Then, parallel and ray foliations are well defined
3. Behaviour of *one periodic trajectory* is reduced to the behaviour of a *finite number of petals* in a corresponding parallel foliation by contraction (if the Flower Conjecture holds…)

4. To prove the Flower Conjecture, we give a finite list of topological obstructions, and exclude the cases one by one from this list (2.1 and 2.2 - Pacman and Ear, we excluded together).



5. Hence Flower conjecture holds.



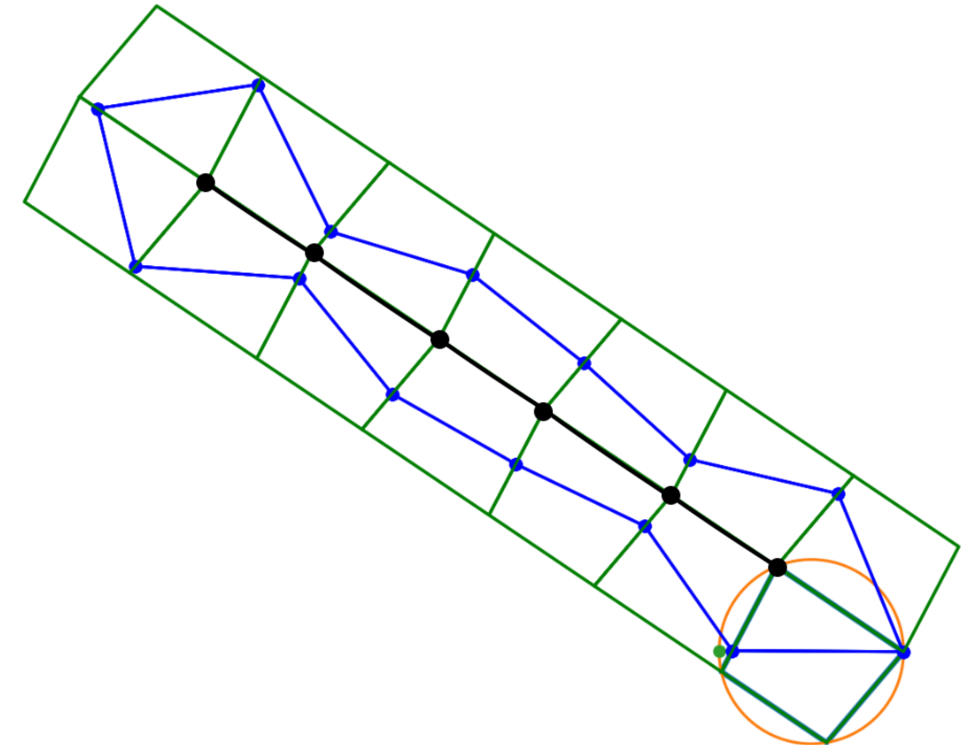


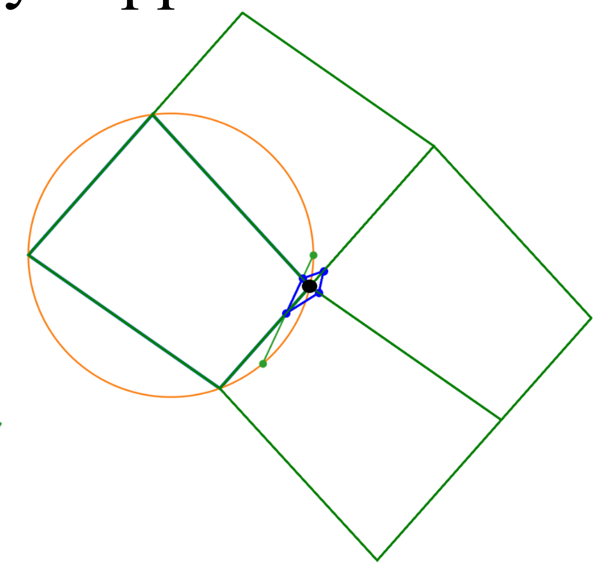
Generalizations and further questions

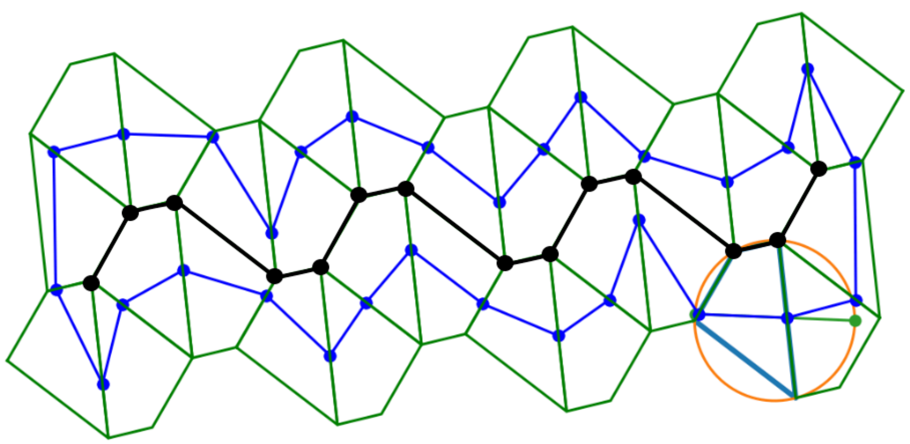
— Density property

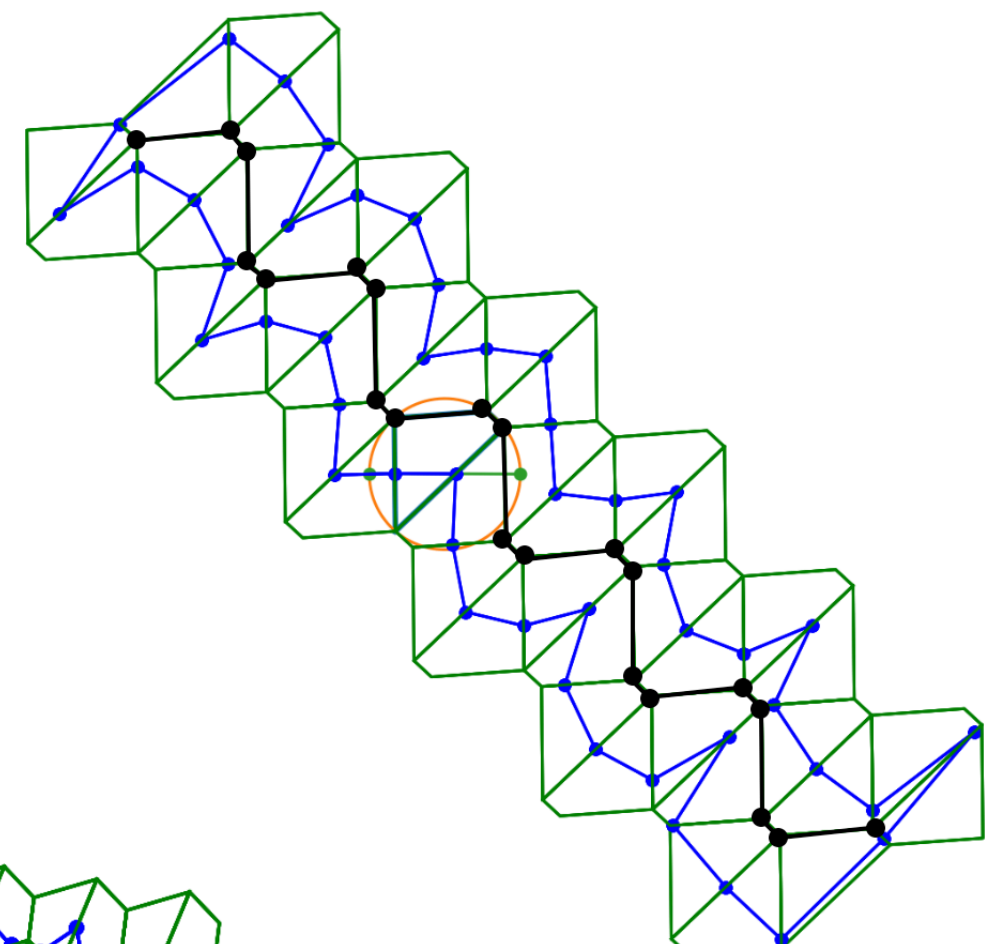
— Cyclic quadrilaterals : ?

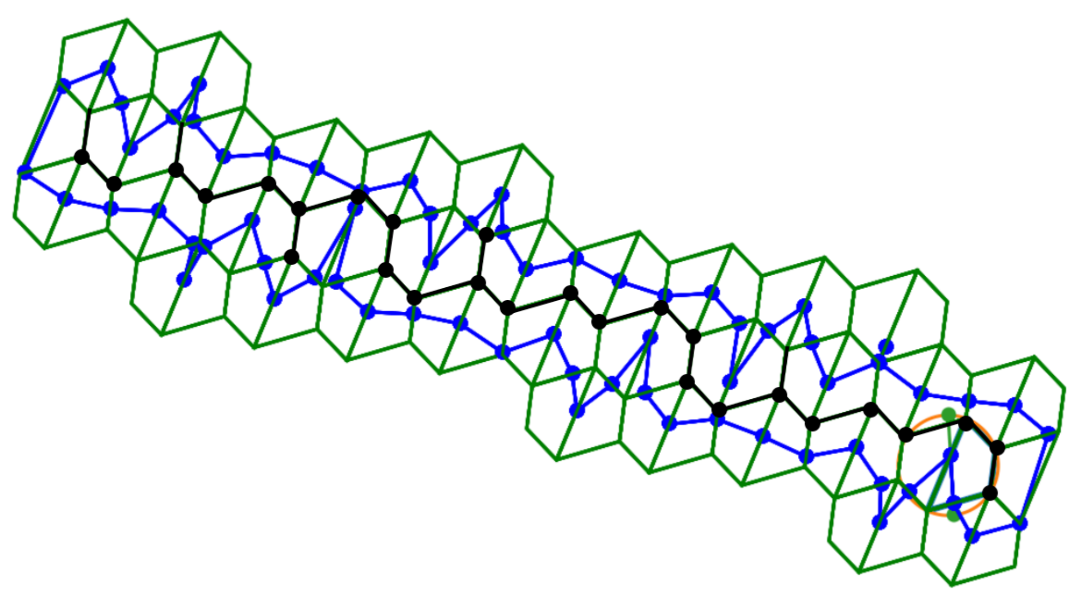
— Symbolic dynamics of fully flipped IET ?





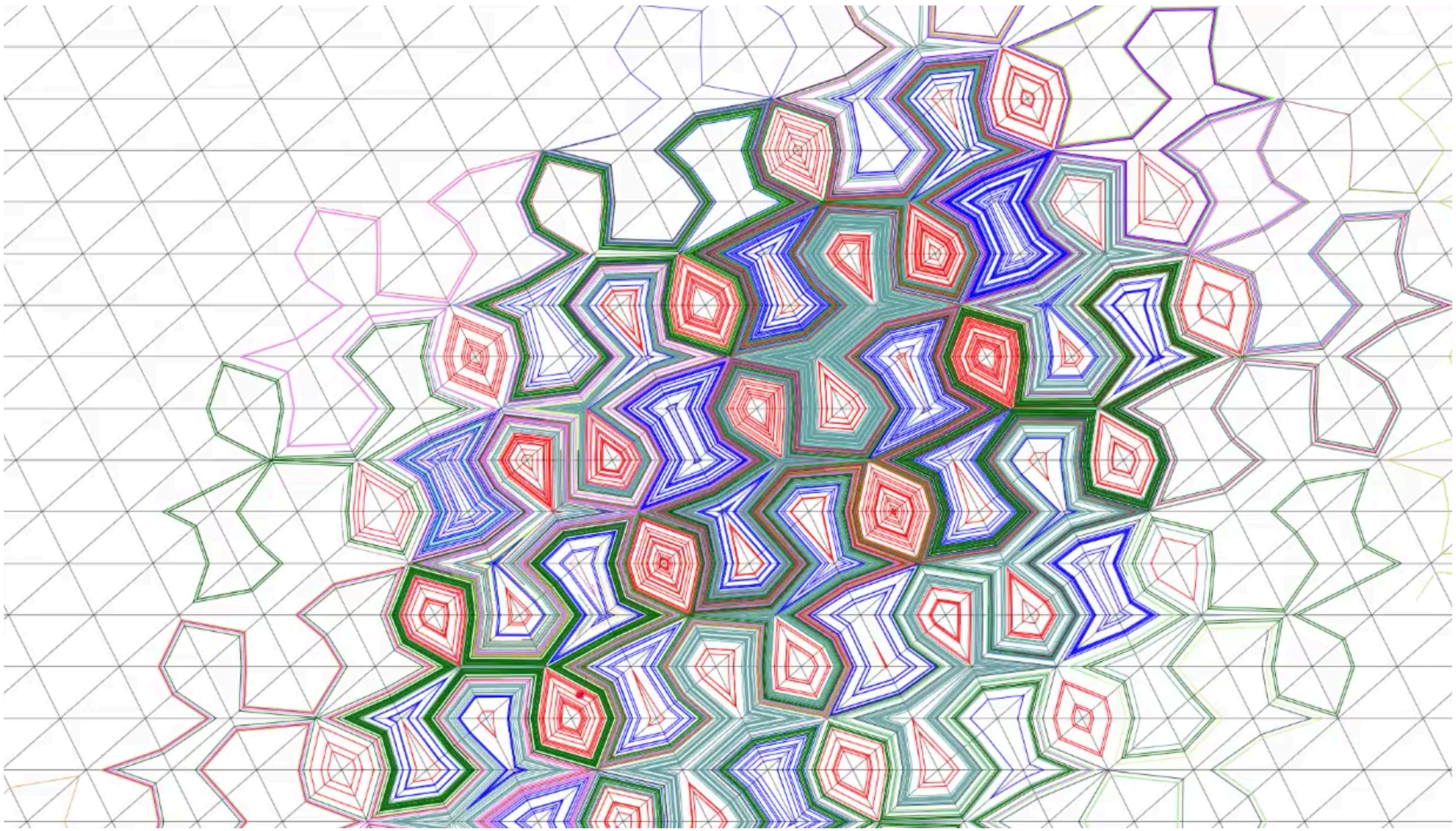






A last video: parallel foliation in motion

Thm. (O. P.-R.) There exists a triangle tiling billiard trajectories passing by all tiles in a tiling (and converging, up to rescaling, to the Rauzy fractal). This trajectory is included in a parallel foliation where all other trajectories are periodic.



© picture by Ofir David

Vielen Dank für die Aufmerksamkeit!

Note for the reader. Dear reader, this is a condensed version of the slides showed at my talk at *Swiss Knots*.
In order to access the videos showed in the talk, please contact me directly at olga@pa-ro.net
And also please follow the news — possibly and hopefully, we will soon create a full video illustrating the proof of the Tree Conjecture (ongoing collaboration with Ofir David).
In order to access the paper, you can find it on arxiv by following the link:
O. Paris-Romaskevich, *On a proof of the Tree Conjecture for triangle tiling billiards*